ON THE VOLUME OF K-SEMISTABLE FANO MANIFOLDS

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ABSTRACT. We prove that the volume of an n-dimensional K-semistable Fano manifold that is not $\mathbb{P}^n$ is at most $2n^n$. Moreover, the equality holds only if $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ or X is a smooth quadric hypersurface $Q \subset \mathbb{P}^{n+1}$. Our proof is based on a new connection between K-semistability and minimal rational curves. More generally, we show that the volume of a K-semistable Fano manifold with a minimal rational curve of degree d is bounded above by the volume of $\mathbb{P}^{d-1} \times \mathbb{P}^{n-d+1}$.

CONTENTS

1. INTRODUCTION

The study of K-(poly)stable Fano varieties is a very active research area. Thanks to the resolution of the Yau-Tian-Donaldson conjecture, we know that a Fano variety admits a Kähler-Einstein metric (resp. a unique Kähler-Einstein metric) if and only if it is K-polystable (resp. K-stable). A Fano variety is K-semistable if and only if it degenerates (via a special test configuration) to a K-polystable

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Fano variety. There are well-established valuative criterion for K-(semi)stability ([Fuj19, Li17]) and powerful methods to test it (see [AZ22]).

In this paper, we study the anticanonical volume (or degree) of K-semistable Fano varieties. For a Fano variety X, its (anticanonical) volume $\mathrm{vol}(X)$ is defined to be the self-intersection number $(-K_X)^n$. It is known that the volume of an $n-$dimensional Fano manifold can exceed the volume of $\mathbb{P}^n$, even among toric Fano manifolds (see [Deb01, Section 5.11]). In a major advance in the study of K-stability, Fujita [Fuj18] proved that the volume of an n-dimensional K-semistable Fano manifolds X satisfies $\mathrm{vol}(X) \leq (n+1)^n = \mathrm{vol}(\mathbb{P}^n)$ and the first equality holds if and only if $X \cong \mathbb{P}^n$. It was later generalized to possibly singular $\mathbb{Q}$-Fano varieties by Liu [Liu18]. The toric Fano case was proved earlier in [BB17]. Moreover, from the boundedness of K-semistable $\mathbb{Q}$-Fano varieties ([Jia20, Corollary 1.2]), we know that the set of volumes of n-dimensional K-semistable $\mathbb{Q}$-Fano varieties is finite away from zero. In this paper, we solve the conjecture on characterizing the second-largest volume for K-semistable Fano manifolds,

Theorem 1.1. *Any K-semistable Fano manifold X that is not $\mathbb{P}^n$ satisfies $\mathrm{vol}(X) = (-K_X)^n \leq 2n^n$ and the equality holds only if $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ or X is a smooth quadric hypersurface $Q \subset \mathbb{P}^{n+1}$. In particular, this holds for any Kähler-Einstein metric manifold with positive Ricci curvature.*

Like the result of Fujita, the last statement could be seen as a result in Kähler differential geometry. However, our proof uses purely algebraic geometry. The above result seems to be first conjectured in [AIM20, Problem 2.6], and is stated as a folklore conjecture in [Zhu25, Conjecture 6.8] which highlights the interesting (but also mysterious) feature that there are two Fano manifolds with second largest volume. Our proof will indeed give a satisfactory explanation of this feature by connecting it to the theory of minimal rational curves on Fano manifolds.

There is a local analogue of this conjecture sometimes called the ODP conjecture (see Conjecture 2.3) for the volume of klt singularity (see [SS17, LX19]). The ODP conjecture would imply that the volume of singular K-semistable Fano varieties is strictly less than $2n^n$ (by Liu's local-to-global volume comparison, see Remark 3.9). Conversely, using Theorem 1.1, we immediately verify the ODP conjecture for the Fano cone over a K-semistable smooth Fano manifold (see Theorem 4.5).

The toric version of Theorem 1.1 has its own interest and is called the "gap hypothesis" in [AB24, Conjecture 3.10]. It implies the sharp bound of the canonical height of canonical model of toric Fano varieties over $\mathrm{Spec}\,\mathbb{Z}$ by [AB24, Lemma 3.8]. Moreover, in the toric case, the ODP conjecture is solved by Moraga-Süß's result ([MS24]) using convex geometric methods. So combined with Theorem 1.1, we indeed have the sharp upper bound for all possibly singular K-semistable toric Fano varieties.

Theorem 1.2. *The second largest volume of n-dimensional K-semistable toric Fano varieties is $2n^n$ and the equality holds only if $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$.*

We remark that Theorem 1.2 was proved in [AB24] for smooth toric Fano manifolds of dimension $n \leqslant 6$ by using the classification of [Obr07] and also for certain singular toric Fano varieties by convex geometric arguments (see [AB24, Lemma 3.9]). Furthermore, using the well-known one-to-one correspondence between Gorenstein toric Fano varieties and reflexive lattice polytopes in $\mathbb{R}^n$, Theorem 1.2 immediately implies a convex geometric statement.

Corollary 1.3. *Suppose $P \subseteq \mathbb{R}^n$ is a n-dimensional reflexive lattice polytope with barycenter at $0 \in \mathbb{R}^n$. Assume P is not unimodularly equivalent to $(n+1)$ times a standard simplex $(n+1)\Delta_n$, then the volume of P with respect to the Lebesgue measure in $\mathbb{R}^n$ satisfies $\mathrm{vol}_{\mathbb{R}^n}(P) \leqslant 2n^n/n!$ and the equality holds if and only if P is unimodularly equivalent to $[0,2] \times (n\Delta_{n-1})$.*

We will prove Theorem 1.1 by exploring a new connection between K-stability and minimal rational curves. Roughly speaking, we are going to estimate the volume by using the valuative criterion to test K-semistability via valuations associated to weighted blowups along minimal rational curves. An interesting new phenomenon we find is that the volume estimates become sharp when we let certain weights go to infinity, and we indeed get the following general result, which to our knowledge is the first result connecting explicitly the theory of K-stability and the theory of minimal rational curves on Fano manifolds.

Theorem 1.4. *Assume that a K-semistable Fano manifold X admits a minimal rational curve $f : \mathbb{P}^1 \to X$ such that $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(d-2)} \oplus \mathcal{O}^{\oplus(n-d+1)}$ for $2 \le d \le n+1$. Then we have $\mathrm{vol}(X) \le \mathrm{vol}(\mathbb{P}^{d-1} \times \mathbb{P}^{n-d+1})$.*

The smooth toric case is special in the sense that there are always smoothly embedded projective spaces with trivial normal bundles (see [CFH14, Ara06]). So the toric Fano case can also be considered as an example of the following direct generalization of Fujita's result.

Theorem 1.5. *Let X be an n-dimensional K-semistable Fano manifold, and $Z \subset X$ a codimension r smooth complex submanifold. Assume that the normal bundle of Z inside X is trivial and set $d = (-K_X)^{n-r} \cdot Z$. Then we have the inequality*

$$(-K_X)^n \leqslant (r+1)^r \cdot \binom{n}{r} \cdot d = (-K_{\mathbb{P}^r \times Z})^n. \tag{1}$$

Moreover, the equality holds if and only if X is biholomorphic to $\mathbb{P}^r \times Z$. In particular Z is also a K-semistable Fano manifold.

We sketch the organization of this paper. In the next section, we recall basic knowledge about some key concepts used in later sections: K-stability, Seshadri constant, and minimal rational curves. In section 3, we prove Theorem 1.5, and Theorem 1.1 in the case the Fano manifold contains a minimal rational curve with a trivial normal bundle. We also deduce Theorem 1.2 and Corollary 1.3. In section 4, we prove Theorem 1.4 to achieve the full proof of Theorem 1.1. We end the paper by giving some related examples and finding (K-semistable) Fano manifolds with minimal volumes.

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2. Preliminaries

We will work over $\mathbb{C}$. Unless otherwise specified, all varieties are assumed to be normal and projective. A $\mathbb{Q}$-Fano variety X is a variety with at worst klt singularities such that the anti-canonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. A singularity $x \in X$ consists of a variety X and a closed point $x \in X$. A singularity $x \in X$ is called klt if X is klt near a neighborhood of x. A $\mathbb{R}$-valuation over singularity x is a valuation $v\colon K(X)^* \to \mathbb{R}$ center at x (namely, for all $f \in \mathcal{O}_{X,x}$, we have $v(f) \geqslant 0$ and $v(f) > 0$ if and only if $f \in \mathfrak{m}_x$) and $v|_{\mathbb{C}^*} = 0$. The set of all such valuations is denoted by $\mathrm{Val}_{X,x}$.

2.1. K-stability.

K-stability. K-stability, first introduced by Tian ([Tia97]) and later reformulated algebraically by Donaldson ([Don02]), is an algebro-geometric notion to characterize the existence of Kähler-Einstein metrics on Fano varieties. In this subsection, we recall some notions in K-stability theory that are relevant to our paper, and refer to [Xu24] for a detailed exposition of K-stability theory.

We say that a prime divisor E is over X if there exists a proper birational morphism $\mu\colon Y \to X$ such that Y is normal and E is a prime divisor on Y. We define the log discrepancy of the divisor E over X as

$$A(E) = A_X(E) := 1 + \mathrm{coeff}_E(K_Y - \mu^* K_X).$$

The volume of an $\mathbb{R}$-Cartier divisor D is defined as $\mathrm{vol}_X(D) := \limsup_{m\to+\infty} \frac{h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^n/n!}$. Note that the limsup is actually a limit and the volume function is continuous in the big cone $\mathrm{Big}(X)$ ([Laz04, Section 2.2.C]). Define the S-invariant

$$S(E) := S(-K_X; E) := \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}(-K_X) - xE)\,\mathrm{d}x,$$

where, for the simplicity of notation, we just write $\mathrm{vol}(-K_X - xE)$ for $\mathrm{vol}(\mu^*(-K_X) - xE)$. We have the following valuative criterion for K-semistability,

Theorem 2.1. [Fuj19, Li17] *A Fano variety X is K-semistable if and only if $A(E) - S(E) \geqslant 0$ for every divisor E over X.*

Another equivalent way to characterize K-semistability is via the δ-invariant (also known as stability threshold), we recall $\delta(X) := \inf_{E/X} \frac{A(E)}{S(E)}$ where the infinum is taking over all divisors E over X ([FO18, BJ20]). Then X is K-semistable if and only if $\delta(X) \geqslant 1$ by the valuative criterion.

One may define the log discrepancy $A_X(v)$ for any valuation $v \in \mathrm{Val}_{X,x}$, see [JM12, Section 5.1]. For a klt singularity $x \in (X, \Delta)$, we always have $A_X(v) > 0$ for any valuation $v \in \mathrm{Val}_{X,x}$. Denote $\mathrm{Val}^*_{X,x} = \{v \in \mathrm{Val}_{X,x} \mid A_{(X,\Delta)}(v) < +\infty\}$. The volume of a valuation $v \in \mathrm{Val}_{X,x}$ is defined as

$$\mathrm{vol}_{X,x}(v) := \limsup_{m \to +\infty} \frac{l(\mathcal{O}_{X,x}/\mathfrak{a}_m(v))}{m^n/n!},$$

where $\mathfrak{a}_m(v)$ denotes the valuation ideals: $\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{X,x} \mid v(f) \geq m\}$. The first named author introduced the following invariant for singularity ([Li18]), which plays a key role in the study of local K-stability:

Definition 2.2. *Suppose $x \in X$ is a klt singularity, for a valuation $v \in \mathrm{Val}_{X,x}$, we define the normalized volume*

$$\widehat{\mathrm{vol}}_X(v) := \begin{cases} A_X(v)^n \cdot \mathrm{vol}(v) & \text{if } A_X(v) < +\infty \\ +\infty & \text{if } A_X(v) = +\infty. \end{cases}$$

The local volume of $x \in X$ is defined as $\widehat{\mathrm{vol}}(x, X) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}_X(v)$.

It was proved that for any n-dimensional klt singularity $x \in X$, we have $\widehat{\mathrm{vol}}(x, X) \leqslant n^n$, with the equality holds if and only if $x \in X$ is a smooth point ([LX19]). The following conjecture is known as the ODP conjecture ([SS17, LX19]).

Conjecture 2.3. *(ODP conjecture) The second largest local volume of an n-dimensional klt singularity is $2(n-1)^n$, with equality if and only if $x \in X$ is an ordinary double point.*

For later purposes, we formulate a calculus lemma and an estimate of volume. Let $\mathscr{F}$ denote the collection of continuous functions $\phi : [0, \infty) \to [0, \infty)$ that is piecewise smooth, strictly increasing and surjective. In particular, such a function ϕ satisfies

$$\lim_{x \to 0} \phi(x) = 0, \quad \lim_{x \to +\infty} \phi(x) = +\infty. \tag{2}$$

For any $\phi \in \mathscr{F}$, consider the following function of $V \in [0, \infty)$:

$$F^\phi(V) := \frac{1}{V} \int_0^{\phi^{-1}(V)} (V - \phi(x))\, \mathrm{d}x.$$

Lemma 2.4. *(1) For any $\phi \in \mathscr{F}$, the function $F^\phi(V)$ is also an element of $\mathscr{F}$.*

(2) If $\psi \in \mathscr{F}$ satisfies $(\phi^{-1})'(V) \geq (\psi^{-1})'(V)$ for any $V \in (0, \infty)$, then $F^\phi(V) \geq F^\psi(V)$ for any $V \in (0, \infty)$. If moreover the first strict inequality holds for $V \geq V_1$, then $F^\phi(V) > F^\psi(V)$ for $V \geq V_1$. As a consequence, we get $(F^\phi)^{-1}(A) < (F^\psi)^{-1}(A)$ for any $A \in (0, \infty)$ with strict inequality if $A > F^\psi(V_1)$.

