A guided tour to normalized volume

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Dedicated to Gang Tian's Sixtieth Birthday with admiration

Abstract. This is a survey on the recent theory on minimizing the normalized volume function attached to any klt singularities.

Contents

1. Introduction

The development of algebraic geometry and complex geometry has interwoven in the history. One recent example is the interaction between the theory of higher dimensional geometry centered around the minimal model program (MMP), and the existence of 'good' metrics on algebraic varieties. Both subjects have major steps forward, whose influences are beyond the subjects themselves, spurring out new progress in topics once people could not imagine. In this note, we will discuss a 'local stability theory' of singularities, which in our opinion provides an excellent example on the philosophy that there are many unexpected connections underlying these two different topics.

Ever since the starting of the theory of MMP in higher dimensions (that is, the dimension is at least three), people understand that a feature of such a theory is that we need to deal with singular varieties. Then it becomes very nature to investigate this class of singularities for people working on the MMP. To deal with singular varieties in complex geometry is a more recent trend, and it significantly improves people's knowledge on the existence of interesting metrics, even in situations which people originally only want to study smooth varieties.

It becomes clear now, Kawamata log terminal (klt) singularities form an exceptionally important class of singularities for many reasons: it is the natural class of singular varieties for people to inductively prove deep results in the MMP; it is the class of singularities appearing on degenerations in many natural settings and it carries properties which globally Fano varieties have.

What we want to survey here is a rather new theory on klt singularities. The picture consists of two closely related parts: firstly, we want to establish a

structure which provides a canonically determined degeneration to a stable log Fano cone from each klt singularity; secondly, to construct the degeneration, we need a valuation which minimizes the normalized volume function on the 'non-archimedean link', and since such minimum is a deep invariant defined for all klt singularities, we want to explore more properties of this invariant, including calculating it in many cases.

1.1. History

The first prototype of the local stability theory underlies in [MSY06, MSY08]. They find that the existence of Ricci-flat cone metric on an affine variety with a good action by a torus group T is closely related to the normalized volume minimizing problem. In our language, they concentrate on the valuations induced by the vectors in the Reeb cone provided by the torus action. Later a systematic study of K-stability in the setting of Sasaki geometry is further explored in [CS18, CS15].

Consider klt singularities which appear on the Gromov-Hausdorff (GH) limit of Käher-Einstein Fano manifolds. At the first sight, we do not know more algebraic structure for these singularities. Nevertheless, by looking at the metric tangent cone, it is shown in [DS17], built on the earlier works in [CCT02, DS14, Tia13], that the metric tangent cone of such singularities is an affine T-variety with a Ricci-flat cone metric. Furthermore, [DS17] gives a a two-step degeneration description of the metric tangent cone. They further conjecture that this two-step degeneration should only depend on the algebraic structure of the singularity, but not the metric.

Then in [Li18a], the normalized volume function on the 'non-archimedean link' of a given klt singularity is defined, and a series of conjectures on normalized volume function are proposed. This attempt is not only to algebrize the work in [DS17] without invoking the metric, but it is also of a completely local nature. Since then, the investigation on this local stability theory points to different directions.

In [Blu18], the existence of a minimum (opposed to only infimum) which was conjectured in [Li18a] is affirmatively answered. The proof uses the properness estimates in [Li18a] and the observation in [Liu18] that the minimizer can be computed by the minimal normalized multiplicities, and then skillfully uses the techniques from the study of asymptotic invariants (see [Laz04]). Later in [BL18], lower semicontinuity of the volume of singularities are also established using this circle of ideas.

In [Li17, LL19], the case of a cone singularity over a Fano variety is intensively studied, and it was found if we translate the minimizing question for the canonical valuation into a question on the base Fano varieties, what appears is the sign of the β-invariant developed in [Fuj18, Fuj16, Li17].

Built on the previous study of cone singularities, implementing the ideas circled around the MMP in birational geometry, an effective process of degenerating a general singularity to a cone singularity is established in [LX16], provided the minimizer is a divisorial valuation. In [LX17a], a couple of conjectural properties are added to complete the picture proposed in [Li18a], and now the package is called

'*stable degeneration conjecture*', see Conjecture 4.1. The investigation in [LX16] is also extended in [LX17a] to the case when the minimizer is a quasi-monomial valuation with a possibly higher rational rank, where the study involves a considerable amount of new techniques. As a corollary, the first part of Donaldson-Sun's conjecture in [DS17] is answered affirmatively in [LX17a]. Later the work is extended in [LWX18] and a complete solution of Donaldson-Sun's conjecture is found.

Applications to global questions, especially the existence of KE metrics on Fano varieties, are also explored. In [Liu18], built on the work of [Fuj18], an inequality to connect the local volume and the global one is proved. Then in [SS17, LX17b], via the approach of the 'comparison of moduli', complete moduli spaces parametrizing explicit Fano varieties with a KE metric are established by studying the local constraint posted by the lower bound of the local volumes.

1.2. Outline

In the note, we will survey a large part of the results mentioned above. From the perspective of techniques, there are three closely related ways to think about the volume of a singularity: the infima of the normalized volume of valuations, of the normalized multiplicity of primary ideals or of the volume of models. The viewpoint using valuations gives the most canonical picture, e.g. the stable degeneration conjecture, but there are less techniques available to directly study the space. The viewpoint using ideals is flexible for many purposes, e.g. taking degenerations. Moreover, though usually working on a single ideal does not give too much advantage over others, working on a graded sequence of ideals really enables one to use the powerful theory on asymptotical invariants for such setting. The third viewpoint of using models allows us to apply the machinery from the MMP theory, and it is the key to degenerate the underlying singularities into cone singularities. The interplay among these three circle of techniques is fruitful, and we expect further insight can be made in the future.

In Section 2, we give the definition of the function of the normalized volumes and sketch the basic properties of its minimizer, including the existence. In Section 3, we discuss the theory on searching for Sasaki-Einstein metrics on a Fano cone singularities. The algebraic side, namely the K-stability notions on a Fano cone plays an important role as we try to degenerate any klt singularity to a K-semistable Fano cone. Such an attempt is formulated in the stable degeneration conjecture, which is the focus of Section 4. In Section 5, we present some applications, including the torus equivariant K-stability (Section 5.1), a solution of Donaldson-Sun's conjecture (Section 5.2) and the K-stability of cubic threefolds (Section 5.3). In the last Section 6, we discuss many unsolved questions, which we hope will lead to some future research. Some of them give new approaches to attack existing problems.

2. Definitions and first properties

2.1. Definitions

In this section, we give the definition of the normalized volume $\widehat{\mathrm{vol}}_{(X,D),x}(v)$ (or abbreviated as $\widehat{\mathrm{vol}}(v)$ if there is no confusion) for a valuation v centered on a klt singularity $x \in (X, D)$ as in [Li18a]. It consists of two parts: the volume $\mathrm{vol}(v)$ (see Definition 2.1) and the log discrepancy $A_{X,D}(v)$ (see Definition 2.2).

Let X be a reduced, irreducible variety defined over $\mathbb{C}$. A *real valuation* of its function field $K(X)$ is a non-constant map $v\colon K(X)^\times \to \mathbb{R}$, satisfying:

- $v(fg) = v(f) + v(g)$;
- $v(f + g) \geq \min\{v(f), v(g)\}$;
- $v(\mathbb{C}^*) = 0$.

We set $v(0) = +\infty$. A valuation v gives rise to a valuation ring $\mathcal{O}_v := \{f \in K(X) \mid v(f) \geq 0\}$. We say a real valuation v is *centered at* a scheme-theoretic point $\xi = c_X(v) \in X$ if we have a local inclusion $\mathcal{O}_{\xi,X} \hookrightarrow \mathcal{O}_v$ of local rings. Notice that the center of a valuation, if exists, is unique since X is separated. Denote by Val_X the set of real valuations of $K(X)$ that admits a center on X. For a closed point $x \in X$, we denote by $\mathrm{Val}_{X,x}$ the set of real valuations of $K(X)$ centered at $x \in X$. It's well known that $v \in \mathrm{Val}_X$ is centered at $x \in X$ if $v(f)$ for any $f \in \mathfrak{m}_x$.

For each valuation $v \in \mathrm{Val}_{X,x}$ and any integer m, we define the valuation ideal $\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{x,X} \mid v(f) \geq m\}$. Then it is clear that $\mathfrak{a}_m(v)$ is an $\mathfrak{m}_x$-primary ideal for each $m > 0$.

Given a valuation $v \in \mathrm{Val}_X$ and a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_X$, we may evaluate $\mathfrak{a}$ along v by setting $v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a} \cdot \mathcal{O}_{c_X(v),X}\}$. It follows from the above definition that if $\mathfrak{a} \subset \mathfrak{b} \subset \mathcal{O}_X$ are nonzero ideals, then $v(\mathfrak{a}) \geq v(\mathfrak{b})$. Additionally, $v(\mathfrak{a}) > 0$ if and only if $c_X(v) \in \mathrm{Cosupp}(\mathfrak{a})$. We endow Val_X with the weakest topology such that, for every ideal $\mathfrak{a}$ on X, the map $\mathrm{Val}_X \to \mathbb{R} \cup \{+\infty\}$ defined by $v \mapsto v(\mathfrak{a})$ is continuous. The subset $\mathrm{Val}_{X,x} \subset \mathrm{Val}_X$ is endowed with the subspace topology. In some literatures, the space $\mathrm{Val}_{X,x}$ is called the *non-archimedean link* of $x \in X$. When $X = \mathbb{C}^2$, the geometry of $\mathrm{Val}_{X,x}$ is understood well (see [FJ04]). For higher dimension, its structure is much more complicated but can be described as an inverse limit of dual complexes (see [JM12, BdFFU15]).

Let $Y \xrightarrow{\mu} X$ be a proper birational morphism with Y a normal variety. For a prime divisor E on Y, we define a valuation $\mathrm{ord}_E \in \mathrm{Val}_X$ that sends each rational function in $K(X)^\times = K(Y)^\times$ to its order of vanishing along E. Note that the center $c_X(\mathrm{ord}_E)$ is the generic point of $\mu(E)$. We say that $v \in \mathrm{Val}_X$ is a *divisorial valuation* if there exists E as above and $\lambda \in \mathbb{R}_{>0}$ such that $v = \lambda \cdot \mathrm{ord}_E$.

Let $\mu : Y \to X$ be a proper birational morphism and $\eta \in Y$ a point such that Y is regular at η. Given a system of parameters $y_1, \cdots, y_r \in \mathcal{O}_{Y,\eta}$ at η and $\alpha = (\alpha_1, \cdots, \alpha_r) \in \mathbb{R}^r_{\geq 0} \setminus \{0\}$, we define a valuation v_α as follows. For $f \in \mathcal{O}_{Y,\eta}$ we can write it as $f = \sum_{\beta \in \mathbb{Z}^r_{\geq 0}} c_\beta y^\beta$, with $c_\beta \in \widehat{\mathcal{O}_{Y,\eta}}$ either zero or unit. We set

$$v_\alpha(f) = \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}.$$

A *quasi-monomial valuation* is a valuation that can be written in the above form.

Let $(Y, E = \sum_{k=1}^N E_k)$ be a log smooth model of X, i.e. $\mu : Y \to X$ is an isomorphism outside of the support of E. We denote by $\mathrm{QM}_\eta(Y, E)$ the set of all quasi-monomial valuations v that can be described at the point $\eta \in Y$ with respect to coordinates $(y_1, \cdots, y_r)$ such that each y_i defines at η an irreducible component of E (hence η is the generic point of a connected component of the intersection of some of the divisors E_i). We put $\mathrm{QM}(Y, E) := \bigcup_\eta \mathrm{QM}_\eta(Y, E)$ where η runs over generic points of all irreducible components of intersections of some of the divisors E_i.

Given a valuation $v \in \mathrm{Val}_{X,x}$, its *rational rank* $\mathrm{rat.rank}\, v$ is the rank of its value group. The *transcendental degree* $\mathrm{trans.deg}\, v$ of v is the transcendental degree of the field extension $\mathbb{C} \hookrightarrow \mathcal{O}_v/\mathfrak{m}_v$. The Zariski-Abhyankar Inequality says that

$$\mathrm{trans.deg}\, v + \mathrm{rat.rank}\, v \leq \dim X.$$

A valuation satisfying the equality is called an *Abhyankar valuation*. By [ELS03], we know that a valuation $v \in \mathrm{Val}_X$ is Abhyankar if and only if it is quasi-monomial.

Definition 2.1. Let X be an n-dimensional normal variety. Let $x \in X$ be a closed point. We define the *volume of a valuation* $v \in \mathrm{Val}_{X,x}$ following [ELS03] as

$$\mathrm{vol}_{X,x}(v) = \limsup_{m \to \infty} \frac{\ell(\mathcal{O}_{x,X}/\mathfrak{a}_m(v))}{m^n/n!}.$$

where ℓ denotes the length of the artinian module.

Thanks to the works of [ELS03, LM09, Cut13] the above limsup is actually a limit.

Definition 2.2. Let (X, D) be a klt log pair. We define the *log discrepancy function of valuations* $A_{(X,D)} : \mathrm{Val}_X \to (0, +\infty]$ in successive generality.

(a) Let $\mu : Y \to X$ be a proper birational morphism from a normal variety Y. Let E be a prime divisor on Y. Then we define $A_{(X,D)}(\mathrm{ord}_E)$ as

$$A_{(X,D)}(\mathrm{ord}_E) := 1 + \mathrm{ord}_E(K_Y - \mu^*(K_X + D)).$$

(b) Let $(Y, E = \sum_{k=1}^{N} E_k)$ be a log smooth model of X. Let η be the generic point of a connected component of $E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}$ of codimension r. Let $(y_1, \cdots, y_r)$ be a system of parameters of $\mathcal{O}_{Y,\eta}$ at η such that $E_{i_j} = (y_j = 0)$. Then for any $\alpha = (\alpha_1, \cdots, \alpha_r) \in \mathbb{R}_{\geq 0}^r \setminus \{0\}$, we define $A_{(X,D)}(v_\alpha)$ as

$$A_{(X,D)}(v_\alpha) := \sum_{j=1}^{r} \alpha_j A_{(X,D)}(\mathrm{ord}_{E_{i_j}}).$$

(c) In [JM12], it was showed that there exists a retraction map $r_{Y,E} : \mathrm{Val}_X \to \mathrm{QM}(Y, E)$ for any log smooth model (Y, E) over X, such that it induces a homeomorphism $\mathrm{Val}_X \to \varprojlim_{(Y,E)} \mathrm{QM}(Y, E)$. For any real valuation $v \in \mathrm{Val}_X$, we define

$$A_{(X,D)}(v) := \sup_{(Y,E)} A_{(X,D)}(r_{(Y,E)}(v)).$$

where (Y, D) ranges over all log smooth models over X. For details, see [JM12] and [BdFFU15, Theorem 3.1]. It is possible that $A_{(X,D)}(v) = +\infty$ for some $v \in \mathrm{Val}_X$, see e.g. [JM12, Remark 5.12].

Then we can define the main invariant in this paper. As we mentioned in Section 3, it is partially inspired the definition in [MSY08] for a valuation coming from the Reeb vector field.

Definition 2.3 ([Li18a])**.** Let (X, D) be an n-dimensional klt log pair. Let $x \in X$ be a closed point. Then the *normalized volume function of valuations* $\widehat{\mathrm{vol}}_{(X,D),x} : \mathrm{Val}_{X,x} \to (0, +\infty)$ is defined as

$$\widehat{\mathrm{vol}}_{(X,D),x}(v) = \begin{cases} A_{(X,D)}(v)^n \cdot \mathrm{vol}_{X,x}(v), & \text{if } A_{(X,D)}(v) < +\infty; \\ +\infty, & \text{if } A_{(X,D)}(v) = +\infty. \end{cases}$$

The *volume of the singularity* $(x \in (X, D))$ is defined as

$$\widehat{\mathrm{vol}}(x, X, D) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}_{(X,D),x}(v).$$

Since $\widehat{\mathrm{vol}}(v) = \widehat{\mathrm{vol}}(\lambda \cdot v)$ for any $\lambda \in \mathbb{R}_{>0}$, for any valuation $v \in \mathrm{Val}_{X,D}$ with a finite log discrepancy, we can rescale such that $\lambda \cdot v \in \mathrm{Val}_{X,D}^{=1}$ where $\mathrm{Val}_{X,D}^{=1}$ consists of all valuations $v \in \mathrm{Val}_{X,x}$ with $A_{(X,D)}(v) = 1$.

Remark 2.4. A definition of volume of singularities is also given in [BdFF12]. Their definition is the local analogue of the volume K_X whereas our definition is the one of the volume of $-K_X$. In particular, a singularity has volume 0 in the definition of [BdFF12] if it is log canonical.

2.2. Properties

In this section, we discuss some properties of $\widehat{\mathrm{vol}}$ on $\mathrm{Val}_{X,x}$. We start from the properness and Izumi estimates. As a corollary, we conclude that $\widehat{\mathrm{vol}}(x, X, D)$ is always positive for any klt singularity $x \in (X, D)$.

Theorem 2.5 ([Li18a])**.** *Let* $(x \in (X, D))$ *be a klt singularity. Then there exists positive constants* C_1, C_2 *which only depend on* $x \in (X, D)$ *(but not the valuation* v*) such that the following holds.*

1. *(Izumi-type inequality) For any valuation* $v \in \mathrm{Val}_{X,x}$, *we have*

$$v(\mathfrak{m}_x)\mathrm{ord}_x \leq v \leq C_2 \cdot A_{(X,D)}(v)\mathrm{ord}_x.$$

2. *(Properness) For any valuation* $v \in \mathrm{Val}_{X,x}$ *with* $A_{(X,D)}(v) < +\infty$, *we have*

$$C_1 \frac{A_{(X,D)}(v)}{v(\mathfrak{m}_x)} \leq \widehat{\mathrm{vol}}(v).$$

Note that since $x \in X$ is singular, ord_x in the above inequality might not be a valuation. In other words, for $f, g \in \mathcal{O}_{X,x}$, $\mathrm{ord}_x(fg) \geq \mathrm{ord}_x(f) + \mathrm{ord}_x(g)$ may be a strict inequality.

The above Izumi type inequality is well known when $x \in X$ is a smooth point. In the case of a general klt singularity, it can be reduced to the smooth case after a log resolution and decreasing the constant. Then for the properness, it follows from a more subtle estimate that there exists a positive constant c_2 that

$$\mathrm{vol}(v) \geq c_2 \left(\sup_{\mathfrak{m}_x} \frac{v}{\mathrm{ord}_x} \right)^{1-n} \cdot \frac{1}{v(\mathfrak{m})}.$$

Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \mathbb{Z}}$ be a graded sequence of $\mathfrak{m}_x$-primary ideals. By the works in [LM09, Cut13], the following identities hold true:

$$\mathrm{mult}(\mathfrak{a}_\bullet) := \lim_{m \to +\infty} \frac{\ell(\mathcal{O}_{X,x}/\mathfrak{a}_m)}{m^n/n!} = \lim_{m \to +\infty} \frac{\mathrm{mult}(\mathfrak{a}_m)}{m^n}.$$

In particular, the two limits exist. Note that, by definition, for any $v \in \mathrm{Val}_{X,x}$ and $\mathfrak{a}_\bullet(v) = \{\mathfrak{a}_m(v)\}$, we have $\mathrm{vol}(v) = \mathrm{mult}(\mathfrak{a}_\bullet(v))$.

The following observation on characterizing the normalized volumes by normalized multiplicities provides lots of flexibility in the study as we will see.

Theorem 2.6 ([Liu18])**.** *Let* $(x \in (X, D))$ *be an* n-*dimensional klt singularity. Then we have*

$$\widehat{\mathrm{vol}}(x, X, D) = \inf_{\mathfrak{a}:\, \mathfrak{m}_x\text{-}primary} \mathrm{lct}(X, D; \mathfrak{a})^n \mathrm{mult}(\mathfrak{a}) = \inf_{\mathfrak{a}_\bullet:\, \mathfrak{m}_x\text{-}primary} \mathrm{lct}(X, D; \mathfrak{a}_\bullet)^n \mathrm{mult}(\mathfrak{a}_\bullet).$$

We also set $\mathrm{lct}(X, D; \mathfrak{a}_\bullet)^n \mathrm{mult}(\mathfrak{a}_\bullet) = +\infty$ *if* $\mathrm{lct}(X, D; \mathfrak{a}_\bullet) = +\infty$.

Proof. Firstly, for any $\mathfrak{m}_x$-primary ideal $\mathfrak{a}$, we can take a divisorial valuation $v \in \mathrm{Val}_{X,x}$ computing $\mathrm{lct}(\mathfrak{a})$. In other words, $\mathrm{lct}(\mathfrak{a}) = A_X(v)/v(\mathfrak{a})$. We may rescale v such that $v(\mathfrak{a}) = 1$. Then clearly $\mathfrak{a}^m \subset \mathfrak{a}_m(v)$ for any $m \in \mathbb{N}$, hence $\mathrm{mult}(\mathfrak{a}) \geq \mathrm{vol}(v)$. Therefore, $\mathrm{lct}(\mathfrak{a})^n \mathrm{mult}(\mathfrak{a}) \geq A_X(v)^n \mathrm{vol}(v)$ which implies

$$\widehat{\mathrm{vol}}(x, X, D) \leq \inf_{\mathfrak{a}:\, \mathfrak{m}_x\text{-primary}} \mathrm{lct}(\mathfrak{a})^n \mathrm{mult}(\mathfrak{a}). \tag{1}$$

Secondly, for any graded sequence of $\mathfrak{m}_x$-primary ideals $\mathfrak{a}_\bullet$, we have

$$\mathrm{lct}(\mathfrak{a}_\bullet) = \lim_{m \to \infty} m \cdot \mathrm{lct}(\mathfrak{a}_m)$$

by [JM12, BdFFU15]. Hence

$$\mathrm{lct}(\mathfrak{a}_\bullet)^n \mathrm{mult}(\mathfrak{a}_\bullet) = \lim_{m\to\infty} (m \cdot \mathrm{lct}(\mathfrak{a}_m))^n \frac{\mathrm{mult}(\mathfrak{a}_m)}{m^n} = \lim_{m\to\infty} \mathrm{lct}(\mathfrak{a}_m)^n \mathrm{mult}(\mathfrak{a}_m).$$

As a result,

$$\inf_{\mathfrak{a}\,:\,\mathfrak{m}_x\text{-primary}} \mathrm{lct}(\mathfrak{a})^n \mathrm{mult}(\mathfrak{a}) \le \inf_{\mathfrak{a}_\bullet\,:\,\mathfrak{m}_x\text{-primary}} \mathrm{lct}(\mathfrak{a}_\bullet)^n \mathrm{mult}(\mathfrak{a}_\bullet). \tag{2}$$

Lastly, for any valuation $v \in \mathrm{Val}_{X,x}$, we consider the graded sequence of its valuation ideals $\mathfrak{a}_\bullet(v)$. Since $v(\mathfrak{a}_\bullet(v)) = 1$, we have $\mathrm{lct}(\mathfrak{a}_\bullet) \le A_X(v)$. We also have $\mathrm{mult}(\mathfrak{a}_\bullet(v)) = \mathrm{vol}(v)$. Hence $\mathrm{lct}(\mathfrak{a}_\bullet(v))^n \mathrm{mult}(\mathfrak{a}_\bullet(v)) \le A_X(v)^n \mathrm{vol}(v)$, which implies

$$\inf_{\mathfrak{a}_\bullet\,:\,\mathfrak{m}_x\text{-primary}} \mathrm{lct}(\mathfrak{a}_\bullet)^n \mathrm{mult}(\mathfrak{a}_\bullet) \le \widehat{\mathrm{vol}}(x, X, D). \tag{3}$$

The proof is finished by combining (1), (2), and (3). $\square$

In general we have the following relation between a sequence of graded ideals and the one from a valuation: Let Φ^g be an ordered subgroup of the real numbers $\mathbb{R}$. Let $(R, \mathfrak{m})$ be the local ring at a normal singularity $o \in X$. A Φ^g-graded filtration of R, denoted by $\mathcal{F} := \{\mathfrak{a}^m\}_{m \in \Phi^g}$, is a decreasing family of $\mathfrak{m}$-primary ideals of R satisfying the following conditions:

(i) $\mathfrak{a}^m \ne 0$ for every $m \in \Phi^g$, $\mathfrak{a}^m = R$ for $m \le 0$ and $\cap_{m\ge0}\mathfrak{a}^m = (0)$;

(ii) $\mathfrak{a}^{m_1} \cdot \mathfrak{a}^{m_2} \subseteq \mathfrak{a}^{m_1+m_2}$ for every $m_1, m_2 \in \Phi^g$.

Given such an $\mathcal{F}$, we get an associated order function

$$v = v_{\mathcal{F}} : R \to \mathbb{R}_{\ge 0} \qquad v(f) = \max\{m; f \in \mathfrak{a}^m\} \text{ for any } f \in R.$$

Using the above **(i)-(ii)**, it is easy to verify that v satisfies $v(f+g) \ge \min\{v(f), v(g)\}$ and $v(fg) \ge v(f) + v(g)$. We also have the associated graded ring:

$$\mathrm{gr}_{\mathcal{F}} R = \sum_{m \in \Phi^g} \mathfrak{a}^m / \mathfrak{a}^{>m}, \text{ where } \mathfrak{a}^{>m} = \bigcup_{m' > m} \mathfrak{a}^{m'}.$$

For any real valuation v with valuative group Φ^g, $\{\mathcal{F}^m\} := \{\mathfrak{a}_m(v)\}$ is a Φ^g-graded filtration of R. We will need the following facts.

Lemma 2.7 (see [Tei03, Tei14]**).** *With the above notations, the following statements hold true:*

(1) ([Tei14, Page 8]) *If $\mathrm{gr}_{\mathcal{F}} R$ is an integral domain, then $v = v_{\mathcal{F}}$ is a valuation centered at $o \in X$. In particular, $v(fg) = v(f) + v(g)$ for any $f, g \in R$.*

(2) (Piltant) *A valuation v is quasi-monomial if and only if the Krull dimension of $\mathrm{gr}_v R$ is the same as the Krull dimension of R.*

The existence of a minimizer for $\widehat{\mathrm{vol}}_{(X,D),x}$ was conjectured in the first version of [Li18a] and then proved in [Blu18].

Theorem 2.8 ([Blu18]**).** *For any klt singularity $x \in (X, D)$, there exists a valuation $v_{\min} \in \mathrm{Val}_{X,x}$ that minimizes the function $\widehat{\mathrm{vol}}_{(X,D),x}$.*

Let us sketch the idea of proving the existence of $\widehat{\mathrm{vol}}$-minimizer. We first take a sequence of valuations $(v_i)_{i \in \mathbb{N}}$ such that

$$\lim_{i \to \infty} \widehat{\mathrm{vol}}(v_i) = \widehat{\mathrm{vol}}(x, X, D).$$

Then we would like to find a valuation v^* that is a limit point of the sequence $(v_i)_{i \in \mathbb{N}}$ and then show that v^* is a minimizer of $\widehat{\mathrm{vol}}$.

Instead of seeking a limit point v^* of $(v_i)_{i \in \mathbb{N}}$ in the space of valuations, we consider graded sequences of ideals. More precisely, each valuation v_i induces a graded sequence $\mathfrak{a}_\bullet(v_i)$ of $\mathfrak{m}_x$-primary ideals. By Theorem 2.6, we have

$$\widehat{\mathrm{vol}}(v_i) \geq \mathrm{lct}(\mathfrak{a}_\bullet(v_i))^n \mathrm{mult}(\mathfrak{a}_\bullet(v_i)) \geq \widehat{\mathrm{vol}}(x, X, D).$$

Therefore, once we find a graded sequence of $\mathfrak{m}_x$-primary ideals $\tilde{\mathfrak{a}}_\bullet$ that is a 'limit point' of the sequence $(\mathfrak{a}_\bullet(v_i))_{i \in \mathbb{N}}$, a valuation v^* computing $\mathrm{lct}(\tilde{\mathfrak{a}}_\bullet)$ will minimizes $\widehat{\mathrm{vol}}$. The existence of such 'limits' relies on two ingredients: the first is an asymptotic estimate to control the growth for $\mathfrak{a}_k(v_i)$ for a fixed k; once the growth is controlled, we can apply the generic limit construction.

Proof. For simplicity, we will assume $D = 0$. More details about log pairs can be found in [Blu18, Section 7].

Let us choose a sequence of valuations $v_i \in \mathrm{Val}_{X,x}$ such that

$$\lim_{i \to \infty} \widehat{\mathrm{vol}}(v_i) = \widehat{\mathrm{vol}}(x, X).$$

Since the normalized volume function is invariant after rescaling, we may assume that $v_i(\mathfrak{m}) = 1$ for all $i \in \mathbb{N}$ where $\mathfrak{m} := \mathfrak{m}_x$. Our goal is to show that the family of graded sequences of $\mathfrak{m}$-primary ideals $(\mathfrak{a}_\bullet(v_i))_{i \in \mathbb{N}}$ satisfies the following conditions:

(a) For every $\epsilon > 0$, there exists positive constants M, N so that

$$\mathrm{lct}(\mathfrak{a}_m(v_i))^n \mathrm{mult}(\mathfrak{a}_m(v_i)) \leq \widehat{\mathrm{vol}}(x, X) + \epsilon \text{ for all } m \geq M \text{ and } i \geq N.$$

(b) For each $m, i \in \mathbb{N}$, we have $\mathfrak{m}^m \subset \mathfrak{a}_m(v_i)$.
(c) There exists $\delta > 0$ such that $\mathfrak{a}_m(v_i) \subset \mathfrak{m}^{\lfloor m\delta \rfloor}$ for all $m, i \in \mathbb{N}$.

Part (b) follows easily from $v_i(\mathfrak{m}) = 1$. Hence $\mathrm{vol}(v_i) \leq \mathrm{mult}(\mathfrak{m}) =: B$. For part (c), we need to use Theorem 2.5. By Part (2), there exists a positive constant C_1 such that

$$A_X(v) \leq C_1^{-1} \cdot v(\mathfrak{m}) \widehat{\mathrm{vol}}(v) \text{ for all } v \in \mathrm{Val}_{X,x}.$$

Let $A := C_1^{-1} \sup_{i \in \mathbb{N}} \widehat{\mathrm{vol}}(v_i)$, then $A_X(v_i) \leq A$ for any $i \in \mathbb{N}$. By Theorem 2.5(1), then there exists a positive constant C_2 such that

$$v(f) \leq C_2 \cdot A_X(v) \mathrm{ord}_x(f) \text{ for all } v \in \mathrm{Val}_{X,x} \text{ and } f \in \mathcal{O}_{X,x}.$$

In particular, $v_i(f) \leq C_2 A \cdot \mathrm{ord}_x(f)$ for all $i \in \mathbb{N}$ and $f \in \mathcal{O}_{X,x}$. Thus by letting $\delta := (C_2 A)^{-1}$ we have $\mathfrak{a}_m(v_i) \subset \mathfrak{m}^{\lfloor m\delta \rfloor}$ which proves part (c).

The proof of part (a) relies on the following result on uniform convergence of multiplicities of valuation ideals.

Proposition 2.9 ([Blu18])**.** *Let* $(x \in X)$ *be an* n-*dimensional klt singularity. Then for* $\epsilon, A, B, r \in \mathbb{R}_{>0}$, *there exists* $M = M(\epsilon, A, B, r)$ *such that for every valuation* $v \in \mathrm{Val}_{X,x}$ *with* $A_X(v) \leq A$, $\mathrm{vol}(v) \leq B$, *and* $v(\mathfrak{m}) \geq 1/r$, *we have*

$$\mathrm{vol}(v) \leq \frac{\mathrm{mult}(\mathfrak{a}_m(v))}{m^n} < \mathrm{vol}(v) + \epsilon \text{ for all } m \geq M.$$

Proof. The first inequality is straightforward. When the point is smooth, the second inequality uses the inequality that for the graded sequence of ideals $\{\mathfrak{a}_\bullet\}$, there exists a k such that for any m and l

$$\mathfrak{a}_{ml} \subseteq \mathfrak{a}_{m-k}^l.$$

The proof of such result uses the multiplier ideal, see [ELS03]. For isolated klt singularity, then an estimate of a similar form in [Tak06] says

$$\mathcal{J}_X^{l-1} \cdot \mathfrak{a}_{ml} \subset \mathfrak{a}_{m-k}^l \tag{4}$$

suffices, where $\mathcal{J}_X$ is the Jacobian ideal of X. Finally, in the general case, an argument using (4) and interpolating $\mathcal{J}_X$ and a power of $\mathfrak{m}$ gives the proof. See [Blu18, Section 3] for more details. $\square$

To continue the proof, let us fix an arbitrary $\epsilon \in \mathbb{R}_{>0}$. Since $A_X(v_i) \leq A$, $\mathrm{vol}(v_i) \leq B$, and $v_i(\mathfrak{m}) = 1$ for all $i \in \mathbb{N}$, Proposition 2.9 implies that there exists $M \in \mathbb{N}$ such that

$$\frac{\mathrm{mult}(\mathfrak{a}_m(v_i))}{m^n} \leq \mathrm{vol}(v_i) + \epsilon/(2A^n) \text{ for all } i \in \mathbb{N}.$$

We also have $\mathrm{lct}(\mathfrak{a}_m(v_i)) \leq A_X(v_i)/v_i(\mathfrak{a}_m(v_i)) \leq m \cdot A_X(v_i)$. Let us take $N \in \mathbb{N}$ such that $\widehat{\mathrm{vol}}(v_i) \leq \widehat{\mathrm{vol}}(x, X) + \epsilon/2$ for any $i \geq N$. Therefore,

$$\begin{aligned}
\mathrm{lct}(\mathfrak{a}_m(v_i))^n \mathrm{mult}(\mathfrak{a}_m(v_i)) &\leq A_X(v_i)^n (\mathrm{vol}(v_i) + \epsilon/(2A^n)) \\
&= \widehat{\mathrm{vol}}(v_i) + \epsilon \cdot A_X(v_i)^n/(2A^n) \\
&= \widehat{\mathrm{vol}}(v_i) + \epsilon/2 \\
&\leq \widehat{\mathrm{vol}}(x, X) + \epsilon.
\end{aligned}$$

So part (a) is proved.

Finally, (b) and (c) guarantee that we can apply a generic limit type construction (cf. [Blu18, Section 5]). Then (a) implies that a 'limit point' $\tilde{\mathfrak{a}}_\bullet$ of the sequence $(\mathfrak{a}_\bullet(v_i))_{i \in \mathbb{N}}$ satisfies that $\mathrm{lct}(\tilde{\mathfrak{a}}_\bullet)^n \mathrm{mult}(\tilde{\mathfrak{a}}_\bullet) \leq \widehat{\mathrm{vol}}(x, X)$. Thus a valuation v^* computing the log canonical threshold of $\tilde{\mathfrak{a}}_\bullet$, whose existence follows from [JM12], necessarily minimizes the normalized volume. $\square$

Theorem 2.10 ([LX17b])**.** *Let* $x \in (X, D)$ *be an* n-*dimensional klt singularity. Then* $\widehat{\mathrm{vol}}(x, X, D) \leq n^n$ *and the equality holds if and only if* $x \in X \setminus \mathrm{Supp}(D)$ *is a smooth point.*

Using the fact that we can specialize a graded sequence of ideals preserving the colength, and the lower semi-continuous of the log canonical thresholds, we easily get the inequality part of Theorem 2.10. Then the equality part gives us a characterization of the smooth point using the normalized volume. The following Theorem 2.11 on the semicontinuity needs a more delicate analysis. We conjecture that the normalized volume function is indeed constructible (see Conjecture 6.6).

Theorem 2.11 ([BL18])**.** *Let* $\pi : (\mathcal{X}, D) \to T$ *together with a section* $t \in T \mapsto x_t \in \mathcal{X}_t$ *be a* $\mathbb{Q}$*-Gorenstein flat family of klt singularities. Then the function* $t \mapsto \widehat{\mathrm{vol}}(x_t, \mathcal{X}_t, D_t)$ *is lower semicontinuous with respect to the Zariski topology.*

Now we introduce a key tool that the minimal model program provides to us to understand minimizing the normalized volume. For more discussions, see Section 4.3.

Definition 2.12 (Kollár component, [Xu14])**.** Let $x \in (X, D)$ be a klt singularity. We call a proper birational morphism $\mu : Y \to X$ provides a Kollár component S, if μ is isomorphic over $X \setminus \{x\}$, and $\mu^{-1}(x)$ is an irreducible divisor S, such that $(Y, S + \mu_*^{-1}D)$ is purely log terminal (plt) and $-S$ is $\mathbb{Q}$-Cartier and ample over X.

Theorem 2.13 ([LX16])**.** *We have the identity:*

$$\widehat{\mathrm{vol}}(x, X, D) = \inf_S \{\widehat{\mathrm{vol}}(\mathrm{ord}_S) \mid \text{for all Kollár components } S \text{ over } x\}. \tag{5}$$

For the explanation of proof, see the discussions for (27) in Section 4.3.

3. Stability in Sasaki-Einstein geometry

To proceed the study of normalized volumes, we will introduce the concept of K-stability. This is now a central notion in complex geometry, which serves as an algebraic characterization of the existence of some 'canonical metrics'.

In the local setting, such problem on an affine T-variety X with a unique fixed point x was first considered in [MSY08]. We can then varies the Reeb vector field $\xi \in \mathfrak{t}_{\mathbb{R}}^+$, and call such a structure (X, ξ) is a Fano cone if X only has klt log terminal singularities. The name is justified since if $\xi \in \mathfrak{t}_{\mathbb{Q}}^+$, let $\langle \xi \rangle$ be the $\mathbb{C}^*$ generated by ξ, then $X \setminus \{x\}/\langle \xi \rangle$ is a log Fano variety.

In [MSY08], the relation between the existence of Sasaki-Einstein metric along (X, ξ) and the K-stability of (X, ξ), a mimic of the absolute case, was explored. A key observation in [MSY08] is that we can define a normalized volume function $\widehat{\mathrm{vol}}_X(\xi)$ for $\xi \in \mathfrak{t}_{\mathbb{R}}^+$, and among all choices of ξ the one minimizing $\widehat{\mathrm{vol}}_X(\cdot)$ gives 'the most stable' direction.

Then an important step to advance such a picture is made in [CS18, CS15] by extending the definition of K-stability notions on (X, ξ) allowing degenerations, and showing that there is a Sasaki-Einstein metric along an isolated Fano cone singularity (X, ξ) if and only of (X, ξ) is K-polystable, extending the solution of the

Yau-Tian-Donaldson's conjecture in the Fano manifold case (see [CDS15, Tia15]) to the cone case.

In this section, we will briefly introduce these settings.

3.1. T-varieties

We first introduce the basic setting using T-varieties. For general results of T-varieties, see [AIP$^+$12].

Assume $X = \mathrm{Spec}_{\mathbb{C}}(R)$ is an affine variety with $\mathbb{Q}$-Gorenstein klt singularities. Denote by T the complex torus $(\mathbb{C}^*)^r$. Assume X admits a good T-action in the following sense.

Definition 3.1 (see [LS13, Section 4]**).** Let X be a normal affine variety. We say that a T-action on X is *good* if it is effective and there is a unique closed point $x \in X$ that is in the orbit closure of any T-orbit. We shall call x (sometimes also denoted by o_X) the vertex point of the T-variety X.

Let $N = \mathrm{Hom}(\mathbb{C}^*, T)$ be the co-weight lattice and $M = N^*$ the weight lattice. We have a weight space decomposition of the coordinate ring of X:

$$R = \bigoplus_{\alpha \in \Gamma} R_\alpha \text{ where } \Gamma = \{\alpha \in M \mid R_\alpha \neq 0\}.$$

The action being good implies $R_0 = \mathbb{C}$, which will always be assumed in the below. An ideal $\mathfrak{a}$ is called homogeneous if $\mathfrak{a} = \bigoplus_{\alpha \in \Gamma} \mathfrak{a} \cap R_\alpha$. Denote by $\sigma^\vee \subset M_{\mathbb{Q}}$ the cone generated by Γ over $\mathbb{Q}$, which will be called the *weight cone* or the *moment cone*. The cone $\sigma \subset N_{\mathbb{R}}$, dual to $\sigma^\vee$, is the same as the following conical set

$$\mathfrak{t}_{\mathbb{R}}^+ := \{\, \xi \in N_{\mathbb{R}} \mid \langle \alpha, \xi \rangle > 0 \text{ for any } \alpha \in \Gamma \setminus \{0\}\}.$$

Motivated by notations from Sasaki geometry, we will introduce:

Definition 3.2. With the above notations, $\mathfrak{t}_{\mathbb{R}}^+$ will be called the Reeb cone of the T-action of X. A vector $\xi \in \mathfrak{t}_{\mathbb{R}}^+$ will be called a Reeb vector on the T-variety X.

To adapt this definition into our setting in Section 2.1, for any $\xi \in \mathfrak{t}_{\mathbb{R}}^+$, we can define a valuation

$$\mathrm{wt}_\xi(f) = \min_{\alpha \in \Gamma} \{\langle \alpha, \xi \rangle \mid f_\alpha \neq 0\}.$$

It is easy to verify that $\mathrm{wt}_\xi \in \mathrm{Val}_{X, o_X}$. The rank of ξ, denoted by $\mathrm{rk}(\xi)$, is the dimension of the subtorus T_ξ (as a subgroup of T) generated by $\xi \in \mathfrak{t}$. The following lemma can be easily seen.

Lemma 3.3. *For any $\xi \in \mathfrak{t}_{\mathbb{R}}^+$, wt_ξ is a quasi-monomial valuation of rational rank equal to the rank of ξ. Moreover, the center of wt_ξ is o_X.*

We recall the following structure results for any T-varieties.

Theorem 3.4 (see [AIP$^+$12, Theorem 4]**).** *Let $X = \operatorname{Spec}(R)$ be a normal affine variety and suppose $T = \operatorname{Spec}(\mathbb{C}[M])$ has a good action on X with the weight cone $\sigma^\vee \subset M_\mathbb{Q}$. Then there exist a normal projective variety Y and a polyhedral divisor $\mathfrak{D}$ such that there is an isomorphism of graded algebras:*

$$R \cong H^0(X, \mathcal{O}_X) \cong \bigoplus_{u \in \sigma^\vee \cap M} H^0\big(Y, \mathcal{O}(\mathfrak{D}(u))\big) =: R(Y, \mathfrak{D}).$$

In other words, X is equal to $\operatorname{Spec}_\mathbb{C}\big(\bigoplus_{u \in \sigma^\vee \cap M} H^0(Y, \mathcal{O}(\mathfrak{D}(u)))\big)$.

In the above definition, a polyhedral divisor $\mathfrak{D} : u \to \mathfrak{D}(u)$ is a map from $\sigma^\vee$ to the set of $\mathbb{Q}$-Cartier divisors that satisfies:

1. $\mathfrak{D}(u) + \mathfrak{D}(u') \le \mathfrak{D}(u + u')$ for any $u, u' \in \sigma^\vee$;
2. $u \mapsto \mathfrak{D}(u)$ is piecewisely linear;
3. $\mathfrak{D}(u)$ is semiample for any $u \in \sigma^\vee$, and $\mathfrak{D}(u)$ is big if u is in the relative interior of $\sigma^\vee$.

Here Y is projective since from our assumption

$$H^0(Y, \mathcal{O}_Y) = R^T = R_0 = \mathbb{C}$$

(see [LS13]). We collect some basic results about valuations on T-varieties.