Proof. First assume that $\phi(x)$ is smooth and strictly increasing on $[0, \infty)$. Then its inverse function $\phi^{-1}(x)$ is also smooth, strictly increasing and satisfies (2). Set $G(V) = \int_0^{\phi^{-1}(V)} (V - \phi(x)) \mathrm{d}x$ so that $F(V) = F^\phi(V) = G(V)/V$. Then G is a smooth function satisfying $G(0) = 0$. Its derivative is equal to $G'(V) = \int_0^{\phi^{-1}(V)} 1\,\mathrm{d}x = \phi^{-1}(V)$. We can then calculate the derivative of $F(V)$ at any $V \in (0, \infty)$:

$$F'(V) = \frac{G' \cdot V - G}{V^2} = \frac{\int_0^{\phi^{-1}(V)} \phi(x)\mathrm{d}x}{V^2} = \frac{H(V)}{V^2} > 0$$

with $H(V) = H^\phi(V) = \int_0^{\phi^{-1}(V)} \phi(x)\mathrm{d}x$. So we know that $F(V)$ is a strictly increasing function of $V \in [0, \infty)$. Using L'Hospital's rule, we easily see that F satisfies the limit conditions in (2). So $F \in \mathscr{F}$.

Note that $H(0) = 0$ and $H'(V) = \phi(\phi^{-1}(V))(\phi^{-1})'(V) = V(\phi^{-1})'(V)$. By integrating the assumed inequality twice, we get the inequality $F^\phi(V) \geq F^\psi(V)$. For the last statement, note that under the assumption of strict inequality, we know that if $A > F^\psi(V_1)$ then $(F^\psi)^{-1}(A) > V_1$ which implies: $F^\phi((F^\psi)^{-1}(A)) > F^\psi((F^\psi)^{-1}(A)) = A$. So we conclude that $(F^\psi)^{-1}(A) > (F^\phi)^{-1}(A)$ when $A > F^\psi(V_1)$.

If ϕ is only piecewise smooth, we can just carry out the above argument piecewise on each interval of smoothness. The same argument applies to the second statement too. □

Corollary 2.5. *Assume $(X, L := -K_X)$ is a K-semistable $\mathbb{Q}$-Fano variety. Let E be a divisor over X that satisfies the estimate: there exists $\phi \in \mathscr{F}$ such that for any $x \in [0, \infty)$,*

$$\mathrm{vol}(L - xE) \geqslant V - \phi(x). \tag{3}$$

There exists a unique solution $T = T(A_X(E))$ to the equation

$$(T - A_X(E))\phi(T) = \Phi(T) \tag{4}$$

where $\Phi(x) = \int_0^x \phi(t)\mathrm{d}t$ is the primitive function of $\phi(x)$ with $\Phi(0) = 0$. Moreover, we have an estimate of the volume:

$$V = (-K_X)^n \leqslant \phi(T) = (F^\phi)^{-1}(A(E)). \tag{5}$$

The equality $V = \phi(T)$ holds if and only if the equality holds in (3) *for $x \in [0, T]$. In this case, T is the pseudo-effective threshold of the valuation* ord_E *and in particular* $\mathrm{vol}(L - TE) = 0$.

Proof. We have an estimate:

$$\begin{aligned} 0 &\leqslant A_X(E) - \frac{1}{V}\int_0^{+\infty} \mathrm{vol}(L - xE)\mathrm{d}x \leqslant A_X(E) - \frac{1}{V}\int_0^{\phi^{-1}(V)} (V - \phi(x))dx \\ &= A_X(E) - F^\phi(V). \end{aligned}$$

By the above lemma, $F^\phi(V)$ is a strictly increasing function that diverges to $+\infty$ as $V \to +\infty$. So there exists a unique V^* that satisfies $A(E) = F^\phi(V^*)$ which is equivalent to the equality:

$$(\phi^{-1}(V^*) - A(E))V^* = \int_0^{\phi^{-1}(V^*)} \phi(x)\mathrm{d}x.$$

Setting $T = \phi^{-1}(V^*)$, we see that T satisfies $(T - A(E))\phi(T) = \Phi(T)$. □

Example 2.6 (Fujita [Fuj18]). *Assume that $\phi(x) = x^n$ and $A(E) = n$. The $\Phi(x) = \frac{x^{n+1}}{n+1}$. The equation* (4) *has the solution $T = n + 1$ and $\phi(T) = (n + 1)^n$.*

Example 2.7. *Assume that $\phi(x) = 2nx^{n-1}$ and $A(E) = n - 1$. The $\Phi(x) = 2x^n$. The equation* (4) *has the solution $T = n$ and $\phi(T) = 2n^n$.*

2.2. **Seshadri Constant.** Let X be a normal projective variety and L an ample $\mathbb{Q}$-Cartier divisor on X. Let $Z \subset X$ be a nonsingular closed subvariety of X of codimension r. The Seshadri constant of L at Z is defined as $\epsilon(L, Z) \coloneqq \sup\{t \in \mathbb{R}_{>0} \mid \pi^*L - tE \text{ is ample}\}$, where $\pi\colon \mathrm{Bl}_Z X \to X$ is the blowup of X along Z and $E \cong \mathbb{P}(N_{Z/X}^\vee)$ is the exceptional divisor. When $X = \mathbb{P}^r \times Z$ and Z is identified with $\{p\} \times Z$ for a fixed $p \in \mathbb{P}^r$, $(\mathrm{Bl}_Z X, E)$ are the same as $((\mathrm{Bl}_p\mathbb{P}^r) \times Z, E_p \times Z)$ where $E_p \cong \mathbb{P}^{r-1}$ is the exceptional divisor of the blow up $\mathrm{Bl}_p\mathbb{P}^r \to \mathbb{P}^r$. We see that $\epsilon(-K_{\mathbb{P}^r\times Z}, Z) = r+1$.

The next proposition generalizes a result of [LZ18] to higher dimensional subvariety Z with trivial normal bundle, which says there is a gap between r and $r+1$ for all the possible value of $\epsilon(-K_X, Z)$.

Proposition 2.8. *Suppose X is a $\mathbb{Q}$-Fano variety of dimension n. If there exists a nonsingular subvariety $Z \subset X$ of codimension $r \geqslant 2$ with trivial normal bundle $N_{Z/X} = \mathcal{O}_Z^{\oplus r}$ and $\epsilon(-K_X, Z) > r$, then $X \cong \mathbb{P}^r \times Z$.*

Proof. We follow the argument as in [LZ18, Theorem 2]. For simplicity, we denote $\epsilon \coloneqq \epsilon(-K_X, Z) > r$. Let $\pi\colon \hat{X} \coloneqq \mathrm{Bl}_Z X \to X$ be the blowup of X along Z and E is the exceptional divisor. We denote $B \coloneqq \pi^*(-K_X) - \epsilon E$. From the definition of $\epsilon(-K_X, Z)$, we know B is a nef divisor. We have $K_{\hat{X}} = \pi^*K_X + (r-1)E$, so $B - K_{\hat{X}} = 2(\pi^*(-K_X) - \frac{\epsilon+r-1}{2}E)$ is nef and big since $(\epsilon+r-1)/2 < \epsilon$. Then by Kawamata's basepoint-free theorem ([KM98, Theorem 3.3]), we know B is indeed semi-ample. Then there exists a fibration $\Phi\colon \hat{X} \to Y \subseteq \mathbb{P}(H^0(\hat{X}, kB))$ induced by the complete linear series $|kB|$ for some $k \gg 0$ and Y is a closed subvariety. Next, let m be an integer such that mB is Cartier. Note that

$$mB - E - K_{\hat{X}} = (m+1)\left(\pi^*(-K_X) - \frac{m\epsilon + r}{m+1}E\right)$$

is an ample divisor since $(m\epsilon + r)/(m + 1) < \epsilon$. Then, by Kodaira's vanishing theorem, we get $H^1(\hat{X}, mB - E) = 0$. Therefore, the natural map $H^0(\hat{X}, mB) \to H^0(E, mB|_E)$ is surjective for all $m > 0$ such that mB is Cartier. We conclude that $\Phi|_E\colon E \to Y$ is a closed embedding. Since $\hat{X}$ has klt singularites, then it has rational singularities ([KM98, Theorem 5.22]), so in particular it is Cohen-Macaulay. Since $B = -K_{\hat{X}} + (r - 1 - \epsilon)E \sim_{\Phi,\mathbb{Q}} 0$, we get $-K_{\hat{X}} \sim_{\Phi,\mathbb{Q}} \lambda E$ where $\lambda - r + 1 + \epsilon > 1$. By [LZ18, Lemma 8], we have $\Phi\colon \hat{X} \to Y$ is not birational. Thus, Φ must be a fiber type contraction.

Since we have already shown $\Phi|_E\colon E \to Y$ is a closed embedding, we conclude that $\Phi|_E\colon E \to Y$ is in fact an isomorphism, so $Y \cong E \cong \mathbb{P}(N_{Z/X}^\vee) \cong Z \times \mathbb{P}^{r-1}$. The general fiber of Φ is a smooth rational curve. By a similar argument as in [LZ18, Lemma 6], we can show $\Phi\colon \hat{X} \to Y$ is a smooth $\mathbb{P}^1$-fibration. Note that $s = \Phi|_E^{-1}\colon Y \to E$ gives a section of Φ, then there exists a rank 2 vector bundle $\mathcal{E}$ over Y such that $\hat{X} = \mathbb{P}_Y(\mathcal{E})$. By [Har77, Proposition V.2.6], there exists an invertible sheaf $\mathcal{L}$ on Y and a surjective morphism $\mathcal{E} \to \mathcal{L}$. We denote $\mathcal{K} = \ker(\mathcal{E} \to \mathcal{L})$. Then $\mathcal{O}_Y(-1) \cong s^* N_{E/\hat{X}} \cong \mathcal{L} \otimes \mathcal{K}^{-1}$. We know that the choice of $\mathcal{E}$ is unique up to twisting of an invertible sheaf. So we may assume $\mathcal{K} = \mathcal{O}_Y$, then $\mathcal{L} = \mathcal{O}_Y(-1) \cong p_2^* \mathcal{O}_{\mathbb{P}^{r-1}}(-1)$ with the projection $p_2 \colon Y \to \mathbb{P}^{r-1}$ and we have the following short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{E} \to \mathcal{O}_Y(-1) \to 0.$$

By [Har77, Prop III.6.3], we have

$$\mathrm{Ext}^1(\mathcal{L}, \mathcal{O}_Y) = H^1(Y, \mathcal{L}^\vee) = \bigoplus_{i+j=1} H^i(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(1)) \otimes H^j(Z, \mathcal{O}_Z) = 0,$$

so the short exact sequence actually splits. Therefore, $\mathcal{E} \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)$ and $\hat{X} \cong \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(-1)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{r-1}} \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(-1)) \times Z \cong \mathrm{Bl}_p \mathbb{P}^r \times Z$ is isomorphic to $\mathbb{P}^r \times Z$ blowup along $\{p\} \times Z$. So $X \cong \mathbb{P}^r \times Z$. ☐

2.3. **Minimal rational curves.** Since the celebrated work of Mori introducing the bend-and-break method for constructing rational curves on Fano manifolds ([Mor79]), the theory of rational curves has been developed extensively and has found numerous applications in algebraic geometry, especially in the classification of Fano manifolds with special properties. Here we recall some basic knowledge of rational curves and refer to [Kol96] for detailed expositions.

A free rational curve is represented by a morphism $f : \mathbb{P}^1 \to X$ that satisfies $f^*TX = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 0$. A free rational curve is called minimal (or standard) if

$$f^*TX = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{d-2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d+1)}.$$

Such a morphism f must be an immersion and its image $f(\mathbb{P}^1)$ has at worst nodal singularities. We will also be interested in its normal bundle:

$$N_{f/X} := f^*TX/T\mathbb{P}^1 = \mathcal{O}(1)^{\oplus(d-2)} \oplus \mathcal{O}^{\oplus(n-d+1)}.$$

Any Fano manifold always admits minimal rational curves, which can be obtained from the bend-and-break process starting with a free rational curve (see [Kol96, IV.Theorem 2.10]). Denote by $\mathrm{RatCurves}^n(X)$ the normalization of open subset of $\mathrm{Ch}(X)$ parametrizing integral rational curves. An irreducible component $\mathcal{M}$ of $\mathrm{RatCurves}^n(X)$ is referred to as a family of rational curves on X. The anticanonical degree $\deg(\mathcal{M})$ of the family $\mathcal{M}$ is defined to be $-K_X \cdot C$ for any curve C belonging to the family. This family $\mathcal{M}$ is equipped with a $\mathbb{P}^1$-bundle $p : \mathcal{U} \to \mathcal{M}$ and an evaluation morphism $q : \mathcal{U} \to X$. The family $\mathcal{M}$ is a *dominating* family if the evaluation morphism $q : \mathcal{U} \to X$ is dominant (i.e. has a dense image). This is equivalent to the condition that a general rational curve in $\mathcal{M}$ is free. A dominating family $\mathcal{M}$ is *locally unsplit* if, for a general point $x \in X$, the subfamily $\mathcal{M}_x = p(q^{-1}(x))$ parametrizing curves through x is proper (i.e. compact). Note that $\mathcal{M}$ is a dominating family of rational curves on X that has a minimal anticanonical degree, then $\mathcal{M}$ is a dominating locally unsplit family. But not all dominating locally unsplit family has a minimal anticanonical degree.