Theorem 3.5 (see [AIP$^+$12]**).** *Assume a T-variety X is determined by the data $(Y, \sigma, \mathfrak{D})$ such that Y is a projective variety, where $\sigma = \mathfrak{t}_\mathbb{R}^+ \subset N_\mathbb{R}$ and $\mathfrak{D}$ is a polyhedral divisor.*

1. *For any T-invariant quasi-monomial valuation v, there exist a quasi-monomial valuation $v^{(0)}$ over Y and $\xi \in M_\mathbb{R}$ such that for any $f \cdot \chi^u \in R_u$, we have:*

   $$v(f \cdot \chi^u) = v^{(0)}(f) + \langle u, \xi \rangle.$$

   We will use $(\xi, v^{(0)})$ to denote such a valuation.
2. *T-invariant prime divisors on X are either vertical or horizontal. Any vertical divisor is determined by a divisor Z on Y and a vertex v of $\mathfrak{D}_Z$, and will be denoted by $D_{(Z,v)}$. Any horizontal divisor is determined by a ray ρ of σ and will be denoted by E_ρ.*
3. *Let D be a T-invariant vertical effective $\mathbb{Q}$-divisor. If $K_X + D$ is $\mathbb{Q}$-Cartier, then the log canonical divisor has a representation $K_X + D = \pi^* H + \operatorname{div}(\chi^{-u_0})$ where $H = \sum_Z a_Z \cdot Z$ is a principal $\mathbb{Q}$-divisor on Y and $u_0 \in M_\mathbb{Q}$. Moreover, the log discrepancy of the horizontal divisor E_ρ is given by:*

   $$A_{(X,D)}(E_\rho) = \langle u_0, n_\rho \rangle, \tag{6}$$

   where n_ρ is the primitive vector along the ray ρ.

Sketch of the proof. For the first statement, the case of divisorial valuations follows from [AIP$^+$12, Section 11]. It can be extended to the case of quasi-monomial valuations by the same proof. Note also that any T-invariant quasimonomial valuation can be approximated by a sequence of T-invariant divisorial valuations. The second statement is in [PS11, Proposition 3.13]. The absolute case (e.g. without

boundary divisor D) for the third statement is from [LS13, Section 4] whose proof also works for the case of log pairs.. $\qquad\square$

We will specialize the study of general affine T-varieties to case that the log pair is klt. Assume X is a normal affine variety with $\mathbb{Q}$-Gorenstein klt singularities and a good T-action. Let D be a T-invariant vertical divisor. Then there is a nowhere-vanishing T-equivariant section s of $m(K_X + D)$ where m is sufficiently divisible. The following lemma says that the log discrepancy of wt_ξ can indeed be calculated in a similar way as in the toric case (the toric case is well-known). Moreover, it can be calculated by using the weight of T-equivariant pluri-log-canonical sections. The latter observation was first made in [Li18a].

Lemma 3.6. *Using the same notion as in the Theorem 3.5, the log discrepancy of* wt_ξ *is given by:* $A_{(X,D)}(\mathrm{wt}_\xi) = \langle u_0, \xi \rangle$*. Moreover, let s be a T-equivariant nowhere-vanishing holomorphic section of $|-m(K_X + D)|$, and denote $\mathcal{L}_\xi$ the Lie derivative with respect to the holomorphic vector field associated to ξ. Then $A_{(X,D)}(\xi) = \lambda$ if and only if*

$$\mathcal{L}_\xi(s) = m\lambda s \quad for \quad \lambda > 0.$$

As a consequence of the above lemma, we can formally extend $A_{(X,D)}(\xi)$ to a linear function on $\mathfrak{t}_\mathbb{R}$:

$$A_{(X,D)}(\eta) = \langle u_0, \eta \rangle. \tag{7}$$

for any $\eta \in \mathfrak{t}_\mathbb{R}$. By Lemma 3.6, $A_{(X,D)}(\eta) = \frac{1}{m}\mathcal{L}_\eta s/s$ where s is a T-equivariant nowhere-vanishing holomorphic section of $|-m(K_X + D)|$.

Definition 3.7 (Log Fano cone singularity). Let (X, D) be an affine pair with a good T action. Assume (X, D) is a normal pair with klt singularities. Then for any $\xi \in \mathfrak{t}_\mathbb{R}^+$, we call the triple (X, D, ξ) a *log Fano cone* structure that is polarized by ξ. We will denote by $\langle \xi \rangle$ the sub-torus of T generated by ξ. If $\langle \xi \rangle \cong \mathbb{C}^*$, then we call (X, D, ξ) to be quasi-regular. Otherwise, we call it irregular.

In the quasi-regular case, we can take the quotient $(X \setminus \{x\}, D \setminus \{x\})$ by the $\mathbb{C}^*$-group generated by ξ in the sense of Seifert $\mathbb{C}^*$-bundles (see [Kol04]), and we will denote by $(X, D)/\langle \xi \rangle$, which is a log Fano variety, because of the assumption that (X, D) is klt at x (see [Kol13, Lemma 3.1]).

3.2. K-stability

In this section, we will discuss the K-stability notion of log Fano cones, which generalizes the K-stability of log Fano varieties originally defined by Tian and Donaldson. For irregular Fano cones, such a notion was first defined in [CS18].

Definition 3.8 (Test configurations). Let (X, D, ξ_0) be a log Fano cone singularity and T a torus containing $\langle \xi_0 \rangle$.

A T-equivariant test configuration (or simply called a test configuration) of (X, D, ξ_0) is a quadruple $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with a map $\pi : (\mathcal{X}, \mathcal{D}) \to \mathbb{C}$ satisfying the following conditions:

(1) $\pi : \mathcal{X} \to \mathbb{C}$ is a flat family and $\mathcal{D}$ is an effective $\mathbb{Q}$-divisor such that $\mathcal{D}$ does not contain any component X_0, the fibres away from 0 are isomorphic to (X, D) and $\mathcal{X} = \mathrm{Spec}(\mathcal{R})$ is affine, where $\mathcal{R}$ is a finitely generate flat $\mathbb{C}[t]$ algebra. The torus T acts on $\mathcal{X}$, and we write $\mathcal{R} = \bigoplus_\alpha \mathcal{R}_\alpha$ as decomposition into weight spaces.

(2) η is a holomorphic vector field on $\mathcal{X}$ generating a $\mathbb{C}^*(= \langle \eta \rangle)$-action on $(\mathcal{X}, \mathcal{D})$ such that π is $\mathbb{C}^*$-equivariant where $\mathbb{C}^*$ acts on the base $\mathbb{C}$ by the multiplication (so that $\pi_* \eta = t\partial_t$ if t is the affine coordinate on $\mathbb{C}$) and there is a $\mathbb{C}^*$-equivariant isomorphism $\phi : (\mathcal{X}, \mathcal{D}) \times_\mathbb{C} \mathbb{C}^* \cong (X, D) \times \mathbb{C}^*$.

(3) The torus T-action commutes with η. The holomorphic vector field ξ_0 on $\mathcal{X} \times_\mathbb{C} \mathbb{C}^*$ (via the isomorphism ϕ) extends to a holomorphic vector field on $\mathcal{X}$ which we still denote to be ξ_0.

In most our study, we only need to treat the case that test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ of (X, D, ξ_0) satisfies that

(4) $K_\mathcal{X} + \mathcal{D}$ is $\mathbb{Q}$-Cartier and the central fibre (X_0, D_0) is klt

In other words, we will mostly consider special test configurations (see [LX14, CS15]).

Condition (1) implies that each weight piece $\mathcal{R}_\alpha$ is a flat $\mathbb{C}[t]$-module. So X and X_0 have the same weight cone and Reeb cone with respect to the fiberwise T-action.

A test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is called a product one if there is a T-equivariant isomorphism $(\mathcal{X}, \mathcal{D}) \cong (X, D) \times \mathbb{C}$ and $\eta = \eta_0 + t\partial_t$ where η_0 is a holomorphic vector field on X that preserves D and commutes with ξ_0. In this case, we will denote $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ by

$$(X \times \mathbb{C}, D \times \mathbb{C}, \xi_0; \eta) =: (X_\mathbb{C}, D_\mathbb{C}, \xi_0; \eta).$$

In [MSY08], only such test configurations are considered.

Definition 3.9 (K-stability). For any special test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ of (X, D, ξ_0) with central fibre (X_0, D_0, ξ_0), its generalized Futaki invariant is defined as

$$\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) := \frac{D_{-T_{\xi_0}(\eta)} \mathrm{vol}_{X_0}(\xi_0)}{\mathrm{vol}_{X_0}(\xi_0)}$$

where we denote

$$T_{\xi_0}(\eta) = \frac{A(\xi_0)\eta - A(\eta)\xi_0}{n}. \tag{8}$$

Since the generalized Futaki invariant defined above only depends on the data on the central fibre, we will also denote it by $\mathrm{Fut}(X_0, D_0, \xi_0; \eta)$.

We say that (X, D, ξ_0) is K-semistable, if for any special test configuration, $\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ is nonnegative.

We say that (X, D, ξ_0) is K-polystable, if it is K-semistable, and any special test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ with $\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) = 0$ is a product test configuration.

In the above definition, we used the notation (8) and the directional derivative:

$$D_{-T_{\xi_0(\eta)}} \mathrm{vol}_{X_0}(\xi_0) := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathrm{vol}_{X_0}(\xi_0 - \epsilon T_{\xi_0(\eta)}).$$

Recall that the $\pi_*\eta = t\partial_t$. Then the negative sign in front of $T_{\xi_0}(\eta)$ in the above formula is to be compatible with our later computation. Using the rescaling invariance of the normalized volume, it is easy to verify that the following identity holds:

$$D_{-T_{\xi_0}\eta} \mathrm{vol}_{X_0}(\xi_0) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \widehat{\mathrm{vol}}_{X_0}(\mathrm{wt}_{\xi_0-\epsilon\eta}) \cdot \frac{1}{nA(\xi_0)^{n-1}}, \tag{9}$$

where $A(\xi_0) = A_{(X_0,D_0)}(\mathrm{wt}_{\xi_0})$. As a consequence, we can rewrite the Futaki invariant of a special test configuration as:

$$\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) := D_{-\eta}\widehat{\mathrm{vol}}_{X_0}(\mathrm{wt}_{\xi_0}) \cdot \frac{1}{nA(\xi_0)^{n-1} \cdot \mathrm{vol}_{X_0}(\xi_0)}. \tag{10}$$

One can show that, up to a constant, the above definition of $\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ coincides with the one in [CS18, CS15] defined using index characters. For convenience of the reader, we recall their definition. It is enough to define the Futaki invariant for the central fibre which we just denote by X. For any $\xi \in \mathfrak{t}_{\mathbb{R}}^+$, the index character $F(\xi, t)$ is defined by:

$$F(\xi, t) := \sum_{\alpha \in \Gamma} e^{-t\langle \alpha, \xi \rangle} \dim_{\mathbb{C}} R_\alpha. \tag{11}$$

Then there is a meromorphic expansion for $F(\xi, t)$ as follows:

$$F(\xi, t) = \frac{a_0(\xi)(n-1)!}{t^n} + \frac{a_1(\xi)(n-2)!}{t^{n-1}} + O(t^{2-n}). \tag{12}$$

One always has the identity $a_0(\xi) = \mathrm{vol}(\xi)/(n-1)!$.

Definition 3.10 (see [CS18]). For any $\eta \in \mathfrak{t}_{\mathbb{R}}$, define:

$$\begin{aligned}
\mathrm{Fut}_{\xi_0}(X, \eta) &= \frac{1}{n-1}D_{-\eta}(a_1(\xi_0)) - \frac{1}{n}\frac{a_1(\xi_0)}{a_0(\xi_0)}D_{-\eta}a_0(\xi_0) \\
&= \frac{a_0(\xi_0)}{n-1}D_{-\eta}\left(\frac{a_1}{a_0}\right)(\xi_0) + \frac{a_1(\xi_0)D_{-\eta}a_0(\xi_0)}{n(n-1)a_0(\xi_0)}.
\end{aligned}$$

This is a complicated expression. But in [CS15, Proposition 6.4], it was showed that, when X is $\mathbb{Q}$-Gorenstein log terminal, there is an identity $a_1(\xi)/a_0(\xi) = A(\xi)(n-1)/2$ for any $\xi \in \mathfrak{t}_{\mathbb{R}}^+$ (by using our notation involving log discrepancies). Note that the rescaling properties $a_0(\lambda\xi) = \lambda^{-n}a_0(\xi)$ and $a_1(\lambda\xi) = \lambda^{-(n-1)}a_1(\xi)$ which imply $\mathrm{Fut}_{\xi_0}(X, \xi_0) = 0$. If we denote $\eta' = \eta - \frac{A(\eta)}{A(\xi_0)}\xi_0$, then we get:

$$\mathrm{Fut}_{\xi_0}(X, \eta) = \mathrm{Fut}_{\xi_0}(X, \eta') = \frac{A(\xi_0)}{2n}D_{-\eta'}a_0(\xi_0) = \frac{1}{2(n-1)!}D_{-T_{\xi_0}(\eta)}\mathrm{vol}(\xi_0). \tag{13}$$

So the definition in [CS18, CS15] differs from our notation by a constant $2(n-1)!/\mathrm{vol}_X(\xi_0)$.

Remark 3.11. More precisely, our notation differs from that in [CS18] by a sign. Our choice of minus sign for $-\eta$, besides being compatible with the sign choice in Tian's original definition of K-stability in [Tia97], is made for least two reasons. The first is that the careful calculation in [LX17a, Section 5.2] shows that the limiting slope of the Ding energy along the geodesic ray associated to any special test configuration is indeed the directional derivative of $\mathrm{vol}(\xi)$ along $-\eta$ instead of η. For the second reason, as we stressed in [LX17a, Remark 3.4], for the special test configuration coming from a Kollár component S, the $-\eta$ vector corresponds to ord_S. Since our goal is to compare $\widehat{\mathrm{vol}}(\mathrm{wt}_{\xi_0})$ and $\widehat{\mathrm{vol}}(\mathrm{ord}_S)$, $-\eta$ is the correct choice of sign (see [LX17a, Proof of Theorem 3.5]).

Remark 3.12. In fact, in a calculation, instead of the generalized Futaki invariant, it is the Berman-Ding invariant, denoted by $D^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$, where

$$D^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) := \frac{D_{-T_{\xi_0}(\eta)}\mathrm{vol}_{X_0}(\xi_0)}{\mathrm{vol}(\xi_0)} - (1 - \mathrm{lct}(\mathcal{X}, D; \mathcal{X}_0)).$$

appears more naturally, whenever we know

(D) there exists a nowhere vanishing section $s \in |m(K_{\mathcal{X}} + \mathcal{D})|$ such that we can use it to define $A(\cdot)$ as in the formula in Lemma 3.6.

Then we can similarly define Ding semi(poly)-stable, replacing $\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ by $D^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$. For a special test configuration, since

$$\mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi_0; \eta) = D^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$$

the two notions coincide.

If we specialize the above definitions to the case of quasi-regular log Fano cone (X, D, ξ_0), then we get the corresponding more familiar notions for the log Fano projective pair $(S, B) = (X, D)/\langle \xi_0 \rangle$.

3.3. Sasaki-Einstein geometry

The introduction of normalized volumes in [Li18a] was motivated by the minimization phenomenon in the study of Sasaki-Einstein metrics. The latter was discovered in [MSY06, MSY08] and was motivated by the so called AdS/CFT correspondence from mathematical physics. Here we give a short account on this. For the reader who are mostly interested in the algebraic part of the theory, one can skip this section. The results will only be used in Section 5.2.

Classically, a Sasaki manifold is defined as an odd dimensional Riemannian manifold (M^{2n-1}, g_M) such that metric cone over it, defined as:

$$(X, g_X) := ((M \times \mathbb{R}_{>0}) \cup \{o_X\}, dr^2 + r^2 g_M)$$

is Kähler. It's convenient to work directly on $X = X^\circ \cup o_X$ which is an affine variety with the Kähler metric $\sqrt{-1}\partial\bar{\partial}r^2$. The Reeb vector field of (X°, g_X) is usually defined as $J(r\partial_r)$ where J is the complex structure on X°. The corresponding holomorphic vector field $\xi = r\partial_r - iJ(r\partial_r)$, which we also call Reeb vector field, generates a $T_\xi \cong (\mathbb{C}^*)^{\mathrm{rk}(\xi)}$-action on X where $r(\xi) \geq 1$. For simplicity, we will

denote such a torus by $\langle \xi \rangle$. Moreover the corresponding element in $(\mathfrak{t}_\xi)_\mathbb{R}$, also denoted by ξ is in the Reeb cone: $\xi \in (\mathfrak{t}_\xi)_\mathbb{R}^+$. The volume of ξ is defined to be the volume density of g_X:

$$
\begin{aligned}
\mathrm{vol}(\xi) \; := \; \mathrm{vol}(r^2) &= \frac{1}{(2\pi)^n n!} \int_X e^{-r^2} (\sqrt{-1}\partial\bar{\partial}r^2)^n \\
&= \frac{1}{(2\pi)^n} \int_M (-Jdr) \wedge (-dJdr)^{n-1} \\
&= \frac{(n-1)!}{2\pi^n} \mathrm{vol}(M, g_M) = \frac{\mathrm{vol}(M, g_M)}{\mathrm{vol}(\mathbb{S}^{2n-1})} \\
&= \frac{\mathrm{vol}(B_1(X), g_X)}{\mathrm{vol}(B_1(\underline{0}), g_{\mathbb{C}^n})}.
\end{aligned}
\tag{14}
$$

Here $g_X = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}r^2(\cdot, J\cdot)$ and $g_M = g_X|_M$ are the Riemannian metric on X and M respectively, $\mathbb{S}^{2n-1}$ is the standard unit sphere in $\mathbb{C}^n$ with volume $\mathrm{vol}(\mathbb{S}^{2n-1}) = 2\pi^n/(n-1)!$.

This is well-defined because if two Sasaki metrics have the same Reeb vector field, then their volumes are the same. Indeed, $\omega_1 = \sqrt{-1}\partial\bar{\partial}r_1$ and $\omega_2 = \sqrt{-1}\partial\bar{\partial}r_2$ have the same Reeb vector field if $r_2 = r_1 e^\varphi$ for a function φ satisfying $\mathcal{L}_{r\partial_r}\varphi = \mathcal{L}_\xi\varphi = 0$ (i.e. φ is a horizontal function on M with respect to the foliation defined by $\mathrm{Im}(\xi_0)$). Letting $r_t^2 = r^2 e^{t\varphi}$ and differentiating the volume we get:

$$
\begin{aligned}
C \cdot \frac{d}{dt}\mathrm{vol}(r_t^2) &= \int_X e^{-r_t^2}(-r_t^2\varphi)\sqrt{-1}\partial\bar{\partial}r_t^2)^n + e^{-r_t^2}n\sqrt{-1}\partial\bar{\partial}(r_t^2\varphi) \wedge (\sqrt{-1}\partial\bar{\partial}r_t^2)^{n-1} \\
&= \int_X -e^{-r_t^2}r_t^2\varphi(\sqrt{-1}\partial\bar{\partial}r_t^2)^n + e^{-r_t^2}n\sqrt{-1}\partial r_t^2 \wedge (\varphi\bar{\partial}r_t^2) \wedge (\sqrt{-1}\partial\bar{\partial}r_t^2)^{n-1} \\
&\quad + \int_X e^{-r_t^2}n\sqrt{-1}\partial r_t^2 \wedge (r_t^2\bar{\partial}\varphi) \wedge (\sqrt{-1}\partial\bar{\partial}r_t^2)^{n-1} \\
&= 0.
\end{aligned}
$$

The second equality follows from integration by parts. The last equality follows by substituting $f = r_t^2$ and $f = \varphi$ in to the following identities and using the fact that φ is horizontal (so that $\xi_t(\varphi) = 0$):

$$
n\sqrt{-1}\partial r_t^2 \wedge \bar{\partial}f \wedge (\sqrt{-1}\partial\bar{\partial}r_t^2)^{n-1} = \xi_t(f)(\sqrt{-1}\partial\bar{\partial}r_t^2)^n.
$$

One should compare this to the fact that two Kähler metrics in the same Kähler class have the same volume.

The Reeb vector field associated to a Ricci-flat Kähler cone metric satisfies the minimization principle in [MSY08]. To state it in general, we assume X is a T-variety with the Reeb cone $\mathfrak{t}_\mathbb{R}^+$ with respect to T and recall the variation formulas of volumes of Reeb vector fields from [MSY08]. For any $\xi \in \mathfrak{t}_\mathbb{R}^+$, we can find a radius function $r : X \to \mathbb{R}_+$ such that $\mathrm{vol}(\xi)$ is given by the formula (14).

Lemma 3.13. *The first order derivative of* $\mathrm{vol}_X(\xi)$ *is given by:*

$$D\mathrm{vol}(\xi) \cdot \eta_1 = \frac{1}{(2\pi)^n(n-1)!} \int_X \theta_1 e^{-r^2}(\sqrt{-1}\partial\bar\partial r^2)^n, \tag{15}$$

where $\theta_i = \eta_i(\log r^2)$. *The second order variation of* $\mathrm{vol}_X(\xi)$ *is given by:*

$$D^2\mathrm{vol}(\xi)(\eta_1, \eta_2) = \frac{n+1}{(2\pi)^n(n-1)!} \int_X \theta_1\theta_2 e^{-r^2}(\sqrt{-1}\partial\bar\partial r^2)^n.$$

Now we fix a $\xi_0 \in \mathfrak{t}_{\mathbb{R}}^+$ and a radius function $r : X \to \mathbb{R}_+$ (by using equivariant embedding of X into $\mathbb{C}^N$ for example), we define:

Definition 3.14. $PSH(X, \xi_0)$ is the set of bounded real functions φ on X° that satisfies:

(1) $\varphi \circ \tau = \varphi$ for any $\tau \in \langle \xi_0 \rangle$, the torus generated by ξ_0;
(2) $r_\varphi^2 := r^2 e^\varphi$ is a proper plurisubharmonic function on X.

To write down the equation of Ricci-flat Kähler-cone equation, we fix a T-equivariant no-where vanishing section $s \in H^0(X, mK_X)$ as in the last section and define an associated volume form on X:

$$dV_X := \left((\sqrt{-1})^{mn^2} s \wedge \bar s \right)^{1/m}. \tag{16}$$

Definition 3.15. We say that $r_\varphi^2 := r^2 e^\varphi$ where $\varphi \in PSH(X, \xi_0)$ is the radius function of a Ricci-flat Kähler cone metric on (X, ξ_0) if φ is smooth on X^{reg} and there exists a positive constant $C > 0$ such that

$$(\sqrt{-1}\partial\bar\partial r_\varphi^2)^n = C \cdot dV, \tag{17}$$

where the constant C is equal to:

$$C = \frac{\int_X e^{-r_\varphi^2}(\sqrt{-1}\partial\bar\partial r_\varphi^2)^n}{\int_X e^{-r_\varphi^2} dV_X} = \frac{(2\pi)^n n! \mathrm{vol}(\xi_0)}{\int_X e^{-r_\varphi^2} dV_X}.$$

Motivated by standard Kähler geometry, one defines the Monge-Ampère energy $E(\varphi)$ using either its variations or the explicit expression on the link $M := X \cap \{r = 1\}$:

$$\delta E(\varphi) \cdot \delta\varphi = -\frac{1}{(n-1)!(2\pi)^n \mathrm{vol}(\xi_0)} \int_X \delta\varphi e^{-r_\varphi^2}(\sqrt{-1}\partial\bar\partial r_\varphi^2)^n.$$

Then the equation (16) is the Euler-Lagrange equation of the following Ding-Tian-typed functional:

$$D(\varphi) = E(\varphi) - \log\left(\int_X e^{-r_\varphi^2} dV_X \right).$$

This follows from the identity:

$$\delta D(\varphi) \cdot \delta\varphi = \frac{1}{(2\pi)^n(n-1)!\mathrm{vol}(\xi_0)} \int_X \delta\varphi e^{-r_\varphi^2}(\sqrt{-1}\partial\bar{\partial}r_\varphi^2)^n - \frac{\int_X r_\varphi^2 \delta\varphi e^{-r_\varphi^2} dV_X}{\int_X e^{-r_\varphi^2} dV_X}$$

$$= n \int_X e^{-r_\varphi^2}\delta\varphi \left(\frac{(\sqrt{-1}\partial\bar{\partial}r_\varphi^2)^n}{(2\pi)^n n!\mathrm{vol}(\xi_0)} - \frac{dV_X}{\int_X e^{-r_\varphi^2} dV_X} \right).$$

Compared with the weak Kähler-Einstein case, it is expected that the regularity condition in the above definition is automatically satisfied. With this regularity assumption, on the regular part X^{reg}, both sides of (17) are smooth volume forms and we have $r_\varphi\partial_{r_\varphi} = 2\mathrm{Re}(\xi_0)$ or, equivalently, $\xi_0 = r_\varphi\partial_{r_\varphi} - iJ(r_\varphi\partial_{r_\varphi})$. Moreover, taking $\mathcal{L}_{r_\varphi\partial_{r_\varphi}}$ on both sides gives us the identity $\mathcal{L}_{r_\varphi\partial_{r_\varphi}} dV = 2n\, dV$. Equivalently we have:

$$\mathcal{L}_{\xi_0} s = mn \cdot s,$$

where $s \in |-mK_X|$ is the chosen T-equivariant non-vanishing holomorphic section. By Lemma 3.6, this implies $A_X(\mathrm{wt}_{\xi_0}) = n$ (see [HS17, LL19] for this identity in the quasi-regular case). The main result of [MSY08] can be stated as follows.

Theorem 3.16. *If (X, ξ_0) admits a Ricci-flat Kähler cone metric, then $A_X(\xi_0) = n$ and wt_{ξ_0} obtains the minimum of vol on $\mathfrak{t}_{\mathbb{R}}^+$.*

The following result partially generalizes Berman's result on K-polystability of Kähler-Einstein Fano varieties to the more general case of Ricci-flat Fano cones. Together with Theorem 4.6, it is used to show a generalization the minimization result [MSY08]: the valuation wt_{ξ_0} minimizes $\widehat{\mathrm{vol}}$ where ξ_0 is the Reeb vector field of the Ricci-flat Fano cone.

Theorem 3.17 (see [CS15, LX16, LX17a]). *Assume (X, ξ_0) admits a Ricci-flat Kähler cone metric. Then $A_X(\mathrm{wt}_{\xi_0}) = n$ and (X, ξ_0) is K-polystable among all special test configurations.*

Proof. Fix any smooth Kähler cone metric $\sqrt{-1}\partial\bar{\partial}r^2$ on X. Any special test configuration determines a geodesic ray $\{r_t^2 = r^2 e^{\varphi_t}\}_{t>0}$ of Kähler cone metrics. Denote $D(t) = D(\varphi_t)$. Then we have the following formula:

$$\lim_{t\to 0} \frac{D(t)}{-\log|t|^2} = \frac{D_{-\eta}\mathrm{vol}(\xi_0)}{\mathrm{vol}(\xi_0)} - (1 - \mathrm{lct}(\mathcal{X}, \mathcal{X}_0)) = D^{\mathrm{NA}}(\chi, \xi_0; \eta), \qquad (18)$$

which is a combination of two ingredients:

1. The Fano cone version of an identity from Kähler geometry which combined with (15) gives the formula:

$$\lim_{t\to 0} \frac{E(\varphi_t)}{-\log|t|^2} = \frac{D_{-\eta}\mathrm{vol}(\xi_0)}{\mathrm{vol}(\xi_0)}. \qquad (19)$$

2. $G(\varphi_t)$ is subharmonic in t (cone version of Berndtsson's result) and its Lelong number at $t = 0$ is given by $1 - \mathrm{lct}(\mathcal{X}, \mathcal{X}_0)$ (cone version of Beman's result).

The other key result is the cone version of Berndtsson's subharmonicity and uniqueness result, which was used to characterize the case of vanishing Futaki invariant. □

Remark 3.18. The argument in [LX17a] gives a slightly more general result: Assume (X, ξ_0) admits a Ricci-flat Käler cone metric, then $A_X(\mathrm{wt}_{\xi_0}) = n$ and (X, ξ_0) is Ding-polystable among $\mathbb{Q}$-Gorenstein test configurations (see Remark 3.12).

4. Stable degeneration conjecture

In this section, we give a conjectural description of minimizers for general klt singularities, and explain various parts of the picture that we can establish.

4.1. Statement

For a klt singularity $x \in (X, D)$, one main motivation to study the minimizer v of $\widehat{\mathrm{vol}}_{(X,D),x}$ is to establish a 'local K-stability' theory, guided by the local-to-global philosophy mentioned in the introduction. In particular, we propose the following conjecture for all klt singularities.

Conjecture 4.1 (Stable Degeneration Conjecture, [Li18a, LX17a]). *Given any arbitrary klt singularity $x \in (X = \mathrm{Spec}(R), D)$, there is a unique minimiser v up to rescaling. Furthermore, v is quasi-monomial, with a finitely generated associated graded ring $R_0 =_{\mathrm{defn}} \mathrm{gr}_v(R)$, and the induced degeneration*

$$(X_0 = \mathrm{Spec}(R_0), D_0, \xi_v)$$

is a K-semistable Fano cone singularity. (See below for the definitions.)

Let us explain the terminology in more details: First by the grading of R_0, there is a $T \cong \mathbb{C}^r$-action on X_0 where r is the rational rank of v, i.e. the valuative semigroup Φ of v generates a group $M \cong \mathbb{Z}^r$. Moreover, since the valuation v identifies M to a subgroup of $\mathbb{R}$ and sends Φ into $\mathbb{R}_{\geq 0}$, it induces an element in the Reeb cone ξ_v. If R_0 is finitely generated, then [LX17a] shows that we can embed $(x \in X) \subset (0 \in \mathbb{C}^N)$ and find an rational vector $\xi \in \mathfrak{t}_{\mathbb{R}}^+ \cap N_{\mathbb{Q}}$ sufficiently close to ξ_v such that the $\mathbb{C}^*$-action generated by ξ degenerates X to X_0 with a good action. We denote by o (or o_{X_0}) the unique fixed point on X_0. Furthermore, the extended Rees algebra yielding the degeneration does not depend on the choice of ξ. So we can define D_0 as the degeneration of D.

Conjecture 4.1, if true, would characterize deep properties of a klt singularity. Various parts are known, see Theorem 4.14. However, the entire picture remains open in general.

4.2. Cone case

The study of the case of cone is not merely verifying a special case. In fact, since the stable degeneration conjecture predicts the degeneration of any klt singularities to a cone, understanding the cone case is a necessary step to attack the conjecture. Here we divide our presentations into two case: the rank one case and the

general higher rank case. Although our argument in the higher rank case covers the rank one case with various simplifications, we believe it is easier for reader to first understand the rank one case, as it is equivalent to the more standard K-semistability theory of the base which is a log Fano pair. This connection is made via the theory of β-invariant, which is first introduced in [Fuj18] in terms of ideal sheaves and further developed in [Li17, Fuj16] via valuations.

4.2.1. Rank one case. The rank one Fano cone is just a cone over a log Fano pair. More precisely, let (S, B) be an $(n-1)$-dimensional log Fano pair, and r a positive integer such that $r(K_S + B)$ is Cartier. Then we can consider the minimizing problem of the normalized volume at the vertex of the cone

$$x \in (X, D) = C(S, B; -r(K_S + B)).$$

Such a question was first extensively studied in [Li17]. More precisely, there is a canonical divisorial valuation obtained by blowing up x to get a divisor S_0 isomorphic to S, which yields the degeneration of $x \in (X, D)$ to itself with ξ being the natural rescaling vector field from the cone structure. Therefore, the stable degeneration conjecture predicts $v_{S_0} = \mathrm{ord}_{S_0}$ is a minimizer of $\widehat{\mathrm{vol}}_{(X,D),x}$ if and only if (S, B) is K-semistable, and this is confirmed in [Li17, LL19, LX16].

Theorem 4.2. *The valuation v_{S_0} is a stabilizer of $\widehat{\mathrm{vol}}_{(X,D),x}$ if and only if (S, B) is K-semistable. Moreover, $\widehat{\mathrm{vol}}(S_0) < \widehat{\mathrm{vol}}(E)$ for any other divisor E over x.*

In the below, we will sketch the ideas of two slightly different proofs of Theorem 4.2.

In the first approach, we carry out a straightforward calculation as follows: Given a compactified nontrivial special test configuration $(\mathcal{S}, \mathcal{B})$ of (S, B), then we obtain a valuation v^* by restricting the divisorial valuation of the special fiber S_0 to $K(S) \subset K(S \times \mathbb{A}^1)$, which is a multiple of some divisorial valuation (cf. [BHJ17]). Such a valuation v^* pull backs a valuation v_X^* on $K(X)$. Then we define a $\mathbb{C}^*$-valuation on $K(X)$ by $v_\infty(f_m) = v_X^*(f_m) - mr a_S(v^*)$ over X for any $f_m \in H^0(S, -mr(K_S + B))$. In other words, $v_\infty = v_X^* - r a_S(v^*) v_{S_0}$, and we know that the induced filtration on R yields the Duistermaat-Heckman (DH) measure of $(\mathcal{S}, \mathcal{B})$ (see [BHJ17, Definition 3.5]). We define the ray in

$$\left\{ v_t = v_{S_0} + t \cdot v_\infty \in \mathrm{Val}_{X,x} \mid t \in [0, \frac{1}{r a_S(v^*)}) \right\}.$$

Then the key computation in [Li17] is that

$$\frac{d}{dt}\widehat{\mathrm{vol}}(v_t)|_{t=0} \quad = \quad \frac{n}{r^n}(-K_S - B)^{n-1} \cdot \mathrm{Fut}(\mathcal{S}, \mathcal{B}). \tag{20}$$

In fact, if for any valuation v over S, we denote by $R_m = H^0(S, -mr(-K_S - B))$ and define

$$\mathcal{F}_v^x R_m := \{ f \in R_m \mid f \in H^0(S, -mr(-K_S - B) \otimes \mathfrak{a}_x) \},$$

then we easily see

$$\mathfrak{a}_k(v_t) \cap R_m = \mathcal{F}_{v_\infty}^{\frac{k-m}{t}} H^0(S, -mr(-K_S - B)).$$

So

$$
\begin{aligned}
\mathrm{vol}(v_t) &= \lim_k \frac{l_{\mathbb{C}}(R/\mathfrak{a}_k(v_t))}{k^n/n!} \\
&= \lim_{k\to\infty} \frac{n!}{k^n} \sum_{m=0} \left(\dim \mathcal{F}_{v_\infty}^0 R_m - \dim \mathcal{F}_{v_\infty}^{\frac{k-m}{t}} H^0(S, -mr(-K_S - B)) \right) \\
&= - \int_{-\infty}^{\infty} \frac{d\mathrm{vol}(\mathcal{F}_{v_\infty} R^{(x)})}{(1 + tx)^n},
\end{aligned}
\tag{21}
$$

where $\mathcal{F}_{v_\infty} R^{(x)} := \bigoplus_m \mathcal{F}_{v_\infty}^{mx} R_m$ and the last equality is obtained by a change of variables (see Lemma [Li17, Lemma 4.5]).

Since $A(S_0) = \frac{1}{r}$ and $A(v_\infty) = 0$, $A_{v_t} = \frac{1}{r}$, so

$$\widehat{\mathrm{vol}}(v_t) = -(\frac{1}{r})^n \int_{-\infty}^{\infty} \frac{d\mathrm{vol}(\mathcal{F}_{v_\infty} R^{(x)})}{(1 + tx)^n},$$

and this implies that

$$
\begin{aligned}
\frac{d}{dt}\widehat{\mathrm{vol}}(v_t)|_{t=0} &= \frac{n}{r^n} \int_{-\infty}^{\infty} x \cdot d\mathrm{vol}(\mathcal{F}_{v_\infty} R^{(x)}) \\
&= \frac{n}{r^n} \lim_{k\to\infty} \frac{w_k}{kN_k} \\
&= -\frac{1}{r^n}(-K_{\mathcal{S}} - \mathcal{B})^n, \\
&= \frac{n}{r^n}(-K_S - B)^{n-1} \cdot \mathrm{Fut}(\mathcal{S}, \mathcal{B}).
\end{aligned}
$$

where for the second equality we use that v_∞ is the DH measure for $(\mathcal{S}, \mathcal{B})$.

It is not straightforward to reverse the argument to show that (S, B) is K-semistable implies that ord_{S_0} is a minimizer of $\widehat{\mathrm{vol}}_{(X,D),x}$, since a priori there could be more complicated valuations than those induced by central fibres of test configurations. In particular, originally in [Li17], the techniques of 'taking the limit of a sequence of filtered linear systems' developed in [Fuj18] were used in the case when the associated bigraded ring

$$\bigoplus_{m,k} H^0(S, -rm(K_S + B) \otimes \mathfrak{a}_k)$$

is not finitely generated, and this is enough to treat all $\mathbb{C}^*$-equivariant valuations.

In [LX16], after the MMP method was systematically applied, it was shown that

$$\inf_{v\in\mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v) = \{\inf \widehat{\mathrm{vol}}(\mathrm{ord}_S) \mid \mathbb{C}^*\text{-equivariant Kollár components } S\} \tag{22}$$

(see (27) and the discussion below it). Since Kollár components yield special degenerations, therefore, the above arguments can be essentially reversed. See Section 4.2.2.

Remark 4.3. In fact, we establish a one-to-one correspondence between special test configurations of (S, B) (up to a base change) and rays in $\mathrm{Val}_{X,x}$ emanating from v_{S_0} containing a Kollár component (different with v_{S_0}).

$\qquad$ An interesting consequence is that the above argument indeed gives an alternative way to show that K-semistability implies the valuative criterion of K-semistability with β-invariant as in [Fuj16, Li17], but without using the arguments of 'taking a limit of filtered linear systems'.

$\qquad$ The second approach to treat the cone singularity is developed in [LL19] (see also [LX16]). It is shown that K-semistablity of (S, B) is equivalent to that of $(\bar{X}, \bar{D} + (1 - \frac{1}{rn})S_\infty)$, where $(\bar{X}, \bar{D})$ is the projective cone of (X, D) with respect to $-r(K_X + D)$ and $S_\infty (= S)$ is the divisor at the infinity place. This follows from a straightforward Futaki invariant calculation as in [LX16, Proposition 5.3]. Applying the inequality 5.12 to $x \in (\bar{X}, \bar{D} + (1 - \frac{1}{rn})S_\infty)$, we immediately conclude that

$$\widehat{\mathrm{vol}}(x, \bar{X}, \bar{D}) \geq \frac{(-K_S - B)^{n-1}}{r^n} = \widehat{\mathrm{vol}}_{(X,D),x}(\mathrm{ord}_{S_0}). \qquad (23)$$

$\qquad$ To understand better the relation between the K-semistability of $(\bar{X}, \bar{D} + (1 - \frac{1}{rn})S_\infty)$ and of (S, B), we want to present a direct calculation which connects the calculation on β-invariant on $(\bar{X}, \bar{D} + \frac{1}{rn}S_\infty)$ and the one on (S, B).

Lemma 4.4. *Assume β-invariant is nonnegative for any divisorial valuation over S. Denote by $\hat{L} = \mathcal{O}(1) = \mathcal{O}(S_\infty)$ and $\delta = \frac{n+1}{rn}$. For any $\mathbb{C}^*$-invariant divisorial valuation E. We have the following*

$$\beta(E) := A_{(\bar{X}, \bar{D} + (1 - \frac{1}{rn})S_\infty)}(E) - \frac{\delta}{\hat{L}^n} \int_0^{+\infty} \mathrm{vol}(\mathcal{F}_{\mathrm{ord}_E} \hat{R}^{(x)}) dx \geq 0, \qquad (24)$$

where $\hat{R} = \bigoplus_{m=0}^{+\infty} H^0(\bar{X}, m\hat{L})$.

$\qquad$ The key of the proof is to relate the β-invariant for a $\mathbb{C}^*$-invariant valuation v over $\bar{X}$ to the β-invariant of the restriction of v over the base S.

Proof. We have $K_{\bar{X}} + \bar{D} + (1 - \frac{1}{rn})S_\infty = -\frac{n+1}{rn}\hat{L} = -\delta\hat{L}$, and define

$$\mathcal{F}_v^x \hat{R}_m := \{f \in \hat{R}_m | \ f \in \hat{R}_m = \oplus_{0 \leq k \leq m} H^0(S, kr(-K_S - B)) \text{ and } v(f) \geq x\},$$

$\qquad$ For any $\mathbb{C}^*$-invariant divisorial valuation $v = \mathrm{ord}_E$ on $\bar{X}$, there exists $c_1 \in \mathbb{Z}$, $a \geq 0$ and a divisorial valuation ord_F over S such that for any $f \in H^0(S, mr(-K_S - B))$, we have

$$v(t) = c_1; \text{ and } v(f) = a \cdot \mathrm{ord}_F(f) =: \bar{v}(f).$$

We estimate $\beta(E)$ in three cases depending on the signs of a and c_1:

$(a = 0)$: The valuation v is associated to the canonical $\mathbb{C}^*$-action along the ruling of the cone, up to rescaling, then we easily get $\beta(E) = 0$

$(a > 0$ and $c_1 \geq 0)$: Then the center of v is contained in S_∞. In this case we can easily calculate:

$$
\begin{aligned}
\mathrm{vol}(\mathcal{F}\hat{R}^{(x)}) &= \lim_{m \to +\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}^{xm}\hat{R}_m}{m^n/n!} = \lim_{m \to +\infty} \frac{1}{m^n/n!} \sum_{k=0}^{m} \dim_{\mathbb{C}} \mathcal{F}_{\bar{v}}^{xm-c_1(m-k)} R_k \\
&= n \int_0^1 \mathrm{vol}(\mathcal{F}_{\bar{v}} R^{(c_1 + \frac{x-c_1}{\tau})}) \tau^{n-1} d\tau,
\end{aligned}
$$

where the last identity can be proved in the same way as in (21). So we have:

$$
\begin{aligned}
\int_0^{+\infty} \mathrm{vol}(\mathcal{F}\hat{R}^{(x)}) dx &= n \int_0^{+\infty} dx \int_0^1 \mathrm{vol}(\mathcal{F}_{\bar{v}} R^{(c_1 + \frac{x-c_1}{\tau})}) \tau^{n-1} d\tau \\
&= n \int_0^1 \tau^{n-1} d\tau \int_0^{+\infty} \mathrm{vol}(\mathcal{F}_{\bar{v}} R^{(c_1 + \frac{x-c_1}{\tau})}) dx \\
&= n \int_0^1 \tau^{n-1} d\tau \left[H^{n-1} c_1(1 - \tau) + \tau \int_{c_1}^{+\infty} \mathrm{vol}\left(\mathcal{F}_{\bar{v}} R^{(y)}\right) dy \right] \\
&= \frac{c_1}{n+1} + n \int_0^{+\infty} \mathrm{vol}(\mathcal{F}_{\bar{v}} R^{(x)}) dx \int_0^1 \tau^n d\tau \\
&= \frac{c_1}{n+1} + \frac{n}{n+1} \int_0^{+\infty} \mathrm{vol}(\mathcal{F}_{\bar{v}} R^{(x)}) dx.
\end{aligned}
$$

On the other hand, we have $H^{n-1} = \hat{L}^n$ and:

$$
A_{(\bar{X}, \bar{D}+(1-\beta)S_\infty)}(\mathrm{ord}_E) = A_{(S,B)}(\bar{v}) + c_1 - (1-\beta)c_1 = A_{(S,B)}(\bar{v}) + \frac{c_1}{rn}
$$

So we get:

$$
\begin{aligned}
\beta(E) &= A_{(S,B)}(\bar{v}) + \frac{c_1}{rn} - \frac{\frac{n+1}{rn}}{H^{n-1}} \frac{n}{n+1} \left(\frac{c_1}{n+1} + \int_0^{+\infty} \mathrm{vol}(\mathcal{F}_{\bar{v}} R^{(x)}) dx \right) \\
&= A_{(S,B)}(\bar{v}) - \frac{1}{rH^{n-1}} \int_0^{+\infty} \mathrm{vol}(\mathcal{F}_{\bar{v}} R^{(x)}) dx = \beta(\bar{v}),
\end{aligned}
$$

which is non-negative by our assumption.