Let $\mathcal{M}$ be a dominating locally unsplit family of rational curves. By Mori's bend-and-break argument, we know that for a general point $x \in X$, a general curve in $\mathcal{M}_x$ is minimal. In particular, $\deg(\mathcal{M}) \in \{2, 3, \cdots, n+1\}$. Indeed, if a general free rational curve $C = [f]$ in $\mathcal{M}_x$ is not minimal, then there are at least two $\mathcal{O}(2)$ summands in the splitting of f^*TX. We can then fix two points on the curve C and bend-and-break rational curve into a non-integral (reducible) curve which would contradict the locally unsplit property. Following [Miy04, CD15], we set

$$l_X := \min\{\deg \mathcal{M}; \mathcal{M} \text{ is a dominating locally unsplit family of rational curves on } X\}. \tag{6}$$

By the above discussion, it is easy to see that (see [CD15, Remark 4.2]):

$$\begin{aligned} l_X &= \min\{-K_X \cdot C; C \subset X \text{ is a free rational curve on } X\} \\ &= \min\{-K_X \cdot C; C \subset X \text{ is a minimal rational curve on } X\}. \end{aligned} \tag{7}$$

Example 2.9. *To prove our main results, we will need the following important results on classification of Fano manifolds with dominating locally unsplit families of large anticanonical degrees.*

(1) *(*[CMSB02]*)* $l_X = n+1$ *if and only if* $X \cong \mathbb{P}^n$.
(2) *(*[Miy04, CD15, DH17]*)* $l_X = n$ *if and only if* $X \cong Q^n$ *or* X *is the blowup of* $\mathbb{P}^n$ *along subvariety* Y *of degree* $d_Y \in \{1, \ldots, n\}$ *that is contained in a hyperplane.*

Example 2.10 ([HM03]). *On the other extreme, the condition* $l_X = 2$ *is equivalent to the following condition:*

(1) *There exists a minimal rational curve with trivial normal bundle.*
(2) *For a general point* $x \in X$*, there are only finitely many rational curves through* x *which have minimal degree with respect to* K_X^{-1}.
(3) *for a general point* $x \in X$*, there exists a rational curve which has degree 2 with respect to* K_X^{-1}.

There are many examples of such Fano manifolds. For example, except for the projective space $\mathbb{P}^3$ *and 3-dimensional hyperquadric* Q^3*, all Fano 3-folds of Picard number 1 satisfy this condition. Hypersurfaces of* $\mathbb{P}^{n+1}$ *of degree* n *or* $n+1$ *are also examples of such Fano manifolds when* $n \geqslant 3$.

3. Warm-up: case of trivial normal bundles

In this section, we prove the following theorem, which is a special case of Theorem 1.1.

Theorem 3.1. *Let* X *be a K-semistable Fano manifold that contains a rational curve with a trivial normal bundle. Then* $(-K_X)^n \leqslant 2n^n$ *and the equality holds if and only if* $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$.

The rest of this section is devoted to the proof of Theorem 3.1. We start with a proposition that proves a more general version of estimate (1) by incorporating the δ-invariant.

Proposition 3.2. *Assume X is a n-dimensional Fano manifold, and $Z \subset X$ is a codimension r non-singular subvariety with trivial normal bundle $N_{Z/X} = \mathcal{O}_Z^{\oplus r}$, and we set $d = (-K_X)^{n-r} \cdot Z$, then*

$$(-K_X)^n \leqslant \delta(X)^{-r} \cdot (r+1)^r \binom{n}{r} \cdot d = \delta(X)^{-r} \cdot (-K_{\mathbb{P}^r \times Z})^n.$$

Proof. Let $\pi\colon \hat{X} \coloneqq \mathrm{Bl}_Z X \to X$ be the blowup of X along Z with the exceptional divisor E. First, it is clear that the log discrepancy

$$A_X(E) = 1 + \mathrm{coeff}_E(K_{\hat{X}} - \pi^* K_X) = 1 + (r-1) = r.$$

For the simplicity of notation, set $L = -K_X$. We can assume $x \in \mathbb{Q}$ since the volume function $\mathrm{vol}_{\hat{X}}(\pi^* L - xE)$ is continuous. Then we take $k \in \mathbb{N}^*$ sufficiently large such that $kx \in \mathbb{Z}_{>0}$. Note that we have the exact sequence:

$$0 \to H^0(X, kL \otimes \mathcal{I}_Z^{xk}) \to H^0(X, kL) \to H^0(X, kL \otimes \mathcal{O}_{xkZ}) \to \cdots,$$

which implies:

$$h^0(X, kL \otimes \mathcal{I}_Z^{xk}) \geqslant h^0(X, kL) - h^0(X, kL \otimes \mathcal{O}_{xkZ}).$$

Note that the higher direct images $R^i \pi_* \mathcal{O}_{\hat{X}} = 0$ for $i > 0$. By the Leray spectral sequences, we get $H^0(\hat{X}, \pi^*(kL) - xkE) = H^0(X, kL \otimes \mathcal{I}_Z^{xk})$ and $H^0(\hat{X}, \pi^*(kL)) = H^0(X, kL)$. Thus,

$$\mathrm{vol}_{\hat{X}}(\pi^* L - xE) = \limsup_{k \to +\infty} \frac{h^0(X, kL \otimes \mathcal{I}_Z^{xk})}{k^n/n!} \geqslant L^n - \limsup_{k \to +\infty} \frac{h^0(X, kL \otimes \mathcal{O}_{xkZ})}{k^n/n!}.$$

For $j \in \mathbb{Z}_{\geqslant 0}$, we use the exact sequence

$$0 \to \mathcal{I}_Z^j / \mathcal{I}_Z^{j+1} \to \mathcal{O}_X / \mathcal{I}_Z^{j+1} \to \mathcal{O}_X / \mathcal{I}_Z^j \to 0.$$

And $H^1(X, kL \otimes (\mathcal{I}_Z^j / \mathcal{I}_Z^{j+1})) = \oplus H^1(Z, -kK_Z) = 0$ from Kodaira vanishing and the assumption that normal bundle $N_{Z/X}$ is trivial. It is easy to prove by induction that:

$$\begin{aligned} h^0(X, kL \otimes \mathcal{O}_X / \mathcal{I}_Z^{xk}) &= h^0(X, kL \otimes \mathcal{O}_X / \mathcal{I}_Z^{xk-1}) + h^0(X, kL \otimes \mathcal{I}_Z^{xk-1} / \mathcal{I}_Z^{xk}) \\ &= \sum_{j=0}^{xk-1} h^0(X, kL \otimes \mathcal{I}_Z^j / \mathcal{I}_Z^{j+1}). \end{aligned} \tag{8}$$

Using the assumption that $N_{Z/X}$ is trivial and Z is non-singular, by [Har77, Thm II. 8.24], we get:

$$\mathcal{I}_Z^j / \mathcal{I}_Z^{j+1} \cong \mathrm{Sym}^j(\mathcal{I}_Z / \mathcal{I}_Z^2) = \mathrm{Sym}^j(N_{Z/X}^\vee) = \mathcal{O}_Z^{\oplus \binom{r-1+j}{r-1}}.$$

Then the right-hand-side of (8) is given by

$$\sum_{j=0}^{xk-1} h^0(Z, \mathcal{O}_X(kL) \otimes \mathcal{I}_Z^j / \mathcal{I}_Z^{j+1}) = \sum_{j=0}^{xk-1} \binom{j+r-1}{r-1} \cdot h^0(Z, -kK_Z) = \binom{r+xk-1}{r} \cdot h^0(Z, -kK_Z).$$

Then,

$$\begin{aligned}\limsup_{k\to+\infty}\frac{h^0(X,kL)\otimes\mathcal{O}_{xkZ}}{k^n/n!} &= \limsup_{k\to+\infty}\frac{n!}{r!(n-r)!}\cdot\frac{(r+xk-1)!}{(xk-1)!\cdot k^r}\cdot\frac{h^0(Z,-kK_Z)}{k^{n-r}/(n-r)!}\\ &= \binom{n}{r}(-K_Z)^{n-r}\cdot x^r. \qquad (9)\end{aligned}$$

We set $d=(-K_Z)^{n-r}=(-K_X)^{n-r}\cdot Z$. So we get $\mathrm{vol}_{\hat{X}}(\pi^*L-xE)\geqslant L^n-d\binom{n}{r}x^r$. Since $E\cong Z\times\mathbb{P}^{r-1}$ and $\mathcal{O}(-E)|_E=p_2^*\mathcal{O}_{\mathbb{P}^{r-1}}(1)$, one can easily see that the right-hand side equals to the top-intersection number

$$\begin{aligned}(\pi^*L-xE)^n &= \sum_{k=0}^{n}(\pi^*L)^{n-k}\cdot x^k(-E)^k\\ &= (\pi^*L)^n+(-1)^r\binom{n}{r}dx^r\cdot(-1)^{r-1}=L^n-d\binom{n}{r}x^r.\end{aligned}$$

Moreover, we have:

$$\begin{aligned}S(-K_X;E) &= \frac{1}{(-K_X)^n}\int_0^{T_X(E)}\mathrm{vol}_{\hat{X}}(\pi^*(-K_X)-xE)\,\mathrm{d}x\\ &\geqslant \frac{1}{(-K_X)^n}\int_0^{\epsilon}\left((-K_X)^n-d\binom{n}{r}x^r\right)\,\mathrm{d}x\\ &= \epsilon-\frac{1}{(-K_X)^n}\binom{n}{r}\frac{d\cdot\epsilon^{r+1}}{r+1}=\frac{r}{r+1}\epsilon,\end{aligned}$$

where $\epsilon=\left((-K_X)^n/(\binom{n}{r}\cdot d)\right)^{1/r}$. Then,

$$\delta(X)\ \leqslant\ \frac{A_X(E)}{S(-K_X;E)}\leqslant\frac{r}{\frac{r}{r+1}\epsilon}=\frac{r+1}{\left((-K_X)^n/(\binom{n}{r}\cdot d)\right)^{1/r}}.$$

Therefore, $(-K_X)^n\leqslant\delta(X)^{-r}(r+1)^r\cdot\binom{n}{r}\cdot d=\delta(X)^{-r}\cdot(-K_{\mathbb{P}^r\times Z})^n$. ☐

Lemma 3.3. *Under the same notation as the above proposition, we set*

$$\Lambda_Z(L)\coloneqq\{x\in\mathbb{R}_{\geqslant 0}\mid \mathrm{vol}_{\hat{X}}(\pi^*L-xE)=(\pi^*L-xE)^n\}. \qquad (10)$$

Then we have $\epsilon(L,Z)=\max\{t\in\mathbb{R}_{\geqslant 0}\mid x\in\Lambda_Z(L)$ *for all* $x\in[0,t]\}$.