$(a > 0$ and $c_1 < 0)$: In this case, the center of v is at the vertex. As a consequence we have:

$$
\begin{aligned}
A_{(\bar{X}, \bar{D}+(1-\beta)S_\infty)}(v) &= A_{(S,B)}(\bar{v}) + (-c_1) + (\frac{1}{r} - 1)(-c_1) \\
&= A_{(S,B)}(\bar{v}) + \frac{-c_1}{r} \geq A_{(S,B)}(\bar{v}).
\end{aligned}
$$

The similar calculation as in the second case shows that $\beta(E) \geq \beta(\bar{v})$. $\square$

Finally, to show $\widehat{\mathrm{vol}}(S_0) < \widehat{\mathrm{vol}}(E)$ for $E \neq S_0$, in [LX16], it was first proved that if E is a minimizer then it has to be a $\mathbb{C}^*$-equivariant Kollár component. Then

a careful study of the geometry of E using the equality condition in (23) implies $E = S$. This is similar to the analysis for the equality case in [Fuj18, Liu18] where they showed that the K-stable $\mathbb{Q}$-Fano variety with the maximal volume $(n+1)^n$ can only be $\mathbb{CP}^n$. We will leave the discussion on this uniqueness type result to the general case of cones of higher rational ranks, where we take a somewhat different approach, using more convex geometry.

Remark 4.5. It is worthy pointing out that there is another global invariant for an n-dimensional log Fano pair (S, B), defined as

$$\delta(S, B) = \inf_{v \in \mathrm{Val}_S} \frac{A_{(S,B)}(v) \cdot (-K_S - B)^n}{\int_0^\infty \mathrm{vol}(-K_S - B - tv)dt}$$

(see [FO16, BJ17]). δ-invariant shares lots of common properties with the normalized volume. For example, the existence of minimizers were proved using similar strategy. They both have differential geometric meanings. The minimizer of $\widehat{\mathrm{vol}}$ is related to the metric tangent cone (see section 5.2); while the valuation on $K(S)$ yielding $\delta(S, B)$ is related to the the existence of twisted Kähler-Einstein metrics (see [BJ18]).

For a log Fano pair (S, B) and a cone $x \in (X, D) = C(S, B; -r(K_S + B))$, if (S, B) is not K-semistable, or equivalently $\delta = \delta(S, B) < 1$, then we have

$$\widehat{\mathrm{vol}}(x, X, D) \geq \frac{\delta^n \cdot (-K_S - B)^{n-1}}{r^n}.$$

This follows from our second proof by looking at $(\bar{X}, \bar{D} + (1 - \beta)S_\infty)$ and applying the inequality [BJ17, Theorem D] which can be written as

$$(K_{\bar{X}} + \bar{D} + (1 - \beta)S_\infty)^n \leq \frac{(n+1)^n}{n^n} \cdot \widehat{\mathrm{vol}}(x, X, D) \cdot \bar{\delta}^n,$$

where $\bar{\delta} := \delta(\bar{X}, \bar{D} + (1 - \beta)S_\infty)$. We claim $\min\{\bar{\delta}, 1\} = \delta$. In fact, by the argument in [BJ17, Section 7], we know that $\bar{\delta}$ is computed by a $\mathbb{C}^*$-invariant valuation and the claim follows from the calculation in the proof of Lemma 4.4.

4.2.2. Log Fano cone in general. We proceed to investigate a log Fano cone $o \in (X, D, \xi)$ where the torus T could have dimension larger than one. However, we consider not only the valuations in $\mathfrak{t}_{\mathbb{R}}^+(X)$ coming from the torus as in [MSY08] (see Section 3.1) but all valuations in $\mathrm{Val}_{X,o}$. Compared to the proof of Theorem 4.2, for the higher rational rank case, we rely more on the construction of Kollár components coming from the birational geometry. More explicitly, we use the relation between special test configurations and Kollár components (see [LX16, 2.3] and [LX17a, 3.1]).

By the results from the MMP (see (27) and the explanation below), to show a valuation is a minimizer in $\mathrm{Val}_{X,x}$, we only need to show its normalized volume is not greater than that of any T-invariant Kollár component. On the other hand, any T-equivariant Kollár component E in $\mathrm{Val}_{X,o}$ yields a special test configuration of $(\mathcal{X}, \mathcal{D}, \xi; \eta)$ of (X, D) such that $-\eta \in \mathfrak{t}_{\mathbb{R}}^+(X_0)$ and the valuation associated to $-\eta$ coincides with ord_E. We denote by (X_0, D_0) the fiber with a cone vertex o. Then

we can compare the volumes as $\widehat{\mathrm{vol}}_X(\xi) = \widehat{\mathrm{vol}}_{X_0}(\xi)$ and $\widehat{\mathrm{vol}}_X(E) = \widehat{\mathrm{vol}}(-\eta)$. Since $\xi, -\eta \in \mathfrak{t}_{\mathbb{R}}^+(X_0)$ we reduce the question to the set up of [MSY08] on X_0. Then we only need to each time treat one degeneration X_0 and try to understand how to pass properties between X_0 and X.

With this strategy, we can show the following generalization of Theorem 4.2.

Theorem 4.6 ([LX17a]). *Let* $x \in (X, D, \xi)$ *be a log Fano cone singularity. Then* v_ξ *is a minimizer of* $\widehat{\mathrm{vol}}_{(X,D),x}$ *if and only if* (X, D, ξ) *is K-semistable. In such case,* $\widehat{\mathrm{vol}}(v_\xi) < \widehat{\mathrm{vol}}(v)$ *for any quasi-monomial valuation* v *if* v *is not a rescaling of* v_ξ.

If (X, D, ξ) is K-semistable, then for each special test configuration $(\mathcal{X}, \mathcal{D}, \xi; \eta)$, on X_0, we can consider the ray $\xi_t = \xi - t\eta$ for $t \in [0, \infty)$. We know

$$\frac{d}{dt} \widehat{\mathrm{vol}}_{(X_0,D_0),o}(v_{\xi_t})|_{t=0} = c \cdot \mathrm{Fut}(\mathcal{X}, \mathcal{D}, \xi; \eta) \geq 0.$$

Moreover, when $(X_0, D_0, o) = (X_0, \emptyset, o)$ is an isolated singularity, it was shown in [MSY08] that $\widehat{\mathrm{vol}}(v_{\xi_t})$ is a convex function. We obtain a stronger result for any log Fano cone (X_0, D_0, ξ_0) (see Section 4.2.3). In particular, we conclude that $\widehat{\mathrm{vol}}(v_{\xi_t})$ is an increasing function of t, and its limit is $\widehat{\mathrm{vol}}(-\eta)$, thus the inequality in the following relation holds true:

$$\widehat{\mathrm{vol}}_{(X,D),x}(\xi) = \widehat{\mathrm{vol}}_{(X_0,D_0),o}(\xi) \leq \widehat{\mathrm{vol}}_{(X_0,D_0),o}(-\eta) = \widehat{\mathrm{vol}}_{(X,D),x}(E).$$

The first identity consists of two identities: $A_{(X,D)}(v_\xi) = A_{(X_0,D_0)}(v_\xi)$ and $\mathrm{vol}_X(v_\xi) = \mathrm{vol}_{X_0}(v_\xi)$, which essentially follow from the flatness of T-equivariant test configuration (see [LX17a, Lemma 3.2]). The last identity is because $v_{-\eta} = \mathrm{ord}_E$.

This argument is reversible since we can indeed attach to any special test configuration such a set of valuations (see Remark 4.3): if we consider the valuation w_t obtained by considering the vector field ξ_t as a valuation on $K(\mathcal{X})$ and then take its restriction on $K(X)$. The corresponding degeneration induces the test configuration. See [LX16, 6] and [LX17a, 4.2] for more details.

4.2.3. Uniqueness. We have seen the convexity of the normalized volume function in the Reeb cone plays a key role. In [MSY08], the strict convexity on the normalized function is established for the valuation varying inside the Reeb cone for an isolated singularity. This is the kind of property we need for the uniqueness of the minimizer of a K-semistable Fano cone singularity (X, D, ξ). However, as we do not know the associated graded ring of other minimizer is finitely generated, we can not degenerate two minimizers into the Reeb cone. Thus we need develop a technique to deal with valuations outside the Reeb cone.

The idea of the argument in [LX17a, Section 3.2] is to use the theory of Newton-Okounkov bodies which was first developed in [LM09, KK12]) and in the local setting in [Cut13, KK14]. This is a theory which realizes the volumes in algebraic geometry with an asymptotic nature to the Euclidean volumes of some

convex bodies in $\mathbb{R}^n$. So our aim is to apply the Newton-Okounkov body construction to translate the normalized volume of valuations into the volume of convex bodies, and then invoke a convexity property of the volumes functions known in the latter setting.

To start, we first need to set a valuation $\mathbb{V}$ with $\mathbb{Z}^n$-valued valuation, which sends the elements in R to the lattice points inside a convex region $\tilde{\sigma}$, so that later we can realize the normalized volumes of valuations as the volume of subsets in $\tilde{\sigma}$.

For any fixed $T \cong (\mathbb{C}^*)^r$-equivariant quasi-monomial valuation μ, we know it is of the form $(\xi_\mu, v^{(0)})$ where $\xi_\mu \in M_\mathbb{R}$ and $v^{(0)}$ is a quasi-monomial valuation over $K(Y)$, such that for any function $f \in R_u$,

$$\mu(f) = \langle \xi_\mu, u \rangle + v^{(0)}(f)$$

(see Theorem 3.5(1)). We fix a lexicographic order on $\mathbb{Z}^r$ and define for any $f \in R$,

$$\mathbb{V}_1(f) = \min\{u; f = \sum_u f_u \text{ with } f_u \neq 0\} = \mathbb{V}_1(f),$$

i.e., the first factor $\mathbb{V}_1$ comes from the toric part of μ.

We extend this $\mathbb{Z}^r$-valuation $\mathbb{V}_1$ to become a $\mathbb{Z}^n$-valued valuation in the following way: Denote $u_f = \mathbb{V}_1(f) \in \sigma^\vee$ and f_{u_f} the corresponding nonzero component. Define $\mathbb{V}_2(f) = v^{(0)}(f_{u_f})$. Because $\{\beta_i\}$ are $\mathbb{Q}$-linearly independent, we can write $\mathbb{V}_2(f) = \sum_{i=1}^s m_i^* \beta_i$ for a uniquely determined $m^* := m^*(f_{u_f}) = \{m_i^* := m_i^*(f_{u_f})\}$. Moreover, the Laurent expansion of f has the form:

$$f_{u_f} = z_1^{m_1^*} \ldots z_s^{m_s^*} \chi_{m^*}(z'') + \sum_{m \neq m^*} z_1^{m_1} \ldots z_s^{m_s} \chi_m(z''). \tag{25}$$

Then $\chi_{m^*}(z'')$ in the expansion of (25) is contained in $\mathbb{C}(Z)$, where on some model of Y, we have $Z = \{z_1 = 0\} \cap \ldots \{z_s = 0\} = D_1 \cap \cdots \cap D_s$ is the center of $v^{(0)}$.

Extend the set $\{\beta_1, \ldots, \beta_s\}$ to $d = n - r$ $\mathbb{Q}$-linearly independent positive real numbers $\{\beta_1, \ldots, \beta_s; \gamma_1, \ldots, \gamma_{d-s}\}$. Define $\mathbb{V}_3(f) = w_\gamma(\chi_{m^*}(z''))$ where w_γ is the quasi-monomial valuation with respect to the coordinates z'' and the $(d-s)$ tuple $\{\beta_1, \ldots, \beta_s; \gamma_1, \ldots, \gamma_{d-s}\}$.

Now we assign the lexicographic order on

$$\mathbb{G} := \mathbb{Z}^r \times G_2 \times G_3 \cong \mathbb{Z}^r \times \mathbb{Z}^s \times \mathbb{Z}^{n-r-s}$$

and define $\mathbb{G}$-valued valuation:

$$\mathbb{V}(f) = (\mathbb{V}_1(f), \mathbb{V}_2(f_{u_f}), \mathbb{V}_3(\chi_{m^*})). \tag{26}$$

Let $\mathcal{S}$ be the valuative semigroup of $\mathbb{V}$. Then $\mathcal{S}$ generates a cone $\tilde{\sigma}$ which is the one we are looking for. We also let $P_1 : \mathbb{R}^n \to \mathbb{R}^r$, $P_2 : \mathbb{R}^n \to \mathbb{R}^s$ and $P = (P_1, P_2) : \mathbb{R}^n \to \mathbb{R}^{r+s}$ be the natural projections. Then $P_1(\tilde{\sigma}) = \sigma \subset \mathbb{R}^r$.

To continue, we consider how to construct some subsets $\Delta_{\tilde{\Xi}_t} \subset \tilde{\sigma}$ whose Euclidean volume is the same as the normalized volumes of the valuations. For any $\xi \in \text{int}(\sigma)$, denote by wt_ξ the valuation associated to ξ. We can connect wt_ξ

and μ by a family of quasi-monomial valuations: $\mu_t = ((1-t)\xi + t\xi_\mu, tv^{(0)})$ defined as

$$\mu_t(f) = tv^{(0)}(f) + \langle u, (1-t)\xi + t\xi_\mu \rangle \quad \text{for any } f \in R_u.$$

So the vertical part of μ_t corresponds to the vector $\Xi_t := ((1-t)\xi + t\xi_\mu, t\beta) \in \mathbb{R}^{r+s}$. Extend Ξ_t to $\tilde{\Xi}_t := (\Xi_t, 0) \in \mathbb{R}^n$ and define the following set:

$$\Delta_{\tilde{\Xi}_t} = \left\{ y \in \tilde{\sigma}; \langle y, \tilde{\Xi}_t \rangle \leq 1 \right\} = \left\{ y \in \tilde{\sigma}; \langle P(y), \Xi_t \rangle \leq 1 \right\}.$$

Because $\widehat{\mathrm{vol}}$ is rescaling invariant, we can assume $A_{(X,D)}(v) = A_{(X,D)}(\xi) = 1$. Then by the T-invariance of v_t, we easily get:

$$A(v_t) = tA(v^{(0)}) + A_{(X,D)}((1-t)\xi + t\zeta) = tA_{(X,D)}(v) + (1-t)A_{(X,D)}(\xi) \equiv 1.$$

The Newton-Okounkov body theory implies that we have

$$\widehat{\mathrm{vol}}(v_t) = \mathrm{vol}(v_t) = \mathrm{vol}(\Delta_{\tilde{\Xi}_t}).$$

To finish the uniqueness argument, now we only need to look at the convex geometry of $\Delta_{\tilde{\Xi}_t}$. We note that $\tilde{\Xi}_t$ is linear with respect to t, and each region $\Delta_{\tilde{\Xi}_t}$ is cut out by a hyperplane H_t on the convex cone $\tilde{\sigma}$. Moreover, all H_t passes through a fixed point. A key result from convex geometry then shows that $\phi(t) := \mathrm{vol}(\Delta_{\tilde{\Xi}_t})$ is strictly convex as a function of $t \in [0, 1]$ (see [MSY06, Gig78]). By the assumption $\phi(0) = \mathrm{vol}(v_0) = \widehat{\mathrm{vol}}(\mathrm{wt}_\xi)$ is a minimum. So the strict convexity implies

$$\phi(1) = \mathrm{vol}(\Delta_{\tilde{\Xi}_1}) = \widehat{\mathrm{vol}}(v) > \widehat{\mathrm{vol}}(\mathrm{wt}_\xi) = \phi(0).$$

4.3. Results on the general case

To treat the general case, the key idea, suggested by the degeneration conjecture, is to understand how an arbitrary klt singularity can be degenerated to a K-semistable Fano cone singularity. In [LX16], by localizing the setting of [LX14], the following approach of using Kollár components is developed.

From each ideal $\mathfrak{a}$, we can take a dlt modification of

$$f \colon (Y, D_Y) \to (X, D + \mathrm{lct}(X, D; \mathfrak{a}) \cdot \mathfrak{a}),$$

where $D_Y = f_*^{-1}D + \mathrm{Ex}(f)$ and for any component $E_i \subset \mathrm{Ex}(f)$ we have

$$A_{X,D}(E) = \mathrm{lct}(X, D; \mathfrak{a}) \cdot \mathrm{mult}_E f^* \mathfrak{a}.$$

There is a natural inclusion $\mathcal{D}(D_Y) \subset \mathrm{Val}_{X,x}^{=1}$, and using a similar argument as in [LX14], we can show that there exists a Kollár component S whose rescaling in $\mathrm{Val}_{X,x}^{=1}$ contained in $\mathcal{D}(D_Y)$ satisfies that

$$\widehat{\mathrm{vol}}(\mathrm{ord}_S) = \mathrm{vol}^{\mathrm{loc}}(-A_{X,D}(S) \cdot S) \leq \mathrm{vol}^{\mathrm{loc}}(-K_Y - D_Y) \leq \mathrm{mult}(\mathfrak{a}) \cdot \mathrm{lct}^n(X, D; \mathfrak{a}).$$

Here $\mathrm{vol}^{\mathrm{vol}}(\cdot)$ is the local volume of divisors over X as defined in [Ful13]. Then Theorem 2.6 immediately implies that

$$\widehat{\mathrm{vol}}(x, X, D) = \inf\{\widehat{\mathrm{vol}}(\mathrm{ord}_S) \mid S \text{ is a Kollár component over } x\}. \tag{27}$$

Moreover, if $x \in (X, D)$ admits a torus group T-action, then by degenerating to the initial ideals, as the colengths are preserved and the log canonical thresholds may only decrease, the infimum of the normalized multiplicities in Theorem 2.6 can be only run over all T-equivariant ideals. Then the equivariant MMP allows us to make all the above data Y and S be T-equivariant.

In case a minimizer is divisorial, then the above discussion shows that

Lemma 4.7 ([LX16, Blu18])**.** *A divisorial minimizer of* $\widehat{\mathrm{vol}}_{X,D}$ *yields a Kollár component.*

In general, we know that the minimizer is a limit of a rescaling of Kollár components (see [LX16]). So understanding the limiting process is crucial. When the minimizer is quasi-monomial v of rational rank r, i.e., the valuation v is étale locally a monomial valuation with respect to a log resolution $(Y, E) \to X$, then a natural candidates will be the valuations given by taking rational approximations of the monomial coordinates $\alpha \in \mathbb{R}^r_{>0}$.

Our first observation in [LX17a] is using MMP results including the ACC of log canonical thresholds, we could construct a weak log canonical model which extracts divisors whose coordinates are good linear Diophantine approximations of the coordinates of v.

Proposition 4.8. *For any quasi-monomial valuation v computing a log canonical threshold of a graded sequence of ideals, we can find a sequence of divisors $S_1,..., S_r$, such that*

1. *there is a model $Y \to X$ which precisely extracts $S_1,..., S_r$ over x,*
2. *there exists a component Z of $\cap_{i=1}^r S_i$ such that $(Y, E := \sum_{i=1}^r S_i)$ is toroidal around the generic point $\eta(Z)$,*
3. *v is étale locally a monomial valuation over $\eta(Z)$ with respect to (Y, E) (see Section 2.1),*
4. *(Y, E) is log canonical, and $-K_Y - E$ is nef.*

Fix the first model $Y_0 = Y$, then one can construct a sequence of models (Y_j, E_j) satisfying Proposition 4.8 such that a suitable rescaling of the components of E_j become closer and closer to v. To make the notation easier, we rescale v into $\mathrm{Val}_{X,x}^{=1}$. Similarly, we can embed the dual complex of a dlt modification of (Y_j, E_j) into $\mathrm{Val}_{X,x}^{=1}$ (see [dFKX17]). Our construction moreover satisfies that

$$\mathcal{DR}(Y_0, E_0) \supset \mathcal{DR}(Y_1, E_1) \supset \cdots$$

Then the above discussion indeed implies that

Lemma 4.9. *A quasi-monomial minimizer $v \in \mathrm{Val}_{X,x}^{=1}$ can be written as a limit of $c_j \cdot \mathrm{ord}_{S_j} \in \mathcal{DR}(Y_j, E_j)$ where S_i are Kollár components.*

It would be natural to expect that $c_j \cdot \mathrm{ord}_{S_j}$ is indeed contained in the simplex $\sigma_{\eta(Z)} \subset \mathrm{Val}_{X,x}^{=1}$ which corresponds to all the monomial valuations in $\mathrm{Val}_{X,x}^{=1}$ over $\eta(Z)$ with respect to (Y, E). However, for now we can not show it.

If we further assume $R_0 = \mathrm{gr}_v(R)$ is finitely generated, then we have the following

Proposition 4.10. *If $R_0 = \mathrm{gr}_v(R)$ is finitely generated, then $\mathrm{gr}_v(R) \cong \mathrm{gr}_{v_i}(R)$ for any $v_i \in \sigma_{\eta(Z)}$ sufficiently close to v.*

This immediately implies that $(X_0 := \mathrm{Spec}(R_0), D_0)$ is semi-log-canonical (slc). The final ingredient we need is the following,

Proposition 4.11. *Under the above assumptions on (X, D) and its quasi-monomial minimizer v, then ξ_v is a minimizer of (X_0, D_0). In particular,*

$$\widehat{\mathrm{vol}}(x, X, D) = \widehat{\mathrm{vol}}(o, X_0, D_0).$$

Proof. We claim that ξ_v is indeed a minimizer of $\widehat{\mathrm{vol}}_{X_0, D_0}$. If not, we can find a degeneration (Y, D_Y, ξ_Y) induced by an irreducible anti-ample divisor E over $o' \in X_0$ with

$$\widehat{\mathrm{vol}}_Y(\xi_E) = \widehat{\mathrm{vol}}_{X_0}(\mathrm{ord}_E) < \widehat{\mathrm{vol}}_{X_0}(\xi_v) = \widehat{\mathrm{vol}}_Y(\xi_Y).$$

This is clear by our discussion when (X_0, D_0) is klt. The same thing still holds when the model extracting S_j is only log canonical but not plt, which implies that (X_0, D_0) is semi-log-canonical but not klt. In fact, denote by $(X_0^{\mathrm{n}}, D_0^{\mathrm{n}}) \to (X_0, D_0)$ the normalization, then Lemma 4.13 implies that

$$\widehat{\mathrm{vol}}(o', X_0, D_0) := \sum_{o_i \to o'} \widehat{\mathrm{vol}}(o_i, X_0^{\mathrm{n}}, D_0^{\mathrm{n}}) = 0$$

in this case. The argument in [LX17a, Lemma 4.13] then says in this case, we can still extract an equivariant anti-ample irreducible divisor E over $o' \in X_0$ with $\widehat{\mathrm{vol}}(\mathrm{ord}_E)$ arbitrarily small.

Then Lemma 4.12 shows that we can construct a degeneration from (X, D) to (Y, D_Y) and a family of valuations $v_t \in \mathrm{Val}_{X,x}$ for $t \in [0, \epsilon]$ (for some $0 < \epsilon \ll 1$), with the property that

$$\widehat{\mathrm{vol}}_X(v_t) = \widehat{\mathrm{vol}}_Y(\xi_Y - t\eta) < \widehat{\mathrm{vol}}_Y(\xi_Y) = \widehat{\mathrm{vol}}_{X_0}(\xi_v) = \widehat{\mathrm{vol}}_X(v),$$

where for the second inequality, we use again the fact that $\widehat{\mathrm{vol}}_Y(\xi_Y - t \cdot \eta)$ is a convex function in this setting as well. But this is a contradiction.   □

Lemma 4.12. *Let $(x \in X) \subset (0 \in \mathbb{C}^N)$ be a closed affine variety. If $\lambda_1 \in \mathbb{N}^N$ is a coweight of $(\mathbb{C}^*)^N$ which gives an action degenerating X to X_0 when $t \to 0$, and $\lambda_2 \in \mathbb{N}^N$ degenerates X_0 to Y when $t \to 0$, then for $k \in \mathbb{N}$ sufficiently large, $k\lambda_1 + \lambda_2$ degenerates X to Y_0.*

The proof was essentially given in [LX16, section 6] (see also [LWX18, Lemma 3.1]) and uses some argument in the study of toric degenerations (see e.g. [And13, Section 5]).

Lemma 4.13. *If $o \in (X, D)$ is an lc but not klt point, then*

$$\widehat{\mathrm{vol}}(o, X, D) := \inf_{v \in \mathrm{Val}_{X,o}} \widehat{\mathrm{vol}}(v) = 0.$$

Proof. Let $\pi^{\mathrm{dlt}}\colon (X^{\mathrm{dlt}}, D^{\mathrm{dlt}}) \to (X, D)$ be a dlt modfication and pick o'' a preimage of o under π^{dlt}, then $\widehat{\mathrm{vol}}(o'', X^{\mathrm{dlt}}, D^{\mathrm{dlt}}) \geq \widehat{\mathrm{vol}}(o, X, D)$, thus we can assume $(X^{\mathrm{dlt}}, D^{\mathrm{dlt}})$ is dlt $\mathbb{Q}$-factorial.

By specialing a sequence of points, and applying Theorem 2.11, we can assume $o \in (X, D)$ is a point on a smooth variety with a smooth reduced divisor D. Now we can take a weighted blow up of $(1, \epsilon, ..., \epsilon)$ where the first coordinate yields D. Then the exceptional divisor E has its normalized volume

$$\widehat{\mathrm{vol}}(E) = \frac{(n-1)^n \epsilon^n}{\epsilon^{n-1}} = (n-1)^n \epsilon \to 0 \text{ as } \epsilon \to 0.$$

$\square$

This implies that (X_0, D_0) is klt and (X_0, D_0, ξ_v) is a K-semistable Fano cone. To summarize, we have shown Part (a) in the following theorem which characterize what we know about the Stable Degeneration Conjecture 4.1 for a general klt singularity.

Theorem 4.14 ([LX17a, Theorem 1.1]). *Let $x \in (X, D)$ be a klt singularity. Let v be a quasi-monomial valuation in $\mathrm{Val}_{X,x}$ that minimises $\widehat{\mathrm{vol}}_{(X,D)}$ and has a finitely generated associated graded ring $\mathrm{gr}_v(R)$ (which is always true if the rational rank of v is one by Lemma 4.7). Then the following properties hold:*

- (a) *The degeneration $\bigl(X_0 =_{\mathrm{defn}} \mathrm{Spec}\bigl(\mathrm{gr}_v(R)\bigr), D_0, \xi_v\bigr)$ is a K-semistable Fano cone, i.e. v is a K-semistable valuation;*
- (b) *Let v' be another quasi-monomial valuation in $\mathrm{Val}_{X,x}$ that minimises $\widehat{\mathrm{vol}}_{(X,D)}$. Then v' is a rescaling of v.*

Conversely, any quasi-monomial valuation that satisfies (a) above is a minimiser.

Proof. We first show the uniqueness in general, under the assumption that it admits a degeneration (X_0, D_0, ξ_v) given by a K-semistable minimiser v. For another quasi-monomial minimiser v' of rank r', by a combination of the Diophantine approximation and an MMP construction including the application of ACC of log canonical thresholds (see Proposition 4.8), we can obtain a model $f\colon Z \to X$ which extracts r' divisors E_i $(i = 1, ..., r')$ such that $(Z, D_Z =_{\mathrm{defn}} \sum E_i + f_*^{-1} D)$ is log canonical. Moreover, the quasi-monomial valuation v' can be computed at the generic point of a component of the intersection of E_i, along which (Z, D_Z) is toroidal. Then with the help of the MMP, one can show $Z \to X$ degenerates to a birational morphism $Z_0 \to X_0$. Moreover, there exists a quasi-monomial valuation w computed on Y_0 which can be considered as a degeneration of v' with

$$\widehat{\mathrm{vol}}_{X_0}(w) = \widehat{\mathrm{vol}}_X(v') = \widehat{\mathrm{vol}}_X(v) = \widehat{\mathrm{vol}}_{X_0}(\xi_v).$$

Thus $w = \xi_v$ by Section 4.2.3 after a rescaling. Since $w(\mathbf{in}(f)) \geq v'(f)$ and $\mathrm{vol}(w) = \mathrm{vol}(v')$, we may argue this implies

$$\xi_v(\mathbf{in}(f)) = v'(f)$$

(see [LX17a, Section 4.3]). Therefore, v' is uniquely determined by ξ_v.

To show the last statement, we already know it for a cone singularity. For a valuation v on a general singularity X such that the degeneration (X_0, D_0, ξ_v) is K-semistable, since the degeneration to the initial ideal argument implies that $\widehat{\mathrm{vol}}(x, X, D) \geq \widehat{\mathrm{vol}}(o, X_0, D_0)$, then

$$\widehat{\mathrm{vol}}_X(v) = \widehat{\mathrm{vol}}_{X_0}(\xi_v) = \widehat{\mathrm{vol}}(o, X_0, D_0)$$

is equal to $\widehat{\mathrm{vol}}(x, X, D)$.   □

So in other words, the stable degeneration conjecture precisely predicts the following two sets coincide:

$$\left\{ \text{ Minimizers of } \widehat{\mathrm{vol}} \right\} \longleftrightarrow \left\{ \text{ K-semistable valuations } \right\}.$$

Theorem 2.8 and Theorem 4.14 together imply the existence of left hand side and the uniqueness of the right hand side, as well as the direction that any K-semistable valuation is a minimizer.

Finally, let us conclude this section with the two dimensional case.

Theorem 4.15. *Let (X, D, x) be a two-dimensional log terminal singularity. The Stable Degeneration Conjecture 4.1 holds for (X, D). Moreover, if D is a $\mathbb{Q}$-divisor, then the minimizer of $\widehat{\mathrm{vol}}_{(X,D)}$ is always divisorial.*

Proof. We first consider the case when $X = \mathbb{C}^2$. Let v_* be a minimizer and denote $\mathfrak{a}_\bullet = \{\mathfrak{a}_m(v_*)\}_{m \in \mathbb{N}}$. Then it was known that v_* computes the log canonical threshold of $(X, D + \mathfrak{a}_\bullet)$. By similar argument as [JM12], we know that v_* must be quasi-monomial.

If v_* is divisorial, then we know that the associated divisor is a Kollár component. Otherwise, v_* satisfies $\mathrm{rat.rk.}(v_*) = 2$ and $\mathrm{tr.deg.}(v_*) = 0$. From the description of valuations on $\mathbb{C}^2$ using *sequences of key polynomials* (SKP), it was showed that the valuative semigroup Γ of v_* is finitely generated (see [FJ04, Theorem 2.28]). Since the residual field of v_* is $\mathbb{C}$, we know that $\mathrm{gr}_{v_*} R \cong \mathbb{C}[\Gamma]$, which is finitely generated. By [LX17a], we know that v_* is indeed the unique minimizer of $\widehat{\mathrm{vol}}$ (up to scaling) which is a K-semistable valuation.

If D is a $\mathbb{Q}$-divisor and v_* is not divisorial, then the pair (X_0, D_0) is a $\mathbb{Q}$-Gorenstein toric pair with $\mathbb{Q}$-boundary toric divisor and the associated Reeb vector field ξ_{v_*} solves the convex geometric problem. But in dimension 2 case (i.e. on the plane), it is easy to see that the corresponding convex geometric problem as discussed in section 4.2.3 for toric valuations always has a rational solution. This is a contradiction to v_* being non-divisorial.

More generally, we know that $X = \mathbb{C}^2/G$ where G is a finite group acting on $\mathbb{C}^2$ without pseudo-reflections. Consider the covering $(\mathbb{C}^2, \tilde{D}, 0) \to (X, D, x)$. Then by the above discussion, there exists a unique minimizer v_* of $\widehat{\mathrm{vol}}_{(\mathbb{C}^2, \tilde{D}, 0)}$. In particular, v_* is invariant under the G-action. So it descends to a minimizer of $\widehat{\mathrm{vol}}_{(X, D, x)}$ which is quasi-monomial and has a finitely generated associated graded ring.

$\square$

5. Applications

In this section, we give some applications of the normalized volume. We have seen that the normalized volume question of a cone singularity is closely related the K-semistability of the base. Another situation where singularities naturally appear is on the limit of smooth Fano manifolds.

5.1. Equivariant K-semistability of Fano

An interesting application of the minimizing theory is to treat the equivariant K-semistability.

Definition 5.1. A log Fano pair (S, B) with a G-action is called G-equivariant K-semistable, if for any G-equivariant test configuration $(\mathcal{S}, \mathcal{B})$, the generalized Futaki invariant $\mathrm{Fut}(\mathcal{S}, \mathcal{B}) \geq 0$. We can similarly define G-equivariant K-polystability.

The notion of usual K-(semi,poly)stability trivially implies the equivariant one. It is a natural question to ask whether they are equivalent, and if it is confirmed it will reduce the problem of verifying K-stability into a much simpler ones if the log Fano pair carries a large symmetry. When S is smooth and $B = 0$, this is proved in [DS16], using an analytic argument. Here we want to explain how our approach can give a proof of such an equivalence when $G = T$ is a torus group.

The key is the fact we obtain in (22) and (27) : let $x \in (X, D)$ be a klt singularity which admits a T action for a torus group T, then

$$\inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v) = \{\inf(\widehat{\mathrm{vol}}(\mathrm{ord}_S))|\ \ T\text{-equivariant Kollár components } S\}. \quad (28)$$

So if (S, B) is not K-semistable, by Theorem 4.2, we know that over the cone $x \in (X, D)$, the valuation ord_{S_∞} obtained by the canonical blow up does not give a minimizer. By (28), there exists a T-equivariant valuation v such that $\widehat{\mathrm{vol}}(v) < \widehat{\mathrm{vol}}(\mathrm{ord}_{S_\infty})$. So we can find a T-equivariant Kollár component S such that $\widehat{\mathrm{vol}}(\mathrm{ord}_S) < \widehat{\mathrm{vol}}(\mathrm{ord}_{S_\infty})$. Then arguing as before, we can find a T-equivariant test configuration $(\mathcal{S}, \mathcal{B})$ with $\mathrm{Fut}(\mathcal{S}, \mathcal{B}) < 0$.

To prove a similar statement for K-polystability is more delicate. Assume a K-semistable log pair (S, B) admits a test configuration $(\mathcal{S}, \mathcal{B})$ with $\mathrm{Fut}(\mathcal{S}, \mathcal{B}) = 0$. We still take the cone construction of a K-semistable log Fano pair as before. The special test configuration determines a ray v_t of valuations in $\mathrm{Val}_{X,x}$, emanating from the canonical component $v_0 = \mathrm{ord}_{S_\infty}$. Using the fact that the Futaki invariant is 0, a minimal model program argument shows that this implies for $t \ll 1$, v_t is automatically $\mathbb{C}^*$-equivariant, which immediately implies the test configuration is $\mathbb{C}^*$-equivariant. Therefore we show the following result (also see [CS16] for an earlier attempt).

Theorem 5.2 ([LX16, LWX18]). *The K-semistability (resp. K-polystability) of a log Fano pair (S, B) is equivalent to the T-equivariant K-semistablity (T-equivariant K-polystablity) for any torus group T acting on (S, B).*

For other groups G, e.g. finite groups or general reductive groups, we haven't proved the corresponding result as (28). It is a consequence of the uniqueness part of the stable degeneration conjecture. We also note that in [LX17a], it is proved that quasi-monomial minimizers over a T-equivariant klt singularity are automatically T-invariant.

5.2. Donaldson-Sun's Conjecture

One major application of what we know about the stable degeneration conjecture, formulated in Theorem 4.14, is the solution of [DS17, Conjecture 3.22] (see Conjecture 5.3), which predicts that for a singularity appearing on a Gromov-Hausdorff limit of Kähler-Einstein metrics, its metric tangent cone only depends on the algebraic structure of the singularity. In this section, we briefly explain the idea.

5.2.1. K-semistable degeneration. Let (M_k, g_k) be a sequence of Kähler-Einstein manifolds with positive curvature. Then possibly taking a subsequence, (M_k, g_k) converges in the Gromov-Hausdorff topology to a limit metric space (X, d_∞). By the work of Donaldson-Sun and Tian, X is homeomorphic to a $\mathbb{Q}$-Fano variety. For any point $x \in X$, a metric tangent cone $C_x X$ is defined as a pointed Gromov-Hausdorff limit:

$$C_x X = \lim_{r_k \to 0} \left(X, x, \frac{d_\infty}{r_k} \right). \tag{29}$$

By Cheeger-Colding's theory, $C_x X$ is always a metric cone. By [CCT02], the real codimension of singularity set of $C_x X$ is at least 4 and the regular part admits a Ricci-flat Kähler cone structure. In [DS17], it is further proved that $C_x X$ is an affine variety with an effective torus action. They proved that $C_x X$ is uniquely determined by the metric structure d_∞ and can be obtained in the following steps. In the first step, they defined a filtration $\{\mathcal{F}^\lambda\}_{\lambda \in \mathcal{S}}$ of the local ring $R = \mathcal{O}_{X,o}$ using the limiting metric structure d_∞. Here $\mathcal{S}$ is a set of positive numbers that they called the holomorphic spectrum which depends on the torus action on the metric tangent cone C. In the second step, they proved that the associated graded ring of $\{\mathcal{F}^\lambda\}$ is finitely generated and hence defines an affine variety, denoted by W. In the last step, they showed that W equivariantly degenerates to C. Notice that this process depends crucially on the limiting metric d_∞ on X. They then made the following conjecture.

Conjecture 5.3 (Donaldson-Sun). *Both W and C depend only on the algebraic germ structure of X near x.*

We made the following observations:

1. $\{\mathcal{F}^{\lambda}\}$ comes from a valuation v_0. This is due to the fact that W is a normal variety. More explicitly, since the question is local, we can assume $X = \mathrm{Spec}(R)$ with the germ of $x \in X$, by the work in [DS17], one can embed both X and C into a common ambient space $\mathbb{C}^N$, and v_0 on X is induced by the monomial valuation wt_{ξ_0} where ξ_0 is the linear holomorphic vector field with $2\mathrm{Im}(\xi_0)$ being the Reeb vector field of the Ricci flat Kähler cone metric on C. By this construction, it is clear that the induced valuation by v_0 on W is nothing but wt_{ξ_0}.
2. v_0 is a quasi-monomial valuation. This follows from Lemma 2.7.

More importantly we conjectured in [Li18a] that v_0 can be characterized as the unique minimizer of $\widehat{\mathrm{vol}}_{X,x}$. As a corollary of the theory developed so far, we can already confirm [DS17, Conjecture 3.22] for W.

Theorem 5.4 ([LX17a]). *The semistable cone W in Donaldson-Sun's construction depends on the algebraic structure of (X, x).*

The proof consists of the following steps consisting of analytic and algebraic arguments:

1. By Theorem 3.17, (C, ξ_0) is K-polystable and in particular K-semistable. By Theorem 4.6, wt_{ξ_0} is a minimizer of $\widehat{\mathrm{vol}}_C$.
2. By Proposition 5.5, (W, ξ_0) is K-semistable. By Theorem 4.6 again, wt_{ξ_0} is a minimizer of $\widehat{\mathrm{vol}}_W$. Moreover, by Theorem 4.14, v_0 is a minimizer of $\widehat{\mathrm{vol}}_X$.
3. v_0 is a quasi-monomial minimizer of $\widehat{\mathrm{vol}}_X$ with a finitely generated associated graded ring. By Theorem 4.14, such a v_0 is indeed the unique minimizer of $\widehat{\mathrm{vol}}$ among all quasi-monomial valuations.

The following is an immediate consequence of Theorem 4.6.

Proposition 5.5. *Assume there is a special degeneration of a log-Fano cone (X, D, ξ_0) to (X_0, D_0, ξ_0). Assume that (X_0, D_0, ξ_0) is K-semistable, then (X, D, ξ_0) is also K-semistable, or equivalently, wt_{ξ_0} is the minimizer of $\widehat{\mathrm{vol}}_{(X,D,x)}$.*

Asssume (X, x) lives on a Gromov-Hausdorff limit of Kähler-Einstein Fano manifold. Then we can define the volume density in the sense of Geometric Measure Theory as the following quantity:

$$\Theta(x, X) = \lim_{r \to 0} \frac{\mathrm{Vol}(B_r(x))}{r^{2n} \mathrm{Vol}(B_1(\underline{0}))}. \tag{30}$$

Note that $n^n = \widehat{\mathrm{vol}}(0, \mathbb{C}^n)$. The normalized volumes of klt singularities on Gromov-Hausdorff limits have the following differential geometric meaning:

Theorem 5.6 ([LX17a]). *With the same notation as above, we have the identity:*

$$\frac{\widehat{\mathrm{vol}}(x, X)}{n^n} = \Theta(x, X). \tag{31}$$

Proof. From the standard metric geometry, we have $\Theta(x,X) = \Theta(o_C, C)$. Because C admits a Ricci-flat Kähler cone metric, by Theorem 3.17, (C, ξ_0) is K-semistable. $\widehat{\mathrm{vol}}(x,X) = \widehat{\mathrm{vol}}(o_C, C)$.

On the other hand, since C is a metric cone, from the definition of the volume of $\xi_0 = \frac{1}{2}\left(r\partial_r - iJ(r\partial_r)\right)$ is equal to:

$$\Theta(o_C, C) = \frac{\mathrm{Vol}(C \cap \{r=1\})}{\mathrm{Vol}(S^{2n-1})} = \mathrm{vol}(\xi_0).$$

By Theorem 3.16, $A(\mathrm{wt}_{\xi_0}) = n$ and $\widehat{\mathrm{vol}}(o_C, C) = n^n \mathrm{vol}(\xi_0) = n^n \Theta(o_C, C)$.

$\square$

5.2.2. Uniqueness of polystable degeneration. To confirm Donaldson-Sun's conjecture, we also need to prove the uniqueness of polystable degenerations for K-semistable Fano cones.

Since a Fano cone singularity (C, ξ) with a Ricci-flat Kähler cone metric is aways K-polystable (see [CS15, Theorem 7.1] and also Theorem 3.17), once knowing that W only depends on the algebraic structure of $o \in M_\infty$, an affirmative answer to Conjecture 5.3 follows from the following more general result by letting $(X, D, \xi_0) = (W, \emptyset, \xi_0)$:

Theorem 5.7 ([LWX18])**.** *Given a K-semistable log Fano cone singularity (X, D, ξ_0), there always exists a special test configuration $(\mathcal{X}, \mathcal{D}, \xi_0; \eta)$ which degenerates (X, D, ξ_0) to a K-polystable log Fano cone singularity (X_0, D_0, ξ_0). Furthermore, such (X_0, D_0, ξ_0) is uniquely determined by (X, D, ξ_0) up to isomorphism.*

For the special case of smooth (or $\mathbb{Q}$-Gorenstein smoothable) Fano varieties, this was proved in [LWX14, 7.1] based on analytic results which also show the uniqueness of Gromov-Hausdorff limit for a flat family of Fano Kähler-Einstein manifolds. Our proof of Theorem 5.7 is however a completely new algebraic argument.

We briefly discuss the idea to prove Theorem 5.7 in [LWX18], which heavily depends on the study of normalized volumes as discussed in Section 4.2.