Proof. The argument is similar with [Fuj18, Theorem 2.3(2)]. We denote

$$\gamma=\gamma_Z(L)\coloneqq\max\{t\in\mathbb{R}_{\geqslant 0}\mid x\in\Lambda_Z(L)\text{ for all }x\in[0,t]\}. \qquad (11)$$

When $0\leqslant t\leqslant\epsilon(L,Z)$, we have π^*L-tE is nef, then $\mathrm{vol}_{\hat{X}}(\pi^*L-tE)=(\pi^*L-tE)^n$. Therefore, $\epsilon(L,Z)\leqslant\gamma$. In particular, $\gamma>0$. Now, in order to show $\epsilon(L,Z)\geqslant\gamma$, it suffices to show for any $\eta>0$ sufficiently small such that $\gamma-\eta\in\mathbb{Q}_{>0}$ and $\pi^*L-(\gamma-\eta)E$ is ample. Fix an $\delta\in\mathbb{Q}_{>0}$ such that $\pi^*L-\delta E$ is ample. Take any $t\in\mathbb{Q}_{>0}$ satisfying $t\leqslant\min\{1,(\gamma-\eta)/\delta,\ \eta/(\gamma-\delta)\}$. Then,

$$(\pi^*L-(\gamma-\eta)E)-t(\pi^*L-\delta E)=(1-t)\left(\pi^*L-\frac{\gamma-\eta-t\delta}{1-t}E\right).$$

We set $x_t := (\gamma - \eta - t\delta)/(1-t)$. We take sufficiently large $k \in \mathbb{N}^*$ such that $kx_t \in \mathbb{Z}_{>0}$. From Kodaira's vanishing theorem, we have $H^i(X, kL) = 0$ for $i \geqslant 1$. Then the exact sequence is given by

$$0 \to H^0(X, kL \otimes \mathcal{I}_Z^{kx_t}) \to H^0(X, kL) \to H^0(X, kL \otimes \mathcal{O}_{kx_tZ}) \to H^1(X, kL \otimes \mathcal{I}_Z^{kx_t}) \to 0.$$

And $H^i(X, kL \otimes \mathcal{I}_Z^{kx_t}) = H^{i-1}(X, kL \otimes (\mathcal{O}_X/\mathcal{I}_Z^{kx_t}))$ for $i \geqslant 2$. On the one hand, since $x_t \in \Lambda_Z(L)$ and by the equation (9),

$$\limsup_{k\to+\infty} \frac{h^1(X, kL \otimes \mathcal{I}_Z^{kx_t})}{k^n/n!} = -L^n + \mathrm{vol}_{\hat{X}}(\pi^*L - x_tE) + \limsup_{k\to+\infty} \frac{h^0(X, kL \otimes \mathcal{O}_{kx_tZ})}{k^n/n!} = 0.$$

Then, $h^1(X, kL \otimes \mathcal{I}_Z^{kx_t}) = o(k^n)$. On the other hand, for $j \in \mathbb{Z}_{\geqslant 0}$, by the exact sequence

$$0 \to \mathcal{I}_Z^j/\mathcal{I}_Z^{j+1} \to \mathcal{O}_X/\mathcal{I}_Z^{j+1} \to \mathcal{O}_X/\mathcal{I}_Z^j \to 0.$$

We have $H^i(X, kL \otimes \mathcal{I}_Z^j/\mathcal{I}_Z^{j+1}) = 0$ for $i \geqslant 1$ and $j \geqslant 1$ since $N_{Z/X}$ is trivial. So we get

$$H^i(X, kL \otimes \mathcal{O}_X/\mathcal{I}_Z^{j+1}) \cong H^i(X, kL \otimes \mathcal{O}_X/\mathcal{I}_Z^j)$$

for all $i \geqslant 1$ and $j \geqslant 1$. Then, for $i \geqslant 2$,

$$h^i(X, kL \otimes \mathcal{I}_Z^{kx_t}) = h^{i-1}(X, kL \otimes (\mathcal{O}_X/\mathcal{I}_Z^{kx_t})) = h^{i-1}(X, kL \otimes (\mathcal{O}_X/\mathcal{I}_Z)) = 0.$$

And the higher direct images $R^i\pi_*\mathcal{O}_{\hat{X}} = 0$ for $i > 0$, then by the Leray spectral sequence, we have $H^i(\hat{X}, \pi^*(kL) - kx_tE) \cong H^i(X, kL \otimes \mathcal{I}_Z^{xk_t})$ for $i \geqslant 0$. In particular, we get $h^i(\hat{X}, kL - kx_tE) = o(k^n)$ for $i \geqslant 1$. Then by [dFKL07, Theorem A], we conclude that $\pi^*L - (\gamma - \eta)E$ is ample. This shows $\epsilon(L, Z) \geqslant \gamma$. □

Proof of Theorem 1.5. Since X is assumed to be K-semistable, we know that $\delta(X) \geq 1$ by the valuative criterion (Theorem 2.1). Without loss of generality, we assume the codimension r of Z in X is positive. By the proof of Proposition 3.2, the equality (1) holds if and only if $\delta(X) = 1$ and $\mathrm{vol}_{\hat{X}}(\pi^*(-K_X) - xE) = (\pi^*(-K_X) - xE)^n$ for all $x \in [0, r+1]$, so $\gamma_Z(-K_X) = r+1$ (see (11)). Then by Lemma 3.3, we have $\epsilon(-K_X, Z) = \gamma_Z(-K_X) = r+1$. By Proposition 2.8, we get $X \cong Z \times \mathbb{P}^r$. □

Theorem 3.4. *The second largest volume of n-dimensional K-semistable toric Fano manifold is $2n^n$ and the equality holds only if $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$.*

Proof. When X is a smooth toric Fano manifold, by the work of Chen-Fu-Hwang [CFH14, Corollary 2.5] which is partly based on the work of Araujo [Ara06], we know that there exists submanifold $Z \cong \mathbb{P}^{n-r}$ that is contained in X with trivial normal bundle. By Proposition 3.2 and $\delta(X) \geqslant 1$, we have $(-K_X)^n \leqslant (-K_{\mathbb{P}^r \times Z})^n = (-K_{\mathbb{P}^r \times \mathbb{P}^{n-r}})^n = \binom{n}{r} \cdot (r+1)^r \cdot (n-r+1)^{n-r} =: c_r$. By the following lemma 3.5, for $2 \leq r \leq n-1$, $c_r \leq 2n^n$ with equality holds if and only if $r = 2$ or $r = n-1$ in which case $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$. □

Lemma 3.5. *The sequence $c_r := \binom{n}{r}(r+1)^r(n-r+1)^{n-r}$ with $0 \leq r \leq n$ satisfies $c_r = c_{n-r}$ and $c_0 = (n+1)^n > c_1 = 2n^n > c_2 > \cdots > c_{\lfloor n/2 \rfloor}$.*

Proof. It is clear that $c_r = c_{n-r}$. We calculate:

$$\frac{c_{r+1}}{c_r} = \left(\frac{r+2}{r+1}\right)^{r+1} \cdot \left(\frac{n-r}{n-r+1}\right)^{n-r} = a_{r+1}/a_{n-r}$$

where $a_k = (\frac{k+1}{k})^k = (1+k^{-1})^k$ is an strict increasing sequence for $k \geq 1$ (that converges to e as $k \to +\infty$). So we get $c_{r+1}/c_r < 1$ when $r+1 < n-r$ (or equivalently $2r+1 < n$). The statement then follows easily. □

Proposition 3.6. *Let X be a K-semistable Fano manifold. Assume that $f : \mathbb{P}^1 \to X$ is an immersed rational curve with a non-empty subset of nodal points such that $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}^{n-1}$. Then we have an estimate of the volume $(-K_X)^n < 2n^n$. As a consequence, if there exists a minimal rational curve with a trivial normal bundle on X and $(-K_X)^n = 2n^n$, then the minimal rational curve has no nodal points and X must be isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.*

Proof. The last statement follows from the strict volume inequality and Theorem 1.5. So we just need to prove the inequality $(-K_X)^n < 2n^n$. Denote the ideal sheaf of the nodal curve $C = f(\mathbb{P}^1)$ in X by $\mathcal{I} = \mathcal{I}_C$. Consider the blow up $\mu : \tilde{X} \to X$ along C. By using the same argument as in the proof of Proposition 3.2, we get the estimate:

$$\begin{aligned}\mathrm{vol}(\mu^*(-K_X) - xE) &\geqslant (-K_X)^n - \limsup_{k\to+\infty} \frac{h^0(X, L^k \otimes \mathcal{O}_{kxC})}{k^n/n!} \\ &\geqslant (-K_X)^n - \limsup_{k\to+\infty} \frac{\sum_{y=0}^{xk-1} h^0(C, kL \otimes \mathcal{I}^y/\mathcal{I}^{y+1})}{k^n/n!}.\end{aligned}$$

Since C is a local complete intersection, its conormal sheaf $N^\vee_{C/X} = \mathcal{I}/\mathcal{I}^2$ is locally free and $\mathcal{I}^y/\mathcal{I}^{y+1} = \mathrm{Sym}^y(N^\vee_{C/X})$. So we get:

$$h^0(C, L^k \otimes \mathcal{I}^y/\mathcal{I}^{y+1}) = h^0(C, L^k \otimes \mathrm{Sym}^y(\mathcal{I}/\mathcal{I}^2)) \leqslant h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(dk) \otimes \mathrm{Sym}^y(f^*(\mathcal{I}/\mathcal{I}^2))).$$

There is an inclusion $\mathcal{F} := f^*(\mathcal{I}/\mathcal{I}^2) \hookrightarrow N^\vee_{f/X} = (f^*TX/T\mathbb{P}^1)^\vee$. We know that $\deg(N^\vee_{f/X}) = 2-d$ and $\deg(\mathcal{F}) = \deg(K_X) - \deg \omega_C = -d - 2m + 2$ where $d = -K_X \cdot C$. Since $N^\vee_{f/X}$ is trivial, $d = 2$ and $\mathcal{F} = \mathcal{O}(-a_1) \oplus \mathcal{O}(-a_2) \oplus \cdots \oplus \mathcal{O}(-a_{n-1})$ with $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq 0$ and $\sum_{i=1}^{n-1} a_i = 2m$. It is then easy to get the estimate:

$$h^0(C, L^k \otimes \mathcal{I}^y/\mathcal{I}^{y+1}) \leqslant h^0(\mathbb{P}^1, \mathcal{O}(2k) \otimes \mathrm{Sym}^y(\mathcal{O}(-2) \oplus \mathcal{O}^{\oplus(n-2)})) \tag{12}$$

or

$$h^0(C, L^k \otimes \mathcal{I}^y/\mathcal{I}^{y+1}) \leqslant h^0(\mathbb{P}^1, \mathcal{O}(2k) \otimes \mathrm{Sym}^y(\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}^{\oplus(n-3)})). \tag{13}$$

For the first case (12), we get:

$$\sum_{y=0}^{xk-1} h^0(C, L^k \otimes \mathcal{I}^y/\mathcal{I}^{y+1}) \leq \sum_{y=0}^{xk-1} \sum_{i=0}^{y} (2k - 2i + 1) a_{y,i} \tag{14}$$

where $a_{y,i}=\binom{n-3+y-i}{n-3}$. If $x\leqslant 1$, then by using the explicit expression of binomial coefficient, the right-hand-side of (14) is equal to $b_1k^n/n!+O(k^{n-1})$ where b_1 is given by the following integral (see (19) for a derivation of a more complicated formula):

$$\begin{aligned} b_1 &= n!\int_0^x \mathrm{d}t\int_0^t (2-2z)\frac{(t-z)^{n-3}}{(n-3)!}\mathrm{d}z=\frac{2n!}{(n-2)!}\int_0^x(1-z)(x-z)^{n-2}\mathrm{d}z \\ &= 2nx^{n-1}-2x^n<2nx^{n-1}. \end{aligned}$$

When $x>1$, the right-hand-side of (14) splits into the sum: $\sum_{y=0}^k\sum_{i=0}^y+\sum_{y=k+1}^{xk-1}\sum_{i=0}^k$ and the right-hand-side of (14) is equal to $b_2k^n/n!+O(k^{n-1})$ with

$$\begin{aligned} b_2 &= (2n-2)+n!\int_1^x\mathrm{d}t\int_0^1(2-2z)\frac{(t-z)^{n-3}}{(n-3)!}\mathrm{d}z \\ &= (2n-2)+\frac{2n!}{(n-2)!}\int_0^1(1-z)((x-z)^{n-2}-(1-z)^{n-2})\mathrm{d}z \\ &= \frac{2n!}{(n-2)!}\int_0^1 u(x-1+u)^{n-2}\mathrm{d}u+(2n-2)-(2n-2) \\ &= 2(nx^{n-1}-x^n+(x-1)^n)<2(n-1)x^{n-1}+2. \end{aligned}$$

We set $\psi=2nx^{n-1}$ and

$$\phi(x)=\begin{cases} 2nx^{n-1} & \text{if } 0\le x\le 1\\ 2(n-1)x^{n-1}+2 & \text{if } x>1,\end{cases} \tag{15}$$

so that $\phi(x)<\psi(x)$. It is easy to check that $(\phi^{-1})'(V)\geq(\psi^{-1})'(V)$ and the strict inequality holds when $V>2n$. So we can apply Lemma 2.4 to conclude that $F^\phi(V)\geq F^\psi(V)$ and the strict inequality holds for $V>2n$. From Lemma 2.4 and Corollary 2.5, we get $(-K_X)^n\leq(F^\phi)^{-1}(n-1)<(F^\psi)^{-1}(n-1)=2n^n$.

In the second case (13), we get:

$$\sum_{y=0}^{xk-1}h^0(C,L^k\otimes\mathcal{I}^y/\mathcal{I}^{y+1})\leq\sum_{y=0}^{xk-1}\sum_{i=0}^{y}(2k-i+1)(i+1)\binom{n-4+y-i}{n-4}.$$

Estimating as before, when $x\leqslant 2$, the right-hand-side of (14) is equal to $c_1k^n/n!+O(k^{n-1})$. The coefficient c_1 is equal to:

$$\begin{aligned} c_1 &= n!\int_0^x\mathrm{d}t\int_0^t(2-z)z\frac{(t-z)^{n-4}}{(n-4)!}\mathrm{d}z \\ &= \frac{n!}{(n-3)!}\int_0^x(2-z)z(x-z)^{n-3}\mathrm{d}z=2x^{n-1}(n-x)<2nx^{n-1}. \end{aligned}$$

When $x \geqslant 2$, the right-hand-side of (14) splits into the sum: $\sum_{y=0}^{2k}\sum_{i=0}^{y} + \sum_{y=2k+1}^{xk-1}\sum_{i=0}^{2k}$ and the right-hand-side of (14) is equal to $c_2 k^n/n! + O(k^{n-1})$ with the coefficient c_2 is equal to:

$$\begin{aligned} c_2 &= 2^n(n-2) + n!\int_2^x \mathrm{d}t \int_0^2 (2-z)z\frac{(t-z)^{n-4}}{(n-4)!}\mathrm{d}z \\ &= 2^n(n-2) + \frac{n!}{(n-3)!}\int_0^2 (2-z)z((x-z)^{n-3}-(2-z)^{n-3})\mathrm{d}z \\ &= \frac{n!}{(n-3)!}\int_0^2 (2-z)z(x-z)^{n-3}\mathrm{d}z \\ &= 2nx^{n-1} - 2x^n + (x-2)^n + n(x-2)^{n-1} < (2n-1)x^{n-1} \\ &< (2n-1)x^{n-1} + 2^{n-1}. \end{aligned}$$

We set $\psi(x) = 2nx^{n-1}$ and

$$\phi(x) = \begin{cases} 2nx^{n-1} & \text{if } 0 \le x \le 2 \\ (2n-1)x^{n-1} + 2^{n-1} & \text{if } x \ge 2. \end{cases} \tag{16}$$

It is easy to check that $(\phi^{-1})'(V) \ge (\psi^{-1})'(V)$ and strict inequality holds for $V > 2^n \cdot n$. Then by Lemma 2.4 we get $F^{\phi}(V) \ge F^{\psi}(V)$ with strict inequality when $V > 2^n \cdot n$. So we get $(-K_X)^n \le (F^{\phi})^{-1}(n-1) < (F^{\psi})^{-1}(n-1) = 2n^n$. □

Proof of Theorem 3.1. If X contains a rational curve with non-empty nodal points, then by Proposition 3.6, we have $(-K_X)^n < 2n^n$. So it suffices to consider the case when C is an embedded rational curve, in which case the result follows immediately from Theorem 1.5. □

Proposition 3.7. *Suppose X is a singular n-dimensional K-semistable toric $\mathbb{Q}$-Fano variety. When $n = 2$, then $(-K_X)^2 \leqslant \frac{9}{2}$. When $n \geqslant 3$, then $(-K_X)^n \leqslant \frac{16}{27}(n+1)^n$. In particular, we have $(-K_X)^n < (-K_{\mathbb{P}^1\times\mathbb{P}^n})^n = 2n^n$ holds for all positive integer $n \geqslant 2$.*

Proof. Suppose $x \in X$ lies in the singular locus and (X, x) is a toric singularity. It was proved in [MS24, Theorem 2] that if (X, x) is a n-dimensional $\mathbb{Q}$-Gorenstein toric singularity, then

(1) If $n = 2$, then $\widehat{\mathrm{vol}}(X, x) \leqslant 2$;
(2) If $n \leqslant 3$, then $\widehat{\mathrm{vol}}(X, x) \leqslant \frac{16}{27}n^n$.

By Liu's local-to-global volume comparison ([Liu18]), we have $(-K_X)^n \leqslant (\frac{n+1}{n})^n \cdot \widehat{\mathrm{vol}}(X, x)$. Then when $n = 2$, $(-K_X)^2 \leqslant (\frac{3}{2})^2 \cdot 2 = \frac{9}{2} < 2 \cdot 2^2 = 8 = (-K_{\mathbb{P}^1\times\mathbb{P}^1})^2$. When $n \geqslant 3$,

$$(-K_X)^n \leqslant (\frac{n+1}{n})^n \cdot \frac{16}{27}n^n = \frac{16}{27}\cdot(n+1)^n < 2n^n = (-K_{\mathbb{P}^1\times\mathbb{P}^{n-1}})^n.$$

□

Theorem 3.8. *The second largest volume of n-dimensional K-semistable toric $\mathbb{Q}$-Fano varieties is $2n^n$ and the equality holds only if $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$.*

Proof. When X is a smooth toric Fano manifold, this follows from Theorem 3.4. When X is a strictly singular $\mathbb{Q}$-Fano toric variety, this follows from Proposition 3.7. When the equality $(-K_X)^n = 2n^n$ holds, from Proposition 3.7, we know X must be nonsingular. Then by Theorem 3.4 again, we know $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$. □

Remark 3.9. *We remark that the singular version of Theorem 1.1 (that is, consider all $\mathbb{Q}$-Fano varieties, not just smooth Fano manifolds) follows from ODP conjecture (Conjecture 2.3). Let X be a K-semistable $\mathbb{Q}$-Fano variety and $x \in X$ be a singular point. Assume the ODP conjecture holds, by the same argument of Proposition 3.7, we have $(-K_X)^n \leqslant (\frac{n+1}{n})^n \cdot \widehat{\mathrm{vol}}(X,x) \leqslant (\frac{n+1}{n})^n \cdot 2(n-1)^n = (\frac{n+1}{n})^n \cdot (\frac{n-1}{n})^n \cdot 2n^n < 2n^n$. Since the inequality is strict, it is expected that the equality of Theorem 1.1 would hold only if X is a smooth Fano manifold.*

We conclude this section by a convex-geometric application.

Corollary 3.10. *Suppose $P \subseteq \mathbb{R}^n$ is a n-dimensional reflexive lattice polytope with barycenter at $0 \in \mathbb{R}^n$. Assume P is not unimodularly equivalent to $(n+1)$ times a standard simplex $(n+1)\Delta_n$, then the volume of P with respect to the Lebesgue measure in $\mathbb{R}^n$ satisfies $\mathrm{vol}_{\mathbb{R}^n}(P) \leqslant 2n^n/n!$ and the equality holds if and only if P is unimodularly equivalent to $[0,2] \times (n\Delta_{n-1})$.*

Proof. Here Δ_n denote the standard n-dimensional simplex with vertices $0, e_1, \cdots, e_n$ where e_i are standard lattice basis of $\mathbb{Z}^n$ for $i = 1, \cdots, n$. And two convex bodies P and Q in $\mathbb{R}^n$ are called unimodularly equivalent if there exists an affine lattice automorphism of $\mathbb{Z}^n$ mapping P onto Q. Let X_P be the projective toric variety associated to a reflexive polytope P. Then X_P is a Gorenstein Fano variety ([CLS11, Theorem 8.3.4]). By the result of [WZ04, SZ12, BB13], we know that the barycenter of P is zero if and only if X_P is K-semistable. Since P is not unimodularly equivalent to $(n+1)\Delta_n$, we have $X_P \ncong \mathbb{P}^n$, then Theorem 3.8 immediately implies $(-K_{X_P})^n \leqslant 2n^n$ and the equality holds only if $X_P \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$. Then the reflexive polytope P satisfies $\mathrm{vol}_{\mathbb{R}^n}(P) \leqslant 2n^n/n!$ and $P \cong [0,2] \times (n\Delta_{n-1})$ up to unimodularly equivalent. □

4. Volume estimates and minimal rational curves

4.1. **Proof of Theorem 1.4 and Theorem 1.1.** The calculation in the previous section is based on the test of K-stability via standard blow-ups. In this section, we prove Theorem 1.4 and Theorem 1.1 by considering the test of K-stability via weighted blow-ups along (Zariski open sets of) minimal rational curves.

Proof of Theorem 1.4. Assume that $f : \mathbb{P}^1 \to X$ be minimal rational curve, or equivalently an immersed rational curve that satisfies $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(d-2)} \oplus \mathcal{O}^{\oplus(n-d+1)}$ with $2 \leq d \leq n+1$ and $d = f^*(-K_X) \cdot \mathbb{P}^1 = (-K_X) \cdot f(\mathbb{P}^1)$. The (relative) normal bundle has the splitting:

$$N_{f/X} = f^*TX/T\mathbb{P}^1 = \mathcal{O}(1)^{\oplus(d-2)} \oplus \mathcal{O}^{\oplus(n-d+1)}.$$

Correspondingly, we have the splitting of the conormal bundle:

$$N^{\vee}_{f/X} = \mathcal{O}(-1)^{\oplus(d-2)} \oplus \mathcal{O}^{\oplus(n-d+1)}.$$

Set $Z = f(\mathbb{P}^1)$ which is an irreducible curve with possible nodal singularities. Fix a regular point $x \in Z$, there exists a Zariski open neighborhood U and coordinates $\{z_i; i = 1, \dots, n\}$ such that

(1) $Z \cap U = \{z_1 = \cdots = z_{n-1} = 0\}$;
(2) For any $1 \le i \le d-2$, $\mathrm{d}z_i$ is a local generator of the i-th summand of $\mathcal{O}(-1)^{\oplus(d-2)}$ over $Z \cap U$;
(3) For any $d-1 \le j \le n-1$, $\mathrm{d}z_j$ is a local generator of the j-th summand of $\mathcal{O}^{\oplus(n-d+1)}$ over $Z \cap U$.

For any $\ell \in \mathbb{Z}_{>0}$, we can define a divisorial valuation $v = \mathrm{ord}_E$: for any $f \in \mathcal{O}(U)$,

$$v(f) = \min\{|I| + \ell|J|; f = \sum_{I,J} a_{IJ} z'^I z''^J, a_{IJ} = a_{IJ}(z_n) \neq 0\} \tag{17}$$

where $z' = \{z_1, \dots, z_{d-2}\}$ (resp. $z'' = \{z_{d-1}, \cdots, z_{n-1}\}$), and $z'^I = z_1^{i_1} \cdots z_{d-2}^{i_{d-2}}$ for $I = (i_1, \dots, i_{d-2}) \in \mathbb{N}^{d-2}$ with $|I| = i_1 + \cdots + i_{d-2}$ (resp. $z''^J = z_{d-1}^{j_1} \cdots z_{n-1}^{j_{n-d+1}}$ for $J = (j_1, \cdots, j_{n-d+1}) \in \mathbb{N}^{n-d+1}$ with $|J| = j_1 + \cdots + j_{n-d+1}$).

When restricted to U, the valuation v corresponds to the exceptional divisor E of the weighted blowup of $X \cap U$ along $Z \cap U$ with weights $(\underbrace{1, \dots, 1}_{d-2}, \underbrace{\ell, \cdots, \ell}_{n-d+1})$ (See [QR12] for more a general set-up of weighted blowups). Set $\mathcal{I}_x = \mathcal{I}_x(\mathrm{ord}_E) = \{f \in \mathcal{O}_X; v(f) \ge x\}$. Then we have the exact sequence:

$$0 \to H^0(X, L^k \otimes \mathcal{I}_{xk}) \to H^0(X, L^k) \to H^0(X, L^k \otimes \mathcal{O}_X/\mathcal{I}_{xk}) \to \cdots,$$

which implies:

$$\mathrm{vol}(L - xE) \ge L^n - \limsup_{k \to +\infty} \frac{h^0(X, L^k \otimes \mathcal{O}_X/\mathcal{I}_{xk})}{k^n/n!}.$$

By using the exact sequence:

$$0 \to \mathcal{I}_y/\mathcal{I}_{y+1} \to \mathcal{O}_X/\mathcal{I}_{y+1} \to \mathcal{O}_X/\mathcal{I}_y \to 0,$$

we get inductively the estimates:

$$h^0(X, L^k \otimes \mathcal{O}_X/\mathcal{I}_{xk}) \le \sum_{y=0}^{xk-1} h^0(X, L^k \otimes \mathcal{I}_y/\mathcal{I}_{y+1}). \tag{18}$$

For any $[s] \in H^0(X, L^k \otimes \mathcal{I}_y/\mathcal{I}_{y+1})$, we know that with respect to a local trivialization of L^k, the section s is represented locally over U by a holomorphic function with the following expression modulo terms with $|I|+\ell|J|>y$,

$$\begin{aligned} s &= \sum_{|I|+\ell|J|=y} a_{IJ} z'^I z''^J = \sum_{k=0}^{\lfloor y/\ell\rfloor} \sum_{|I|=y-\ell k, |J|=k} a_{IJ} z'^I z''^J \\ &= \sum_{m=0}^{\lfloor y/\ell\rfloor} \sum_{\substack{|I|+|J|=y-(\ell-1)\lfloor y/\ell\rfloor+m(\ell-1)\\ |I|=y-\ell\lfloor y/\ell\rfloor+m\ell \\ |J|=\lfloor y/\ell\rfloor-m}} a_{IJ} z'^I z''^J. \end{aligned}$$

Assume that W is another Zariski open set such that $W\cap Z$ is non-empty and has at most one nodal point. In particular, there are at most two irreducible components of $W\cap Z$. Fix one irreducible component Z_1 which is smooth, we can choose coordinates (after possibly shrinking W) such that

(1) $W\cap Z_1=\{w_1=\cdots=w_{n-1}=0\}$
(2) For any $1\le i\le d-2$, $\mathrm{d}w_i$ is a local generator of the i-th summand of $\mathcal{O}(-1)^{\oplus(d-2)}$ over $W\cap Z_1$.
(3) For any $d-1\le j\le n-1$, $\mathrm{d}w_j$ is a local generator of the j-th summand of $\mathcal{O}^{\oplus(n-d+1)}$ over $W\cap Z_1$.