Let $(\mathcal{X}^{(i)}, D^{(i)}, \xi_0, \eta^{(i)}), (i = 1, 2)$, be two special test configurations of the log Fano cone (X, D, ξ_0) with the central fibre $(X_0^{(i)}, D_0^{(i)}, \xi_0)$. To show Theorem 5.7, the main step is to show that if $\mathrm{Fut}(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; \eta^{(i)}) = 0, (i = 1, 2)$, then there exists special test configurations $(\mathcal{X}'^{(i)}, \mathcal{D}'^{(i)})$ of $(X_0^{(i)}, D_0^{(i)})$ such that $(\mathcal{X}'^{(i)}, \mathcal{D}'^{(i)})$ have isomorphic central fibres, which we will describe below.

We consider the normalized volume functional defined on the valuation space $\mathrm{Val}_{X,x}$ over the vertex x of the cone X. Then $(\mathcal{X}^{(1)}, \mathcal{D}^{(1)}, \xi_0; \eta^{(1)})$ determines a "ray" of valuations emanating from the toric valuation wt_{ξ_0} and the generalized Futaki invariant $\mathrm{Fut}(\mathcal{X}^{(1)}, \mathcal{D}^{(1)}, \xi_0; \eta^{(1)})$ is the derivative of the normalized volume at wt_{ξ_0} along this ray.

$$(X_0^{(2)}, D_0^{(2)}) \xleftarrow{\quad (\mathcal{X}^{(2)}, \mathcal{D}^{(2)}) \leftarrow \mathcal{Y}_k^{(2)} \leftarrow \mathcal{E}_k^{(2)} \quad} (X, D) \xleftarrow{\quad} Y_k \leftarrow E_k \qquad (32)$$

$$(\mathcal{X}'^{(2)}, \mathcal{D}'^{(2)}) \Bigg\downarrow \qquad\qquad\qquad\qquad \Bigg\downarrow (\mathcal{X}^{(1)}, \mathcal{D}^{(1)}) \leftarrow \mathcal{Y}_k \leftarrow \mathcal{E}_k = E_k \times \mathbb{C}^1$$

$$(X_0', D_0') \xleftarrow{\quad (\mathcal{X}'^{(1)}, \mathcal{D}'^{(1)}) \quad} (X_0^{(1)}, D_0^{(1)}) \xleftarrow{\quad} Y_{k,0} \leftarrow E_k$$

We can approximate ξ_0 by a sequence of integral vectors $\tilde{\xi}_k$ such that $|\tilde{\xi}_k - k\xi_0| \le C$. For $k \gg 1$, the vector $\tilde{\xi}_k - \eta$ corresponds to a Kollár component E_k over X. Our key argument is to show that E_k can be degenerated along $(\mathcal{X}^{(2)}, \mathcal{D}^{(2)})$ to get a model $\mathcal{Y}_k^{(2)} \to \mathcal{X}^{(2)}$ with an exceptional divisor $\mathcal{E}_k^{(2)}$ such that $(\mathcal{Y}_k^{(2)}, \mathcal{E}_k^{(2)}) \times_{\mathbb{C}} \mathbb{C}^* \cong (Y_k, E_k) \times \mathbb{C}^*$ where the isomorphism is compatible with the equivariant isomorphism of the *second* special test configuration. Note that $E_k \times \mathbb{C}^*$ determines a divisorial valuation over $X \times \mathbb{C}^*$ and hence over $(\mathcal{X}^{(2)}, \mathcal{D}^{(2)})$. So the goal is to show that this divisorial valuation can be extracted as the only exceptional divisor over $\mathcal{X}^{(2)}$. By the work in the minimal model program (MMP) (see [BCHM10]), this would be true if there is a graded sequence of ideals $\mathfrak{A}_\bullet$ and a positive real number c_k' such that two conditions are satisfied:

$$(\mathcal{X}^{(2)}, \mathcal{D}^{(2)} + c_k' \mathfrak{A}_\bullet) \text{ is klt} \quad \text{and} \quad A(E_k \times \mathbb{C}; \mathcal{X}^{(2)}, \mathcal{D}^{(2)} + c_k' \mathfrak{A}_\bullet) < 1,$$

where $A(E_k \times \mathbb{C}; \mathcal{X}^{(2)}, \mathcal{D}^{(2)} + c_k' \mathfrak{A}_\bullet)$ is the log discrepancy of (the birational transform of) $E_k \times \mathbb{C}$ with respect to the triple $(\mathcal{X}^{(2)}, \mathcal{D}^{(2)} + c_k' \mathfrak{A}_\bullet)$.

To find such a $\mathfrak{A}_\bullet$, we look at the graded sequence of valuative ideals $\{\mathfrak{a}_\bullet\}$ of ord_{E_k} and its equivariant degeneration along the second special test configuration $(\mathcal{X}^{(2)}, \mathcal{D}^{(2)})$. The resulting graded sequence of ideals over $\mathcal{X}^{(2)}$ will be denoted by $\mathfrak{A}_\bullet$. Using the study in Section 4.2 one can show the assumptions that $(\mathcal{X}^{(1)}, \mathcal{D}^{(1)}; \xi_0)$ is K-semistable and $\mathrm{Fut}(\mathcal{X}^{(1)}, \mathcal{D}^{(1)}, \xi_0; \eta) = 0$ implies

$$f(k) := \widehat{\mathrm{vol}}(E_k) \text{ is of the order } f(0) + O(k^{-2}).$$

This in turn guarantees that we can find c_k' satisfying the above two conditions.

Applying the relative Rees algebra construction to $\mathcal{E}_k^{(2)} \subset \mathcal{Y}_k^{(2)}/\mathbb{C}$, we get a family over $\mathbb{C}^2$, which over $\mathbb{C} \times \{t\}$ is the same as $(\mathcal{X}^{(1)}, \mathcal{D}^{(1)})$ for $t \ne 0$ and gives a degeneration of $(X_0^{(1)}, D_0^{(1)})$ for $t = 0$. On the other hand, over $\{0\} \times \mathbb{C}$, we get a degeneration of $(X_0^{(2)}, D_0^{(2)})$. Therefore, we indeed show that the two special fibers of two special test configurations $(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; \eta^{(i)})$ $(i = 1, 2)$ with $\mathrm{Fut}(\mathcal{X}^{(i)}, \mathcal{D}^{(i)}, \xi_0; \eta^{(i)}) = 0$ will have a common degeneration.

5.3. Estimates in dimension three and K-stability of threefolds

In general, it is not so easy to find the minimizer of $\widehat{\mathrm{vol}}(\cdot)$ for a given singularity. A number of cases have been computed in [Li18a, LL19, LX16, LX17a, LX17b] including quotient singularities, ADE singularities in all dimensions (except 4-dimensional D_4) etc.

Here we study normalized volumes of threefold klt singularities, and then give a global application where we show that all GIT semi-stable (resp. polystable) cubic threefolds are also K-semi-stable (resp. K-polystable). Our main estimate is in Theorem 5.8, which heavily depends on classifications of canonical threefold singularities.

Theorem 5.8 ([LX17b]). *Let $x \in X$ be a 3-dimensional non-smooth klt singularity. Then $\widehat{\mathrm{vol}}(x, X) \le 16$ and the equality holds if and only if it is an A_1 singularity;*

The proof of Theorem 5.8 heavily relies on the classification theory of three dimensional canonical and terminal singularities, developed in the investigation of explicit three dimensional MMP.

The idea goes as follows. Firstly, we reduce to the case of Gorenstein canonical singularity. If $x \in X$ is not Gorenstein, let us take the index one cover $\tilde{x} \in \widetilde{X}$ of $x \in X$. Hence $\tilde{x} \in \widetilde{X}$ is a Gorenstein canonical singularity. If $\tilde{x} \in \widetilde{X}$ is smooth, then $\widehat{\mathrm{vol}}(x, X) = 27/\mathrm{ind}(x, K_X) \le 13.5 < 16$. If $\tilde{x} \in \widetilde{X}$ is not smooth, the a weak version of finite degree formula (Proposition 5.10) implies that $\widehat{\mathrm{vol}}(x, X) < \widehat{\mathrm{vol}}(\tilde{x}, \widetilde{X})$.

Next, let us assume that $x \in X$ is Gorenstein canonical. By [KM98, Proposition 2.36], there exist only finitely many crepant exceptional divisors over X. By [BCHM10], we can extract these divisors simultaneously on a birational model $Y_1 \to X$. If none of these exceptional divisors are centered at x, then [KM98, Theorem 5.34] implies that $x \in X$ is a cDV singularity, hence $\mathrm{lct}(\mathfrak{m}_x) \le 4 - \mathrm{mult}(\mathfrak{m}_x)$ which implies

$$\widehat{\mathrm{vol}}(x, X) \le \mathrm{lct}(\mathfrak{m}_x)^3 \mathrm{mult}(\mathfrak{m}_x) \le (4 - \mathrm{mult}(\mathfrak{m}_x))^3 \mathrm{mult}(\mathfrak{m}_x) \le 16.$$

The equality case can be characterized using the volume of birational models approach in [LX16]. If some crepant exceptional divisor $E_1 \subset Y_1$ is centered at x, then let us run $(Y_1, \epsilon E_1)$-MMP over X for $0 \ll \epsilon < 1$. By [Kol13, 1.35], this MMP will terminate as $Y_1 \dashrightarrow Y \xrightarrow{g} Y'$, where $Y_1 \dashrightarrow Y$ is the composition of a sequence of flips, and $g : Y \to Y'$ contracts the birational transform E of E_1. If $g(E)$ is a curve, then Y' has cDV singularities along $g(E)$ by [KM98, Theorem 5.34]. By choosing a point $y' \in g(E)$, we have

$$\widehat{\mathrm{vol}}(x, X) < \widehat{\mathrm{vol}}(y', Y') \le 16.$$

If $g(E) = y'$ is a point, then we still have $\widehat{\mathrm{vol}}(x, X) < \widehat{\mathrm{vol}}(y', Y')$. Thus it suffices to show $\widehat{\mathrm{vol}}(y', Y') < 16$.

If Y has a singular point $y \in E$, then we know that $y \in Y$ is a cDV singularity. Hence

$$\widehat{\mathrm{vol}}(y', Y') < \widehat{\mathrm{vol}}(y, Y) \le 16.$$

So we may assume that Y is smooth along E. In particular, E is a (possibly non-normal) reduced Gorenstein del Pezzo surface. If E is normal, then classification of such surfaces show that $(-K_E)^2 \le 9$. Thus

$$\widehat{\mathrm{vol}}(y', Y') \le A_{Y'}(\mathrm{ord}_E)^3 \mathrm{vol}(\mathrm{ord}_E) = (-K_E)^2 \le 9 < 16.$$

If E is non-normal, then from Reid's classification [Rei94] either $(-K_E)^2 \leq 4$ or the normalization of E is a Hirzebruch surface. In the former case, we have $\widehat{\mathrm{vol}}(y', Y') \leq 4$. In the latter case, we need to take a general fiber l of E and argue that $\widehat{\mathrm{vol}}_{Y', y'}(\mathrm{ord}_l) \leq 16$.

Here are some intermediate results in proving Theorem 5.8.

Proposition 5.9. *Let $\phi : (Y, y) \to (X, x)$ be a birational morphism of klt singularities such that $y \in \mathrm{Ex}(\phi)$. If $K_Y \leq \phi^* K_X$, then $\widehat{\mathrm{vol}}(x, X) < \widehat{\mathrm{vol}}(y, Y)$.*

See Conjecture 6.4 for more discussions about the following Proposition.

Proposition 5.10. *Let $\pi : (\widetilde{X}, \tilde{x}) \to (X, x)$ be a finite quasi-étale morphism of klt singularities of degree at least 2. Then we have*

$$\widehat{\mathrm{vol}}(x, X) < \widehat{\mathrm{vol}}(\tilde{x}, \widetilde{X}) \leq \deg(\pi) \cdot \widehat{\mathrm{vol}}(x, X).$$

As mentioned [SS17], one main application of the local volume estimate Theorem 5.8 is to the K-stability question of cubic threefolds.

Theorem 5.11 ([LX17b]). *A cubic threefold is K-(poly/semi)stable if and only if it is GIT (poly/semi)stable. In particular, any smooth cubic threefold is K-stable.*

The general strategy to prove Theorem 5.11 is via the comparison of moduli spaces which has first appeared in [MM93] built on the work of [Tia90]. Later it was also applied in [OSS16, SS17].

First, one can construct a proper algebraic space which is a good quotient moduli space with closed points parametrizing all smoothable K-polystable $\mathbb{Q}$-Fano varieties (see e.g. [LWX14, Oda15]). Let M be the closed subspace whose closed points parametrize KE cubic threefolds and their K-polystable limits. By [Tia87], we know that at least one cubic threefold, namely the Fermat cubic threefold, admits a KE metric. Hence M is non-empty. By the Zariski openness of K-(semi)stability of smoothable Fano varieties (cf. [Oda15, LWX14]), the K-moduli space M is birational to the GIT moduli space M^{GIT} of cubic threefolds.

Next, we will show that any K-semistable limit X of a family of cubic threefolds $\{X_t\}$ over a punctured curve is necessarily a cubic threefold. The idea is to control the singularity of X use an inequality from [Liu18] (see Theorem 5.12) between the global volume of a K-semistable Fano variety and the local normalized volume. Since the volume of X is the same as the volume of a cubic 3-fold which is 24, Theorem 5.12 immediately implies that $\widehat{\mathrm{vol}}(x, X) \geq \frac{81}{8}$ for any closed point $x \in X$. The limit X carries a $\mathbb{Q}$-Cartier Weil divisor L which is the flat limit of hyperplane sections in the cubic threefolds X_t. It is clear that $-K_X \sim_{\mathbb{Q}} 2L$ and $(L^3) = 3$, thus once we show that L is Cartier, we can claim that X is a cubic threefold using a result of T. Fujita.

Assume to the contrary that L is not Cartier at some point $x \in X$, then we may take the index 1 cover $(\tilde{x} \in \widetilde{X}) \to (x \in X)$ of L. From the finite degree

formula Theorem 6.5,

$$\widehat{\text{vol}}(\tilde{x}, \widetilde{X}) = \text{ind}(L) \cdot \widehat{\text{vol}}(x, X) \geq 81/4.$$

Hence $\tilde{x} \in \widetilde{X}$ is a smooth point and $\text{ind}(L) = 2$ by Theorem 5.8. Thus $x \in X$ is a quotient singularity of type $\frac{1}{2}(1, 1, 0)$ from the smoothable condition. Then using the local Grothendieck-Lefschetz theorem, we can show that L is indeed Cartier at $x \in X$ which is a contradiction.

So far we have shown that any K-polystable point X in M is a cubic threefold. By an argument of Paul and Tian in [Tia94], we know that any K-(poly/semi)stable hypersurface is GIT (poly/semi)stable. Thus we obtain an injective birational morphism $M \to M^{\text{GIT}}$ between proper algebraic spaces. This implies that M is isomorphic to M^{GIT} which finishes the proof.

Theorem 5.12 ([Liu18])**.** *Let X be an n-dimensional K-semistable Fano variety. Then for any closed point $x \in X$, we have*

$$(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\text{vol}}(x, X).$$

When X is smooth, the above result was first proved in [Fuj18].

6. Questions and Future research

6.1. Revisit stable degeneration conjecture

The following two parts of stable degeneration conjecture, proposed in [Li18a], are still missing.

Conjecture 6.1 (Quasi-monomial). *Let $x \in (X, D)$ be a klt singularity. Any minimizer of $\widehat{\text{vol}}_{(X,D),x}$ is quasi-monomial.*

Conjecture 6.2 (Finite generation). *Let $x \in (X = \text{Spec}(R), D)$ be a klt singularity. Any minimizer of $\widehat{\text{vol}}_{(X,D),x}$ has its associated graded ring $\text{gr}_v(R)$ to be finitely generated.*

Due to the fundamental role of the stable degeneration conjecture, it implies many other interesting properties. We discuss a number of special cases or consequences, with the hope that some of them might be solved first.

One interesting consequence of the uniqueness of the minimizer is the following

Conjecture 6.3 (Group action). *If there is a group G acting on the klt singularity $x \in (X, D)$ such that x is a fixed point, then there exists a G-invariant minimizer.*

Applying this conjecture to a cone singularity, it implies that to test the K-semistability of a log Fano (S, B) with a G-action, we only need to test on G-equivariant test configurations, a fact known for a Fano manifold X and G-reductive.

There are two special cases naturally appearing in contexts. The first one is that when G is a torus group T. It follows the argument in [Blu18] and the techniques of degenerating ideals to their initials, that there is a T-equivariant minimizer. This is the philosophy behind Section 5.1. It also follows from [LX17a] that any quasi-monomial minimizer is T-equivariant.

A more challenging case is when G is a finite group. Indeed, Conjecture 6.3 for finite group G implies the following finite degree formula.

Conjecture 6.4 (Finite degree formula). *If $\pi\colon (y \in Y, D') \to (x \in X, D)$ is a dominant finite morphism between klt singularities, such that $K_Y + D' = \pi^*(K_X + D)$, then*
$$\deg(\pi) \cdot \widehat{\mathrm{vol}}(x, X, D) = \widehat{\mathrm{vol}}(y, Y, D').$$

This is useful when we want to bound the klt singularities $x \in (X, D)$ with a large volume.

Theorem 6.5 ([LX17a]). *Conjecture 6.4 is true when (X, x) is on a Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds.*

Proof. Let $\pi = \pi_X : (Y, y) \to (X, x)$ be a quasi-étale morphism, i.e. π_X is étale in codimension one. Then π_X induces a quasi-étale morphism along the 2-step degeneration of X.

$$\begin{array}{ccc} Y \rightsquigarrow W_Y \rightsquigarrow C_Y & \qquad (33) \\ \downarrow{\pi_X} \quad \downarrow{\pi_W} \quad \downarrow{\pi_C} & \\ X \rightsquigarrow W \rightsquigarrow C & \end{array}$$

We can use the above diagram to prove the degree multiplication formula. Roughly speaking, because C admits a Ricci-flat Kähler cone metric ω_C with radius function r^2 and π_C is quasi-étale, we can pull back it to get $\pi_C^* r^2$ which is also a potential for a weak Ricci-flat Kähler cone metric ω_{C_Y}. By Theorem 3.17, Theorem 4.6 and Theorem 4.14, we know that the Reeb vecotor field associated to ω_C (resp. ω_{C_Y}) induces minimizing valuations of $\widehat{\mathrm{vol}}_X$ (resp. $\widehat{\mathrm{vol}}_Y$). So we get

$$\widehat{\mathrm{vol}}(y, Y) \;\; = \;\; \widehat{\mathrm{vol}}(o_{C_Y}, C_Y) = \deg(\pi_C) \cdot \widehat{\mathrm{vol}}(o_C, C) = \deg(\pi_C) \cdot \widehat{\mathrm{vol}}(x, X).$$

$\square$

Another consequence of the stable degeneration conjecture is the following strengthening of Theorem 2.11.

Conjecture 6.6. *Let $\pi : (\mathcal{X}, \mathcal{D}) \to T$ together with a section $t \in T \mapsto x_t \in \mathcal{X}_t$ be a $\mathbb{Q}$-Gorenstein flat family of klt singularities. Then the function $t \mapsto \widehat{\mathrm{vol}}(x_t, \mathcal{X}_t, \mathcal{D}_t)$ is construtible with respect to the Zariski topology.*

Besides the stable degeneration conjecture, to prove Conjecture 6.6, we also need to know the well expected speculation that K-semistability is an open condition. It is also natural to consider the volume of non-closed point. However, the following conjecture says after the right scaling, it does not contribute more information.

Conjecture 6.7. *If a klt pair (X, D) has a non-closed point η, and let $Z = \overline{\{\eta\}}$ has dimension d. Pick a general closed point $x \in Z$, then*

$$\widehat{\mathrm{vol}}(x, X, D) = \widehat{\mathrm{vol}}(\eta, X, D) \cdot \frac{n^n}{(n-d)^{n-d}}.$$

In fact, combining the argument in [LZ18], for any valuation $v \in \mathrm{Val}_X$ such that its center $Z = \mathrm{Center}_X(v)$ on X is of dimension d and $x \in Z$, denoted by η is the generic point of Z, one can show that

$$\frac{\widehat{\mathrm{vol}}_{(X,D),\eta}(v) \cdot n^n}{(n-d)^{n-d}} \geq \widehat{\mathrm{vol}}(x, X, D).$$

i.e.,

$$\widehat{\mathrm{vol}}(x, X, D) = \inf_v \left\{ \frac{n^n \cdot \widehat{\mathrm{vol}}_{(X,D),\eta}(v)}{(n-d)^{n-d}} \mid x \in Z = \overline{\{\eta\}} = \mathrm{Center}_X(v), \dim(Z) = d \right\}.$$

6.2. Birational geometry study

A different invariant attached to a klt singularities, called the minimal log discrepancy has been intensively studied in the minimal model program, though there are still many deep questions unanswered. We can formulate many similar questions for $\widehat{\mathrm{vol}}$.

6.2.1. Inversion of adjunction. One could look for a theory of the change of the volumes when the klt pair is 'close' to a log canonical singularities, using the inversion of adjunction. We have some results along this line.

Proposition 6.8. *Let $x \in (X, \Delta)$ be an n-dimensional klt singularity. Let D be a normal $\mathbb{Q}$-Cartier divisor containing x such that $(X, D + \Delta)$ is plt. Denote by Δ_D the different of Δ on D. Then*

$$\lim_{\epsilon \to 0+} \frac{\widehat{\mathrm{vol}}(x, X, (1-\epsilon)D + \Delta)}{n^n \epsilon} = \frac{\widehat{\mathrm{vol}}(x, D, \Delta_D)}{(n-1)^{n-1}}.$$

Proof. Using the degeneration argument in [LZ18], we know that

$$\epsilon^{-1}\widehat{\mathrm{vol}}(x, X, (1-\epsilon)D + \Delta) \geq \frac{n^n}{(n-1)^{n-1}}\widehat{\mathrm{vol}}(x, D, \Delta_D).$$

Hence it suffices to show the reverse inequality is true after taking limits. Let us pick an arbitrary Kollár component S over $x \in (D, \Delta_D)$ with valuation ideals $\mathfrak{a}_m := \mathfrak{a}_m(\mathrm{ord}_S)$. Choose m sufficiently divisible so that $\mathfrak{a}_{im} = \mathfrak{a}_m^i$ for any $i \in \mathbb{N}$. Then we know that $\mathrm{lct}(D, \Delta_D; \mathfrak{a}_m) = A_X(\mathrm{ord}_S)/m =: c$. Let $\mathfrak{b}_m$ be the pullback ideal of $\mathfrak{a}_m$ on X. By inversion of adjunction, we have $\mathrm{lct}(X, D + \Delta; \mathfrak{b}_m) = \mathrm{lct}(D, \Delta_D; \mathfrak{a}_m) = c$.