Near $U\cap W\cap Z$, the transition functions from $\mathbf{z}$ and $\mathbf{w}$ must be of the form:

$$z_k = a_k(w_n)w_k + R_2, \quad 1\le k\le n-1$$

where $a_k(w_n)$ is a transition function of $\mathcal{O}_{\mathbb{P}^1}(-1)$ for $1\le k\le d-2$ and of $\mathcal{O}_{\mathbb{P}^1}$ for $d-1\le k\le n-1$, and R_2 consists of terms of degree at least 2 in $w_1,\dots,w_{n-1}$. So near $W\cap Z_1$, we have $s=\sum_{|I|+|J|=y-(\ell-1)\lfloor y/\ell\rfloor} b_{IJ}(w_n)w'^I w''^J + R$ where R are terms of (unweighted) total degree in (w',w'') strictly greater than $y-(\ell-1)\lfloor y/\ell\rfloor$. We can deal with the other possible component of $W\cap Z$ using the same argument. So we see that s induces a well-defined section in $H^0(\mathbb{P}^1, L^k\otimes \mathrm{Sym}^{y-\ell\lfloor y/\ell\rfloor}(\mathcal{O}(-1)^{\oplus(d-2)})\otimes \mathrm{Sym}^{\lfloor y/\ell\rfloor}(\mathcal{O}^{\oplus(n-d+1)}))$. This defines a linear map of vector spaces:

$$H^0(X, L^k\otimes \mathcal{I}_y/\mathcal{I}_{y+1}) \to H^0(\mathbb{P}^1, L^k\otimes \mathrm{Sym}^{y-\ell\lfloor y/\ell\rfloor}(\mathcal{O}(-1)^{\oplus(d-2)})\otimes \mathrm{Sym}^{\lfloor y/\ell\rfloor}(\mathcal{O}^{\oplus(n-d+1)})),$$

whose kernel consists of elements $s\in H^0(X, L^k\otimes\mathcal{I}_y/\mathcal{I}_{y+1})$ that has expansion ($m\ge 1$ now):

$$s=\sum_{m=1}^{\lfloor y/\ell\rfloor} \sum_{\substack{|I|+|J|=y-(\ell-1)\lfloor y/\ell\rfloor+m(\ell-1)\\ |I|=y-\ell\lfloor y/\ell\rfloor+m\ell \\ |J|=\lfloor y/\ell\rfloor-m}} a_{IJ} z'^I z''^J,$$

which by the same argument as above induces a section in $H^0(\mathbb{P}^1, L^k\otimes \mathrm{Sym}^{y-\ell\lfloor y/\ell\rfloor+\ell}(\mathcal{O}(-1)^{\oplus(d-2)})\otimes \mathrm{Sym}^{\lfloor y/\ell\rfloor-1}(\mathcal{O}^{\oplus(n-d+1)}))$. So we inductively get the estimate:

$$\begin{aligned} h^0(X, L^k\otimes\mathcal{I}_y/\mathcal{I}_{y+1}) &\le \sum_{m=0}^{\lfloor y/\ell\rfloor} h^0(\mathbb{P}^1, \mathcal{O}(kd)\otimes \mathrm{Sym}^{y-\ell\lfloor y/\ell\rfloor+m\ell}(\mathcal{O}(-1)^{\oplus(d-2)})\otimes \mathrm{Sym}^{\lfloor y/\ell\rfloor-m}(\mathcal{O}^{\oplus(n-d+1)})) \\ &\le \sum_{m=0}^{\lfloor y/\ell\rfloor} h^0(\mathbb{P}^1, \mathcal{O}(kd)\otimes \mathrm{Sym}^{\ell m}(\mathcal{O}(-1)^{\oplus(d-2)})\otimes \mathrm{Sym}^{\lfloor y/\ell\rfloor-m}(\mathcal{O}^{\oplus(n-d+1)})). \end{aligned}$$

For the last inequality, we used the facts that $y - \ell\lfloor y/\ell \rfloor \geq 0$ and $\mathcal{O}(-1)$ is negative. Then the right-hand-side of (18) is bounded above by

$$I_k(x) := \sum_{y=0}^{xk-1} \sum_{m=0}^{\lfloor y/\ell \rfloor} a_{y,m} h^0(\mathbb{P}^1, \mathcal{O}(kd - \ell m)),$$

where

$$a_{y,m} = \binom{d-3+\ell m}{d-3}\binom{n-d+\lfloor y/\ell \rfloor - m}{n-d}.$$

We will calculate the leading coefficient $b_1 = \lim_{k\to+\infty} I_k(x)/k^n$. First, we assume that $x \leq d$. Then we can expand:

$$\begin{aligned} I_k(x) &= \sum_{y=0}^{xk-1}\sum_{m=0}^{\lfloor y/\ell \rfloor} \frac{(\ell m)^{d-3} + O((\ell m)^{d-4})}{(d-3)!} \frac{(\lfloor y/\ell \rfloor - m)^{n-d} + O((\lfloor y/\ell \rfloor - m)^{n-d-1})}{(n-d)!}(kd - \ell m + 1) \\ &= \left[\frac{k^n}{(d-3)!(n-d)!}\sum_{y=0}^{xk-1}\frac{1}{k}\sum_{m=0}^{\lfloor y/\ell \rfloor}\frac{1}{k}(\ell\frac{m}{k})^{d-3}(\ell^{-1}\frac{y}{k} - \frac{m}{k})^{n-d}(d - \ell\frac{m}{k})\right] + O(k^{n-1}). \end{aligned}$$

As $k \to +\infty$, we see that $I_k(x) = b_1 k^n/n! + O(k^{n-1})$ with b_1 given by the integral:

$$\begin{aligned} & \frac{n!}{(d-3)!(n-d)!}\int_0^x \mathrm{d}t \int_0^{t/\ell} (\ell s)^{d-3}(\ell^{-1}t - s)^{n-d}(d - \ell s)\mathrm{d}s \\ =\ & \frac{n!}{(d-3)!(n-d)!}\ell^{-(n-d+1)}\int_0^x \mathrm{d}t \int_0^t \left(z^{d-3}(d-z)(t-z)^{n-d}\right)\mathrm{d}z \\ =\ & \ell^{-(n-d+1)}\frac{n!}{(d-3)!(n-d+1)!}\int_0^x z^{d-3}(d-z)(x-z)^{n-d+1}\,\mathrm{d}z \\ =\ & \ell^{-(n-d+1)}x^{n-1}\left(dn - (d-2)x\right) =: \phi(x). \end{aligned} \tag{19}$$

It is easy to see that $\phi(x)$ is an increasing function when $x \in [0, \frac{(n-1)d}{(d-2)}]$ with $\phi(d) = \ell^{-(n-d+1)}d^n(n-d+2)$. In particular, $\phi(x)$ is strictly increasing when $x \in [0, d]$ since $d \leq n+1$.

Next, consider the case when $x \geqslant d$. Then we need to split the sum in (18) into two parts:

$$\sum_{y=0}^{dk}\sum_{i=0}^{\lfloor y/\ell \rfloor} a_{y,i}h^0(\mathbb{P}^1, \mathcal{O}(kd - \ell i)) + \sum_{y=dk+1}^{xk-1}\sum_{i=0}^{\lfloor dk/\ell \rfloor} a_{y,i}h^0(\mathbb{P}^1, \mathcal{O}(kd - i\ell)).$$

Similar calculation as above shows that this sum equals $b_2 k^n/n! + O(k^{n-1})$ with b_2 equal to $C \cdot \mathbf{I}$ where $C = \frac{n!}{(d-3)!(n-d)!}\ell^{-(n-d+1)}$ and $\mathbf{I}$ is equal to:

$$\begin{aligned} & \int_0^d \mathrm{d}t \int_0^t z^{d-3}(t-z)^{n-d}(d-z)\mathrm{d}z + \int_d^x \mathrm{d}t \int_0^d z^{d-3}(t-z)^{n-d}(d-z)\mathrm{d}z \\ =\ & C^{-1}\phi(d) + \int_0^d z^{d-3}\frac{1}{n-d+1}\left((x-z)^{n-d+1} - (d-z)^{n-d+1}\right)(d-z)\mathrm{d}z \\ =\ & \frac{1}{n-d+1}\int_0^d z^{d-3}(d-z)(x-z)^{n-d+1}\mathrm{d}z. \end{aligned}$$

We denote $V = L^n = (-K_X)^n$. So we get the estimate $\mathrm{vol}(L - xE) \geq V - \phi(x)$ for any $x \geq 0$ where

$$\phi(x) = \begin{cases} \ell^{-(n-d+1)} x^{n-1}(dn - (d-2)x) & \text{if } 0 \leq x \leq d \\ \frac{\ell^{-(n-d+1)} n!}{(d-3)!(n-d+1)!} \int_0^d z^{d-3}(d-z)(x-z)^{n-d+1}\,\mathrm{d}z & \text{if } x \geq d. \end{cases} \tag{20}$$

Note that the function $\phi = \phi(x)$ belongs to the class $\mathscr{F}$ considered in Lemma 2.4. In other words, it is a continuous, piecewise smooth, strictly increasing function of $x \in [0, \infty)$ satisfying $\lim_{x\to 0}\phi(x) = 0$ and $\lim_{x\to+\infty}\phi(x) = +\infty$. Set

$$\Phi(x) = \int_0^x \phi(t)dt = \begin{cases} \frac{\ell^{-(n-d+1)} x^n}{n+1}(d(n+1) - (d-2)x) & 0 \leq x \leq d \\ \frac{\ell^{-(n-d+1)} n!}{(d-3)!(n-d+2)!} \int_0^d z^{d-3}(d-z)(x-z)^{n-d+2}\mathrm{d}z & x \geq d. \end{cases}$$

Set $T = \sup\{x : V - \phi(x) = 0\}$. If $T \leq d$, then we get $V = \phi(T) \leq \phi(d) = \ell^{-(n-d+1)} d^n(n - d + 2) \leq d^n(n - d + 2)$. Note that the number $d^n(n - d + 2)$ is nothing but the volume of the Fano hypersurface in $\mathbb{P}^{n+1}$ of degree $n - d + 2$ which is always less than $2n^n$ if $2 < d < n + 2$.

So we can assume $T \geq d$. The equation $\Phi(T) = (T - A)\phi(T)$ from (4) with $A = A_X(E) = (d-2) + (n-d+1)\ell$ becomes:

$$\frac{1}{n-d+2}\int_0^d z^{d-3}(d-z)(T-z)^{n-d+2}\,\mathrm{d}z = (T-(d-2)-(n-d+1)\ell)\int_0^d z^{d-3}(d-z)(T-z)^{n-d+1}\,\mathrm{d}z.$$

Even though the explicit solution of T is not available, as $\ell \to +\infty$, the above equation can be written as:

$$\frac{1}{n-d+2} T^{n-d+2}(1 + O(T^{-1})) = (T - (n-d+1)\ell + O(1))T^{n-d+1}(1 + O(T^{-1})).$$

So we get the asymptotic expression for its solution: $T = T(\ell) = (n - d + 2)\ell + O(1)$. So we can calculate that, up to $O(\ell^{-1})$, $\phi(T)$ is given by:

$$\begin{aligned} \phi(T) + O(\ell^{-1}) &= \ell^{-(n-d+1)} \frac{n!}{(d-3)!(n-d+1)!}\int_0^d z^{d-3}(d-z)T^{n-d+1}dz \\ &= \ell^{-(n-d+1)} \frac{n!}{(d-3)!(n-d+1)!}\left(\int_0^1 t^{d-3}(1-t)\mathrm{d}t\right) \cdot d^{d-1}\ell^{n-d+1}(n-d+2)^{n-d+1} \\ &= (n-d+2)^{n-d+1}\frac{n!}{(d-1)!(n-d+1)!}d^{d-1} = \mathrm{vol}(\mathbb{P}^{n-d+1} \times \mathbb{P}^{d-1}). \end{aligned}$$

Letting $\ell \to +\infty$, by Corollary 2.5, we have thus proved $\mathrm{vol}(X) = (-K_X)^n \leq \mathrm{vol}(\mathbb{P}^{d-1} \times \mathbb{P}^{n-d+1})$ and complete the proof of Theorem 1.4. □

Example 4.1. *By using binomial expansion, one can get the following combinatorial expressions for $x \geq d$:*

$$\begin{aligned} \phi(x) &= \ell^{-(n-d+1)} \sum_{j=d-1}^{n} \binom{n}{j} \cdot (j - d + 2) \cdot d^j \cdot (x - d)^{n-j}, \\ \Phi(x) &= \ell^{-(n-d+1)} \sum_{j=d-1}^{n+1} \binom{n+1}{j} \frac{j-d+2}{n+1} d^j \cdot (x-d)^{n+1-j}. \end{aligned}$$

When $d = n + 1$, the above expression becomes: $\phi(x) = (n+1)^n$ (for $x \geq d$) which implies the volume estimate $\mathrm{vol}(X) \leq (n+1)^n = \mathrm{vol}(\mathbb{P}^n)$.

When $d = n$, the above expressions become:

$$\begin{aligned}\phi(x) &= \ell^{-1}\left(n \cdot n^{n-1} \cdot (x-n) + 2n^n\right) = \ell^{-1} n^n (x + 2 - n),\\ \Phi(x) &= \frac{\ell^{-1}}{n+1}\left(\binom{n+1}{2} n^{n-1}(x-n)^2 + (n+1)2n^n(x-n) + 3n^{n+1}\right)\\ &= \frac{\ell^{-1}n^n}{2}\left(x^2 - 2(n-2)x + \frac{n}{(n+1)}(n-1)(n-2)\right).\end{aligned}$$

The equation $(T - (n-2) - \ell)\phi(T) = \Phi(T)$ is asymptotically given by:

$$(T - \ell + O(1))T(1 + O(T^{-1})) = \frac{1}{2}T^2(1 + O(T^{-1}))$$

which implies $T = 2\ell + O(1)$. So $\phi(T) = \ell^{-1}n^n(2\ell + O(1) + 2 - n) \longrightarrow 2n^n$ as $\ell \to +\infty$ and we get an estimate: $\mathrm{vol}(X) \leq 2n^n = \mathrm{vol}(\mathbb{P}^1 \times \mathbb{P}^{n-1}) = \mathrm{vol}(Q^n)$.

Proof of Theorem 1.1. We use the notation l_X from (6)-(7). If $l_X = n + 1$, then as mentioned in Example 2.9 X must be $\mathbb{P}^n$, and if $l_X = n$ then X is either Q^n or the blowup of $\mathbb{P}^n$ along a smooth subvariety Y of degree $d_Y \in \{1, \ldots, n\}$ in a hyperplane. When $d_Y \geq 2$, by Lemma 4.2 below we know $(-K_{\mathrm{Bl}_Y\mathbb{P}^n})^n < 2n^n$. When $d_Y = 1$, by Lemma 4.3 below we know that $\mathrm{Bl}_{\mathbb{P}^{n-2}}\mathbb{P}^n$ is not a K-semistable Fano manifold, so it is excluded. Thus we verified the Theorem in the case $l_X \geq n$. When $l_X < n$, there are minimal rational curves of degree $d \in \{2, \ldots, n-1\}$. For $2 < d < n - 1$, by Theorem 1.4 and Lemma 3.5, $\mathrm{vol}(X) \leq \mathrm{vol}(\mathbb{P}^{d-1} \times \mathbb{P}^{n-d+1}) < 2n^n$. When $d = 2$, by Theorem 3.1 we get $\mathrm{vol}(X) \leq 2n^n$ with equality if and only if $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$. $\square$

Lemma 4.2. *Suppose X is the blowup of $\mathbb{P}^n$ along a subvariety Y of degree $d_Y \in \{1, \cdots, n\}$ that is contained in a hyperplane (See Example 2.9). Then*

$$(-K_{\mathrm{Bl}_Y\mathbb{P}^n})^n = \begin{cases} 2n^n & \text{if } d_Y = 1\\ \frac{d_Y n^n - (n+1-d_Y)^n}{d_Y - 1} =: V_{d_Y} & \text{if } 2 \leq d_Y \leq n.\end{cases}$$

Proof. Denote $d = d_Y$. The normal bundle of Y is given by $N_{Y/\mathbb{P}^n} = \mathcal{O}_Y(1) \oplus \mathcal{O}_Y(d)$. Let $H = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ be the pullback of a hyperplane section of $\mathbb{P}^n$. Then,

$$\begin{aligned}(-K_{\mathrm{Bl}_Y\mathbb{P}^n})^n &= ((n+1)H - E)^n = (n+1)^n + \sum_{k=2}^{n}\binom{n}{k}(n+1)^{n-k}\cdot(-1)^k(H^{n-k}\cdot E^k)\\ &= (n+1)^n - \sum_{k=2}^{n}\binom{n}{k}(n+1)^{n-k}\cdot(s_{k-2}(N_{Y/\mathbb{P}^n})\cdot H^{n-k}|_A)\\ &= (n+1)^n - \sum_{k=2}^{n}\binom{n}{k}(n+1)^{n-k}\cdot(-1)^{k-2}\left(\sum_{i=0}^{k-2} d^i\right)\cdot d,\end{aligned}$$

where $s_{k-2}(N_{Y/\mathbb{P}^n})$ denote the $(k-2)$-th Segre class of normal bundle $N_{Y/\mathbb{P}^n}$. If $d=1$, then $\sum_{i=0}^{k-2} d^i = k-1$. By the combinatorial identity $\binom{n}{k}(k-1) = n\binom{n-1}{k-1} - \binom{n}{k}$,

$$(-K_{\mathrm{Bl}_{\mathbb{P}^{n-2}}\mathbb{P}^n})^n = (n+1)^n - \sum_{k=2}^{n}(-1)^k \left[n\binom{n-1}{k-1} - \binom{n}{k}\right](n+1)^{n-k} = 2n^n.$$

If $d \geqslant 2$, then $\sum_{i=0}^{k-2} d^i = (d^{k-1}-1)/(d-1)$. Then,

$$\begin{aligned}(-K_{\mathrm{Bl}_Y\mathbb{P}^n})^n &= (n+1)^n - \sum_{k=2}^{n}(-1)^k\binom{n}{k}\cdot\left(\frac{d^k-d}{d-1}\right)(n+1)^{n-k} \\ &= \frac{1}{d-1}\left(dn^n - (n+1-d)^n\right).\end{aligned}$$

Note that by L'Hospital's rule $\lim_{d_Y\to 1} V_{d_Y} = 2n^n$ and it is easy to check that $V_{d_Y} < 2n^n$ for $d_Y \geq 2$. □

Lemma 4.3. *Suppose $\pi\colon X \to \mathbb{P}^n$ is the blowup of $\mathbb{P}^n$ along a subvariety $Y \cong \mathbb{P}^{n-2}$ of degree $d_Y = 1$ contained in a hyperplane of $\mathbb{P}^n$, then X is K-unstable.*

Proof. Let $E = \pi^{-1}(\mathbb{P}^{n-2})$ be the exceptional divisor. And let $H = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ be the pullback of a hyperplane section of $\mathbb{P}^n$. Note that the nef cone is $\mathrm{Nef}(X) = \mathbb{R}_{\geqslant 0}[H] + \mathbb{R}_{\geqslant 0}[H-E]$. Then the divisor $-K_X - tE = (n+1)H - (1+t)E$ is nef if and only if $0 \leqslant t \leqslant n$. When $0 \leqslant t \leqslant n$, by the similar computation as in Lemma 4.2,

$$\begin{aligned}\mathrm{vol}_X(-K_X - tE) &= ((n+1)H - (1+t)E)^n = (n+1)^n - \sum_{k=2}^{n}\binom{n}{k}(n+1)^{n-k}(-1-t)^k(k-1) \\ &= (n-t)^{n-1}(2n + (n-1)t).\end{aligned}$$

We see the pseudo-effective threshold $T_X(E)$ equals to the nef threshold n. Then,

$$S(-K_X;E) = \frac{1}{(-K_X)^n}\int_0^n \mathrm{vol}_X(-K_X - tE)\mathrm{d}t = 1 + \frac{n-1}{2(n+1)}.$$

So $A_X(E) - S(-K_X;E) = 1 - (1 + \frac{n-1}{2(n+1)}) = -\frac{n-1}{2(n+1)} < 0$. By the valuative criterion (Theorem 2.1), we conclude that X is K-unstable and destabilized by the exceptional divisor E. □

Remark 4.4. *Note that $X = \mathrm{Bl}_{\mathbb{P}^{n-2}}\mathbb{P}^n$ is a toric Fano variety, so one can also prove lemma 4.3 via toric geometry. The minimal generators of the fan $\Sigma \subseteq \mathbb{R}^n$ is given by $e_1 = (1,0\cdots,0), \cdots, e_n = (0,\cdots,0,1), e_0 = (-1,-1,\cdots,-1), f = (1,1,0,\cdots,0)$. Here the ray generated by f corresponds to the exceptional divisor $E = \pi^{-1}(\mathbb{P}^{n-2})$ by the cone-orbit correspondence. Then the moment polytope with respect to $-K_X$ is given by*

$$P_X = \{(x_1,\cdots,x_n) \in \mathbb{R}^n \mid x_1 + x_2 \geq -1, \sum_{i=1}^{n} x_i \leq 1, x_j \geq -1 \text{ for } j = 1,\cdots,n\}.$$

By calculus, one can compute the barycenter $\mathrm{Bc}(P) = (\frac{n-1}{4(n+1)}, \frac{n-1}{4(n+1)}, -\frac{1}{2(n+1)}, \cdots, -\frac{1}{2(n+1)}) \in \mathbb{R}^n$, which does not coincide with the origin. Thus, X is K-unstable, so this gives another proof of Lemma 4.3. In fact, by the formula $\delta(X,-K_X) = \min_{\rho\subset\Sigma(1)} \frac{1}{\langle \mathrm{Bc}(P), u_\rho\rangle + 1}$ in [BJ20, Section 7]*, we know*

$\delta(X) = \frac{2n+2}{3n+1} < 1$ and it is minimized at $\rho = f = (1, 1, 0 \cdots, 0)$. So the exceptional divisor E in Lemma 4.3 is indeed the δ-minimizer of X.

As an immediate application of Theorem 1.1, we prove the ODP conjecture (Conjecture 2.3) for Fano cones over K-semistable Fano manifolds.

Theorem 4.5. *Suppose $L = r^{-1}(-K_X)$ is an ample line bundle over a n-dimensional smooth K-semistable Fano manifold X for some $r \in \mathbb{N}^*$, then ODP conjecture (Conjecture 2.3) holds for the affine cone $\mathcal{C}(X, L) \coloneqq \mathrm{Spec}(\oplus_{k=0}^{+\infty} H^0(X, kL))$ with the cone vertex o.*

Proof. We denote $\pi\colon \mathrm{Bl}_o\mathcal{C} \to \mathcal{C}$ be the blowup of $\mathcal{C}$ along the cone vertex o. By the result of [Li17, LL19], we know X is K-semistable if and only if the minimizer of normalized volume of $(\mathcal{C}, o)$ is attained by the canonical valuation $v = \mathrm{ord}_E$, which is the divisorial valuation associated to the exceptional divisor $E = \pi^{-1}(o) \cong X$. From [Kol13, Section 3.1], we know

$$K_{\mathrm{Bl}_o\mathcal{C}} + (1-r)E \sim_{\mathbb{Q}} 0 \sim_{\mathbb{Q}} \pi^*(K_{\mathcal{C}}).$$

Then $A_{\mathcal{C}}(\mathrm{ord}_E) = 1 + \mathrm{ord}_E(K_{\mathrm{Bl}_o\mathcal{C}} - \pi^*K_{\mathcal{C}}) = 1 + (r-1) = r$ and $\mathrm{vol}(\mathrm{ord}_E) = L^n$. Then

$$\widehat{\mathrm{vol}}(\mathcal{C}, o) = A_{\mathcal{C}}(\mathrm{ord}_E)^{n+1} \cdot \mathrm{vol}(v) = r^{n+1} \cdot L^n = r(-K_X)^n.$$

Let $i(X) \coloneqq \max\{r \in \mathbb{N}^* \mid -K_X \sim_{\mathbb{Q}} rL \text{ for some ample line bundle } L\}$ be the Fano index of X. Then the above argument shows that $\widehat{\mathrm{vol}}(\mathcal{C}, o) \leqslant i(X) \cdot (-K_X)^n$. We know that for smooth Fano manifold, the Fano index always satisfies $i(X) \leqslant n+1$ and $i(X) = n+1$ if and only if $X \cong \mathbb{P}^n$ and $i(X) = n$ if and only if $X \cong Q \subset \mathbb{P}^{n+1}$ is a smooth quadric hypersurface ([KO73]). If $X \cong \mathbb{P}^n$ and $r = i(\mathbb{P}^n) = n+1$, then $(\mathcal{C}(\mathbb{P}^n, \mathcal{O}(1)), o) \cong (\mathbb{C}^n, 0)$ is a smooth point, which is excluded in the ODP conjecture. If $r < i(\mathbb{P}^n) = n+1$, then there exists some $p \in \mathbb{Z}_{>1}$ such that $rp = i(\mathbb{P}^n) = n+1$. Then $\widehat{\mathrm{vol}}(\mathcal{C}(\mathbb{P}^n, L), o) = r(-K_{\mathbb{P}^n})^n = (n+1)^{n+1}/p < 2n^{n+1}$. So we can assume $X \ncong \mathbb{P}^n$. Then by Theorem 1.1, we have $(-K_X)^n \leqslant 2n^n$ and the equality holds if $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ or X is a smooth quadric hypersurface Q. Combined with Kobayashi-Ochiai's result $i(X) \leqslant n$, we get

$$\widehat{\mathrm{vol}}(\mathcal{C}, o) \leqslant i(X) \cdot (-K_X)^n \leqslant 2n^{n+1}.$$