Let E be an exceptional divisor over X computing $\mathrm{lct}(X, D + \Delta; \mathfrak{b}_m)$. Then E is centered at $x \in X$ since $(X, D + \Delta)$ is plt. For $\epsilon_1 > 0$ sufficiently small, we have that $(X, \Delta + (1 - \epsilon_1)(D + c \cdot \mathfrak{b}_m))$ is a klt pair over which the discrepancy of E is negative. Thus [BCHM10] implies that there exists a proper birational

model $\mu : Y \to X$ which only extracts E. Moreover, $\mu : Y \to X$ is a log canonical modification of $(X, \Delta + D + c \cdot \mathfrak{b}_m)$. Let $\widetilde{D}$ be the normalization of $\mu_*^{-1}D$. Then by adjunction, the lifting morphism $\tilde{\mu} : \widetilde{D} \to D$ is a log canonical (in fact plt) modification of $(D, \Delta_D + c \cdot \mathfrak{a}_m)$. Since $\mathrm{Bl}_{\mathfrak{a}_m} D \to D$ provides a model of the Kollár component S, this is the only log canonical modification of $(D, \Delta_D + c \cdot \mathfrak{a}_m)$. Hence $E|_{\widetilde{D}} = S$ and $(\widetilde{D}, \tilde{\mu}_*^{-1}\Delta_D + E|_{\widetilde{D}})$ is plt. Then by inversion of adjunction, $(Y, \mu_*^{-1}\Delta + \mu_*^{-1}D + E)$ is qdlt and $\mu_*^{-1}D = \widetilde{D}$ is normal. Note that all the constructions so far are independent of the choice of ϵ.

Over the qdlt model $(Y, \mu_*^{-1}\Delta + \mu_*^{-1}D + E)$, we consider a quasi-monomial valuation v_λ of weights 1 and λ along divisors $\widetilde{D}$ and E respectively. By adjunction, we know that $A_{(X,\Delta)}(\mathrm{ord}_E) = A_{(D,\Delta_D)}(\mathrm{ord}_S) + \mathrm{ord}_E(D)$. Hence computation shows that

$$A_{(X,\Delta+(1-\epsilon)D)}(v_\lambda) = \lambda A_{(D,\Delta_D)}(\mathrm{ord}_S) + \lambda\epsilon \cdot \mathrm{ord}_E(D) + \epsilon.$$

Then using the Okounkov body description of the volume (see [LM09, KK12]), we easily see that $\mathrm{vol}(v_\lambda) \le \lambda^{1-n}\mathrm{vol}(\mathrm{ord}_S)$. Hence

$$
\begin{aligned}
\widehat{\mathrm{vol}}_{(X,\Delta+(1-\epsilon)D)}(v_\lambda) &\le \lambda^{1-n}((A_{(D,\Delta_D)}(\mathrm{ord}_S) + \epsilon \cdot \mathrm{ord}_E(D))\lambda + \epsilon)^n \mathrm{vol}(\mathrm{ord}_S) \\
&=: \phi(\lambda).
\end{aligned}
$$

It is easy to see that $\phi(\lambda)$ reaches its minimum at

$$\lambda_0 = \frac{(n-1)\epsilon}{A_{D,\Delta_D}(\mathrm{ord}_S) + \epsilon \cdot \mathrm{ord}_E(D)}.$$

Hence computation shows

$$\epsilon^{-1}\widehat{\mathrm{vol}}_{(X,\Delta+(1-\epsilon)D)}(v_{\lambda_0}) \le \frac{n^n}{(n-1)^{n-1}}(A_{(D,\Delta_D)}(\mathrm{ord}_S) + \epsilon \cdot \mathrm{ord}_E(D))^{n-1}\mathrm{vol}(\mathrm{ord}_S).$$

Thus

$$\limsup_{\epsilon \to 0} \epsilon^{-1}\widehat{\mathrm{vol}}(x, X, \Delta + (1-\epsilon)D) \le \frac{n^n}{(n-1)^{n-1}}\widehat{\mathrm{vol}}_{(D,\Delta_D)}(\mathrm{ord}_S)$$

Since this inequality holds for any Kollár component S over $x \in (D, \Delta_D)$, the proof is finished. $\qquad\square$

When the center is zero dimensional, we also have

Proposition 6.9. *Let $x \in (X, \Delta)$ be a klt singularity. Let $D \ge 0$ be a $\mathbb{Q}$-Cartier divisor such that $(X, \Delta + D)$ is log canonical with $\{x\}$ being the minimal non-klt center. Then there exists $\epsilon_0 > 0$ (depending only on the coefficient of Δ, D and n) and a quasi-monomial valuation $v \in \mathrm{Val}_{X,x}$ such that v computes both $\mathrm{lct}(X, \Delta; D)$ and $\widehat{\mathrm{vol}}(x, X, \Delta + (1-\epsilon)D)$ for any $0 < \epsilon < \epsilon_0$. In particular,*

$$\widehat{\mathrm{vol}}(x, X, \Delta + (1-\epsilon)D) = \widehat{\mathrm{vol}}_{x,(X,\Delta)}(v) \cdot \epsilon^n \text{ for any } 0 < \epsilon < \epsilon_0.$$

Proof. Let $Y^{\mathrm{dlt}} \to X$ be a dlt modification of $(X, \Delta + D)$. Let $K_{Y^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}}$ be the log pull back of $K_X + \Delta + D$. Then by [dFKX17], the dual complex $\mathcal{DR}(\Delta^{\mathrm{dlt}})$ form a natural subspace of $\mathrm{Val}_{X,x}^{=1}$. Any divisorial valuation ord_E computing $\mathrm{lct}(X, \Delta; D)$ corresponds to a rescaling of a valuation in $\mathcal{DR}(\Delta^{\mathrm{dlt}})$. Consider the function $\mathrm{vol}_X :$ $\mathcal{DR}(\Delta^{\mathrm{dlt}}) \to \mathbb{R}_{>0} \cup \{+\infty\}$. Denote by $\mathcal{DR}^\circ(\Delta^{\mathrm{dlt}})$ the open subset of $\mathcal{DR}(\Delta^{\mathrm{dlt}})$ consisting of valuations centered at x. Since $\{x\}$ is the minimal non-klt center of $(X, \Delta + D)$, we know that $\mathcal{DR}^\circ(\Delta^{\mathrm{dlt}})$ is non-empty. By [BFJ14] the function vol is continuous on $\mathcal{DR}(\Delta^{\mathrm{dlt}})$, so we can take a vol-minimizing valuation $v \in \mathcal{DR}^\circ(\Delta^{\mathrm{dlt}})$. Hence v is a minimizer of $\widehat{\mathrm{vol}}$ restricted to $\mathcal{DR}^\circ(\Delta^{\mathrm{dlt}})$.

Assume S is an arbitrary Kollár component over $(X, \Delta + (1 - \epsilon)D)$. Then we have a birational morphism $\mu : Y \to X$ such that $K_Y + \mu_*^{-1}(\Delta + (1 - \epsilon)D) + S$ is plt, and μ is an isomorphism away from x with $S = \mu^{-1}(x)$. Then by ACC of lct [HMX14], we know that there exists ϵ_0 such that $K_Y + \mu_*^{-1}(\Delta + D) + S$ is log canonical whenever $0 < \epsilon < \epsilon_0$. Let v' be an arbitrary divisorial valuation in $\mathcal{DR}^\circ(\Delta^{\mathrm{dlt}})$. Since $K_Y + \mu_*^{-1}(\Delta + D) + S \sim_{\mathbb{Q}} \mu^*(K_X + \Delta + D) + A_{(X,\Delta+D)}(\mathrm{ord}_S)S$, we have

$$0 \le A_{(Y, \mu_*^{-1}(\Delta+D)+S)}(v') = A_{(X,\Delta+D)}(v') - A_{(X,\Delta+D)}(\mathrm{ord}_S) \cdot v'(S).$$

Since $A_{(X,\Delta+D)}(v') = 0$ and $v'(S) > 0$ since $\{x\}$ is the only lc center, we know that $A_{(X,\Delta+D)}(\mathrm{ord}_S) = 0$. Thus a rescaling of ord_S belongs to $\mathcal{DR}(\Delta^{\mathrm{dlt}})$. Then by [LX16] we see that

$$\widehat{\mathrm{vol}}(x, X, \Delta + (1 - \epsilon)D) = \min_{v' \in \mathcal{DR}(\Delta^{\mathrm{dlt}})} \widehat{\mathrm{vol}}_{(X, \Delta+(1-\epsilon)D)}(v')$$

$$= \epsilon^n \min_{v' \in \mathcal{DR}(\Delta^{\mathrm{dlt}})} \mathrm{vol}_X(v') = \widehat{\mathrm{vol}}_{x,(X,\Delta)}(v) \cdot \epsilon^n.$$

$\square$

One should be able to solve the following question using the above techniques.

Question 6.10. Let $x \in (X, \Delta)$ be an n-dimensional klt singularity. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor through x. Let $c = \mathrm{lct}(X, \Delta; D)$, and let W be the minimal log canonical center of $(X, \Delta + cD)$ containing x. By Kawamata's subadjunction, we have $(K_X + \Delta + cD)|_W = K_W + \Delta_W + J_W$, where $(W, \Delta_W + J_W)$ is a generalized klt pair. Denote by $k := \mathrm{codim}_X W$, then is it true that

$$\lim_{\epsilon \to 0+} \epsilon^{-k} \frac{\widehat{\mathrm{vol}}(x, X, \Delta + (1 - \epsilon)cD)}{n^n} \ge \frac{\widehat{\mathrm{vol}}(w, X, \Delta)}{k^k} \cdot \frac{\widehat{\mathrm{vol}}(x, W, \Delta_W + J_W)}{(n - k)^{n-k}}$$

where w is the generic point of W in X and $\widehat{\mathrm{vol}}(x, W, \Delta_W + J_W)$ is similarly defined as for the usual klt pair case in Definition 2.3?

6.2.2. Uniform bound. The following is conjectured in [SS17] (see also [LX17b]).

Conjecture 6.11. *Let $x \in X$ be an n-dimensional singular point, then $\widehat{\mathrm{vol}}(x, X) \le 2(n - 1)^n$.*

The constant $2(n-1)^n$ is the volume of a rational double point. When $n = 3$, it is proved in Theorem 5.8. The implication to the K-stability question of cubic hypersurfaces as in the argument of Theorem 5.11 holds in any dimension.

We also ask whether the following strong property of the set of local volumes holds.

Question 6.12. Fix the dimension n, and a finite set $I \subset [0, 1]$. Is it true that the set $\mathrm{Vol}^{\mathrm{loc}}_{n,I}$ consisting of all possible local volumes of n-dimensional klt singularities $x \in (X, D)$ with (coefficients of $D) \subset I$ has the only accumulation point 0?

Next we give a comparison between local volumes and minimal log discrepancies.

Theorem 6.13. *Let $x \in (X, \Delta)$ be an n-dimensional complex klt singularity. Then there exists a neighborhood U of $x \in X$ such that $(U, \Delta|_U)$ is $(\widehat{\mathrm{vol}}(x, X, \Delta)/n^n)$-lc. Moreover, $\mathrm{mld}(x, X, \Delta) > \widehat{\mathrm{vol}}(x, X, \Delta)/n^n$.*

Proof. If $x \in X$ is not $\mathbb{Q}$-factorial then we may replace X by its $\mathbb{Q}$-factorial modification under which the local volume will increase by [LX17b, Corollary 2.11]. Let Δ_i be any component of Δ containing x. Then [BL18, Theorem 33] implies that $A_{(X,\Delta)}(\Delta_i) \geq \widehat{\mathrm{vol}}(x, X, \Delta)/n^n$. Let E be any exceptional divisor over X such that x is contained in the Zariski closure of $c_X(E)$ and $a(E; X, \Delta) < 0$. Then by [Kol13, Corollary 1.39], there exists a proper birational morphism $\mu : Y \to X$ such that Y is normal, $\mathbb{Q}$-factorial and $E = \mathrm{Ex}(\mu) \supset \mu^{-1}(x)$. Since $K_Y + \mu_*^{-1}\Delta - a(E; X, \Delta)E = \mu^*(K_X + \Delta)$, we know that $(Y, \mu_*^{-1}\Delta - a(E; X, \Delta)E)$ is klt. Let $y \in \mu^{-1}(x)$ be a point, then y lies on E. Hence by [LX17b, Corollary 2.11] and [BL18, Theorem 33] we have

$$\widehat{\mathrm{vol}}(x, X, \Delta) < \widehat{\mathrm{vol}}(y, Y, \mu_*^{-1}\Delta - a(E; X, \Delta)E) \leq A_{(X,\Delta)}(E)n^n.$$

Thus $A_{(X,\Delta)}(E) > \widehat{\mathrm{vol}}(x, X, \Delta)/n^n$ which finishes the proof. $\square$

Next we will discuss application to boundedness generalizing a result by C. Jiang [Jia17, Theorem 1.6].

Corollary 6.14. *Let n be a natural number and c a positive real number. Then the projective varieties X satisfying the following properties:*

- *(X, Δ) is a klt pair of dimension n for some effective $\mathbb{Q}$-divisor Δ,*
- *$-(K_X + \Delta)$ is nef and big,*
- *$\alpha(X, \Delta)^n(-(K_X + \Delta))^n \geq c$,*

form a bounded family.

Proof. By [BJ17, Theorem A and D] (generalizing [Liu18]), for any closed point $x \in X$ we have

$$c \leq \alpha(X, \Delta)^n(-(K_X + \Delta))^n \leq \delta(X, \Delta)^n(-(K_X + \Delta))^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(x, X, \Delta).$$

Hence Theorem 6.13 implies (X, Δ) is $(c/(n+1)^n)$-lc. Therefore, the BAB Conjecture proved by Birkar in [Bir16, Theorem 1.1] implies the boundedness of X. $\square$

Remark 6.15. In the conditions of Corollary 6.14 if we also assume that the co-efficients of Δ are at least ϵ for any fixed $\epsilon \in (0, 1)$, then such pairs (X, Δ) are log bounded. This partially generalizes [Che18, Theorem 1.4]. Besides, all results should hold for $\mathbb{R}$-pairs.

Question 6.16. Is it true that for any n-dimensional klt singularity $x \in X$, we have $\mathrm{mld}(x, X) \geq \widehat{\mathrm{vol}}(x, X)/n^{n-1}$?

6.3. Miscellaneous Questions

6.3.1. Positive characteristics. In this section, we consider a variety X over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. From [Har98, HW02], we know that klt singularities are closely related to strongly F-regular singularities in positive characteristic. Moreover, log canonical thresholds (lct) correspond to F-pure thresholds (fpt) in positive characteristic (see [HW02]). In spirit of Theorem 2.6, we define the F-volume of singularities in characteristic p as follows.

Definition 6.17 ([Liu19]). Let X be an n-dimensional strongly F-regular variety over an algebraically closed field $\Bbbk$ of positive characteristic. Let $x \in X$ be a closed point. We define the F-volume of $(x \in X)$ as

$$\mathrm{Fvol}(x, X) := \inf_{\mathfrak{a}:\, \mathfrak{m}_x\text{-primary}} \mathrm{fpt}(X; \mathfrak{a})^n \mathrm{mult}(\mathfrak{a}).$$

Similar to [dFEM03], Takagi and Watanabe [TW04] showed that if $x \in X$ is a smooth point, then $\mathrm{Fvol}(x, X) = n^n$.

Another interesting invariant of a strongly F-regular singularity $x \in X$ is its F-signature $s(x, X)$, see [SVdB97, HL02, Tuc12]. In [Liu19], we estabilish the following comparison result between the F-volume and the F-signature.

Theorem 6.18 ([Liu19]). *Let* $x \in X$ *be an n-dimensional strongly F-regular singularity. Then*

$$n! \cdot s(x, X) \leq \mathrm{Fvol}(x, X) \leq n^n \min\{1, n! \cdot s(x, X)\}.$$

It would be interesting to study the limiting behavior of F-volumes of mod-p reductions of a klt singularity over characteristic zero when p goes to infinity.

Conjecture 6.19. *Let* $x \in (X, \Delta)$ *be a klt singularity over characteristic* 0. *Let* $x_p \in (X_p, \Delta_p)$ *be its reduction mod* $p \gg 0$, *then*

$$\widehat{\mathrm{vol}}(x, X, \Delta) = \lim_{p \to \infty} \mathrm{Fvol}(x_p, X_p, \Delta_p).$$

Remark 6.20. Together with Theorem 6.18, this will imply that for the reductions (X_p, Δ_p), the F-signature $s(x_p, X_p, \Delta_p)$ has a uniform lower bound as $p \to \infty$, as asked in [CRST18, Question 5.9].

6.3.2. Relation to local orbifold Euler numbers. In [Lan03], Langer introduced local orbifold Euler numbers for general log canonical surface singularities and used it to prove a Miyaoka-Yau inequality for any log canonical surface. In an attempt to understand Langer's inequality using the Kähler-Einstein metric on a log canonical surface, Borbon-Spotti conjectured recently in [BS17] that the volume densities of the singular Kähler-Einstein metrics should match Langer's local Euler numbers (at least for log terminal surface singularities). They verified this in special examples by comparing the known values of both sides. On the other hand, from Theorem 5.6, we know that the normalized volume is equal to the volume density up to a factor $(\dim X)^{\dim X}$ for any point (X, x) that lives on a Gromov-Hausdorff limit of smooth Kähler-Einstein manifolds ([HS17, LX17a]). In view of this connection, one can formulate a purely algebraic problem about two algebraic invariants of the singularities. This problem was already posed by in [BS17] at least in the log terminal case. We formulate the following form by including one of Langer's expectations (see [Lan03, p.381]):

Conjecture 6.21 (see [BS17, p.37]**).** *Let (X, D, x) be a germ of log canonical surface singularity with $\mathbb{Q}$-boundary. Then we have*

$$e_{\mathrm{orb}}(x, X, D) = \begin{cases} \frac{1}{4}\widehat{\mathrm{vol}}(x, X, D), & \text{if } (X, D) \text{ is log terminal ;} \\ 0, & \text{if } (X, D) \text{ is not log terminal.} \end{cases} \quad (34)$$

In [Li18b], it was proved that the above conjecture is true when (X, D, x) is a 2-dimensional log-Fano cone or a log-CY cone. In particular, combined with Langer's calculation, one gets the local orbifold Euler numbers of line arrangements.

Proposition 6.22 ([Lan03, Li18b]**).** *Let $L_1, \ldots, L_n$ be m distinct lines in $\mathbb{C}^2$ passing through 0. Let $D = \sum_{i=1}^m \delta_i L_i$, where $0 \leq \delta_1 \leq \delta_2 \leq \cdots \leq \delta_m \leq 1$. Denote $\delta = \sum_{i=1}^m \delta_i$. Then we have:*

$$e_{\mathrm{orb}}(0, \mathbb{C}^2, D) = \begin{cases} 0 & \text{if} & (\mathbb{C}^2, D, 0) \text{ is not klt ;} \\ (1 - \delta + \delta_m)(1 - \delta_m) & \text{if} & \delta < 2\delta_m; \\ \frac{(2-\delta)^2}{4} & \text{if} & 2\delta_m \leq \delta \leq 2. \end{cases} \quad (35)$$

Here we point out a possible application of Theorem 4.15 (i.e. 2-dimensional case conjecture 4.1) for studying Conjecture 6.21 for any log terminal singularity (x, X, D). First, by Theorem 4.15 there exists a unique Kollár component $S \cong \mathbb{P}^1$ which minimizes the normalized volume. Let $\mu : Y \to X$ be the extraction of S and $\Delta = \mathrm{Diff}_S(D)$. By Theorem 4.14 we know that $(S, \Delta) \cong (\mathbb{P}^1, \sum_i \delta_i p_i)$ is indeed K-semistable (see [LX16, section 6]). Then $\mathscr{F} := \Omega^1(\log(S + D))$ (defined using ramified coverings as in [Lan03]) restricted to S fits into an exact sequence of orbifold sheaves:

$$0 \longrightarrow \Omega_S^1(\log(\Delta)) \to \mathscr{F}|_S \to \mathcal{O}_S \to 0. \quad (36)$$

By [Li18b, Theorem 1.3], we know that $\mathscr{E} := \mathscr{F}|_S$ is slope semistable. Then the generalization of [Wah93, Proposition 3.16] to the logarithmic/orbifold setting

combined together with Langer's work should imply that $e_{\mathrm{orb}}(x, X, D) = \frac{c_1(\mathscr{E})^2}{4(-S\cdot S)_Y}$ which is indeed equal to $\frac{\widehat{\mathrm{vol}}(\mathrm{ord}_S)}{4}$.

6.3.3. Normalized volume function. We have mainly concentrated on the minimizer of the normalized volume function. We can also ask questions on the general behavior of the normalized volume function. For example:

Question 6.23 (Convexity). Let $\sigma \subset \mathrm{Val}_{X,x}$ be a simplex of quasi-monomial valuations. Is it true that $\widehat{\mathrm{vol}}(\cdot)$ is always convex on σ? Is there a more general convexity property for $\widehat{\mathrm{vol}}$ on $\mathrm{Val}_{X,x}$?

Question 6.24. Is the normalized volume a lower semicontinuous function on $\mathrm{Val}_{X,x}$? If this is true, then it would directly imply the existence of minimizer of $\widehat{\mathrm{vol}}$ using the properness estimate in Theorem 2.5.

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