The equality holds if and only if $i(X) = n$ and $(-K_X)^n = 2n^n$, which implies $(\mathcal{C}, o) \cong (\mathcal{C}(Q, \mathcal{O}_Q(1)), o)$ is the ODP singularity (the case when $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ is excluded since the Fano index $i(\mathbb{P}^1 \times \mathbb{P}^{n-1}) = 2$). □

4.2. Examples and questions.

Example 4.6. *Our consideration of weighted blowup is motivated by a special example of a weighted blowup over the quadric hypersurface: $X = Q^n = \{Z_0Z_1 + Z_2Z_3 + Z_4^2 + \cdots + Z_{n+1}^2 = 0\} \subset \mathbb{P}^{n+1}$ for $n \geq 3$. Fix a minimal rational curve which is the line*

$$C \cong \mathbb{P}^1 = \{Z_1 = Z_3 = Z_4 = \cdots = 0\} = \{[t : 0 : s : 0 : 0 : \cdots : 0] \colon [t, s] \in \mathbb{P}^1\} \subset \mathbb{P}^{n+1}.$$

On the affine chart $U_0 = \{Z_0 \neq 0\}$, set $u_i = z_i/z_0$ with $i \neq 0$. Then the equations of X and C become:

$$\begin{aligned} X: &\quad u_1 + u_2u_3 + u_4^2 + \cdots + u_{n+1}^2 = 0 \Longrightarrow u_1 = -u_2u_3 - u_4^2 - \cdots - u_{n+2}^2, \\ C: &\quad u_3 = u_4 = \cdots = u_{n+1} = 0. \end{aligned}$$

In particular, we choose coordinates $\mathbf{u} = \{u_2, u_3, u_4, \cdots, u_{n+1}\}$ as local coordinates on X. Similarly, on the affine chart $U_1 = \{Z_2 \neq 0\}$, if we set $v_j = z_j/z_2$ with $j \neq 2$. The local coordinate is given by $\mathbf{v} = (v_0, v_1, v_4 \cdots, v_{n+1})$. The equations of X, C are given by:

$$\begin{aligned} X: &\quad v_0v_1 + v_3 + v_4^2 + \cdots + v_{n+1}^2 = 0 \Longrightarrow v_3 = -v_0v_1 - v_4^2 - \cdots - v_{n+1}^2, \\ C: &\quad v_1 = v_4 = \cdots = v_{n+1} = 0. \end{aligned}$$

The conormal bundle of $C \setminus \{x\}$ is generated by

$$\{dv_1 = -du_3, dv_4 = u_2^{-1}du_4, \cdots, dv_{n+1} = u_2^{-1}du_{n+1}\}$$

which shows in particular that $N_{C/X}^{\vee} = \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus n-2}$.

Consider the $\mathbb{C}^*$ action on $\mathbb{P}^{n+1}$ with weights $(0, 2, 0, 2, \underbrace{1, \ldots, 1}_{n-3})$ on the homogeneous coordinates. The $\mathbb{C}^*$ action preserves the quadric hypersurface and gives rise to a divisorial valuation whose center on X is the line C. This divisorial valuation can also be obtained via a weighted blowup along the line C. Precisely, in the $\mathbf{u}$-coordinates, we perform the weighted blow along $C \cap U_0 = \{u_3 = u_4 = \cdots = u_{n+1} = 0\}$ with weights $(0, 2, 1, \ldots, 1)$. In the $\mathbf{v}$ coordinates, we also have the weights $(0, 2, 1, \ldots, 1)$. The weighted blowup $\mu : \hat{X} \to X$ in this example is globally defined and the exceptional divisor E is isomorphis to a weighted projective $\mathbb{P}(2, \underbrace{1, \ldots, 1}_{n-2})$-bundle over $\mathbb{P}^1$. We can calculate the volume function $\mathrm{vol}(-K_X - xE) = n^n \mathrm{vol}(H - \xi E)$ where $\xi = \frac{x}{n}$. Interestingly the situation turns out to be different for $n = 3$ and $n \geq 4$. When $n = 3$, the nef threshold (Seshadri constant) of H with respect to E coincides with the pseudo-effective threshold and is equal to 2. We have the expression:

$$\mathrm{vol}(-K_X - xE) = 27 \cdot \mathrm{vol}(H - \xi E) = 27 \cdot \left(2 - \frac{3}{2}\xi + \frac{1}{2}\xi^2\right), \quad 0 \leq \xi \leq 2.$$

When $n \geq 4$, the nef threshold of H with respect to E is 1 while the pseudoeffective threshold of H with respect to E is 2. The volume function consists of two smooth pieces:

$$\begin{aligned} \mathrm{vol}(-K_X - xE) &= n^n \mathrm{vol}(H - \xi E) \\ &= n^n \cdot \begin{cases} 2 - \frac{n}{2}\xi^{n-1} + \frac{n-2}{2}\xi^n, & \xi = \frac{x}{n} \in [0, 1] \\ \frac{n}{2}(2 - \xi)^{n-1} - \frac{n-2}{2}(2 - \xi)^n, & \xi = \frac{x}{n} \in [1, 2]. \end{cases} \end{aligned} \tag{21}$$

In fact, this expression holds for any $n \geq 3$, since when $n = 3$ the second piece and the first piece connect smoothly and become one whole smooth piece. To get the above formula, we can first *calculate the volume function before the nef threshold by calculating the intersection:*

$$\begin{aligned}(\mu^*H-\xi E)^n &= \sum_{k=0}^{n}\binom{n}{k}\mu^*H^{n-k}\cdot\xi^k(-E)^k\\ &= H^n+n\xi^{n-1}\pi^*H\cdot\xi^{n-1}(-E)^{n-1}+\xi^n(-E)^n\\ &= 2-\frac{n}{2}\xi^{n-1}+\xi^n\frac{n-2}{2}\end{aligned}$$

where we used a weighted analogue of the usual intersection formula via Segre classes for projective bundles. It is not surprising that the expression of this piece coincides with the piece in the expression of (20) *when $x=n\xi\le n$. The second piece is more difficult to get. One way to get it is to use the fact that the measure $-\frac{1}{n!}\mathrm{dvol}(H-tE)$ is the Duistermaat-Heckman measure of the previously mentioned Hamiltonian S^1 action (see* [BHJ17]*) and then use the symmetry of its density function around $\xi=1$ in our case. We can also directly calculate the Duistermaat-Heckman measure of S^1-action by using a well-established localization/Fourier formula (*[BGV92, 7.4]*) and for the interest of the reader we will provide such a derivation in appendix.*

Example 4.7. *Let $X\subset\mathbb{P}^{n+1}$ be a Fano hypersurface of degree $1\le b\le n$. Then the minimal rational curve is of splitting type $\mathcal{O}(1)^{n-b}\oplus\mathcal{O}^{b-1}$ so that $d=n-b+2$ (see* [Kol96, Exercise V.4.4]*). The volume of X is given by $V_d=d^n(n-d+2)=(n+2-b)^n\cdot b$. One can easily verify that V_d is strictly smaller than* $\mathrm{vol}(\mathbb{P}^{d-1}\times\mathbb{P}^{n-d+1})$ *except when $b=1$ or $b=2$.*

Question 4.8. *Is there a K-semistable Fano manifold X with $\mathrm{vol}(X)=\mathrm{vol}(\mathbb{P}^{d-1}\times\mathbb{P}^{n-d+1})$ that contains a minimal rational curve of degree $3\le d\le n-1$ but is not isomorphic to $\mathbb{P}^{d-1}\times\mathbb{P}^{n-d+1}$?*

Our results imply that when $d=n$, there is just one such X, i.e. the quadric hypersurface Q^n, while when $d=2$, there is no such X. Moreover, such an X must satisfy $l_X=d$ (see (7)).

We end this paper by finding the minimal possible anti-canonical volume of n-dimensional (K-semistable) Fano manifolds. It is not a well-posed problem for general $\mathbb{Q}$-Fano varieties. Even assuming K-semistablity, there exists a sequence of K-semistable $\mathbb{Q}$-Fano varieties with volume $(-K_X)^n$ tends to zero (See [Jia20, Example 1.4(2)]), so it is in general not possible to determine an optimal positive lower bound. When we restrict to n-dimensional smooth Fano manifold X, the volume $(-K_X)^n$ is always a positive integer, so a priori $(-K_X)^n\geqslant 1$.

Example 4.9. *Suppose $n\geqslant 2$ is an integer. Assume $a_0\leqslant a_1\leqslant\cdots\leqslant a_{n+1}$ are integers and we consider a general degree d hypersurface X_d of weighted projective space $\mathbb{P}(a_0,a_1\cdots,a_{n+1})$. Note that X_d is a Fano variety if $d<\sum_{i=0}^{n+1}a_i$. By the adjunction formula*

$$(-K_X)^n=\frac{d\cdot(\sum_{i=0}^{n+1}a_i-d)^n}{\prod_{i=0}^{n+1}a_i}.$$

If $a_0=\cdots=a_{n-1}=1$, $a_n=2$, $a_{n+1}=n+1$ and $d=2n+2$. Namely, X_d is a general degree $2n+2$ hypersurface in $\mathbb{P}(1^n,2,n+1)$. Then $(-K_{X_{2n+2}})^n=1$. By [IF00, Thm 8.1]*, we know that X_{2n+2} is smooth when n is even. By* [ST24, Thm 1.3]*, we know $\delta(X_{2n+2})\geqslant(n+1)/2>1$, so X_{2n+2} is a smooth K-stable Fano manifold with minimal anti-canonical volume $(-K_{X_{2n+2}})^n=1$ for all even n.*

When n is odd, the general hypersurface $X_{2n+2} \subset \mathbb{P}(1^n, 2, n+1)$ is singular. (After communicating with K. Fujita, we find that this type of example already appears in [KSC04, Theorem 5.22].*)*

Example 4.10. *Let Y be the double cover of $\mathbb{P}^n$ ramified along a degree $2n$ smooth hypersurface D. Then by Hurwitz's formula, $K_Y = \phi^*(K_{\mathbb{P}^n} + \frac{1}{2}D)$. So $(-K_Y)^n = \deg(\phi) \cdot (-K_{\mathbb{P}^n} - \frac{1}{2}D)^n = 2$. By the result of* [LZ22, Zhu21]*, we know K-stability of Y is equivalent to the K-stability of the log Fano pair $(\mathbb{P}^n, \frac{1}{2}D)$. Since we assume D is a smooth hypersurface, then the log Fano pair $(\mathbb{P}^n, (\frac{n+1}{2n} - \varepsilon)D)$ is klt, so it is uniformly K-stable by* [ADL23, Theorem 2.10] *for $0 < \varepsilon \ll 1$. Since $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$, by the interpolation of K-stability (*[ADL24, Prop 2.13]*), we conclude that $(\mathbb{P}^n, \frac{1}{2}D)$ is K-stable. Thus, Y is a smooth K-stable Fano manifold with volume $(-K_Y)^n = 2$ for all dimensions $n \geqslant 2$. Note that Y contains a degree 2 free-immersed rational curve with $n-1$ nodes (see* [Kol96, IV.2.12.3]*).*

We indeed have the following:

Proposition 4.11. *The minimum of anticanonical volumes of n-dimensional Fano manifolds is equal to 1 if n is even and is equal to 2 if n is odd. The minimum volume is obtained by K-stable Fano manifolds.*

This follows from Example 4.9, Example 4.10 and following interesting fact pointed out to us by K. Fujita:

Fact 1. *(*[KSC04, Solution to Exercise 5.23, Page 217]*) Let X be a smooth projective variety of odd dimension n. Then the self-intersection number $(-K_X)^n$ is even.*

Question 4.12. *Classify all Fano manifolds that obtain the minimal volume.*

Appendix: A proof of the formula (21)

We calculate the Duistermaat-Heckman measure via the localization method. Let $\omega = \omega_{\mathrm{FS}}$ denote the restriction of the Fubini-Study metric of $\mathbb{P}^{n+1}$ on $X = Q^n = \{Z_0Z_1 + Z_2Z_3 + Z_4^2 + \cdots + Z_{n+1}^2 = 0\} \subset \mathbb{P}^{n+1}$ for $n \geq 3$. The $\mathbb{C}^*$ action on X is given by: for any $\lambda \in \mathbb{C}^*$,

$$\lambda \circ (Z_0, Z_1, Z_2, Z_3, Z_4, \ldots, Z_{n+1}) = (Z_0, \lambda^2 Z_1, Z_2, \lambda^2 Z_3, \lambda Z_4, \ldots, \lambda Z_{n+1}).$$

The corresponding S^1 action is a Hamiltonian with moment map given by:

$$\mu([Z]) = \frac{2|Z_1|^2 + 2|Z_3|^2 + |Z_4|^2 + \cdots + |Z_{n+1}|^2}{|Z_0|^2 + |Z_1|^2 + \cdots + |Z_{n+1}|^2}.$$

We introduce the equivariant volume functional:

$$V(\theta) = \int_X e^{i\mu(z)\xi} \frac{\omega^n}{n!} = \int_{\mathbb{R}} e^{i\theta\xi}\, \mathrm{DH}(\xi)$$

where $\mathrm{DH}(\xi) = \mu_* \frac{\omega^n}{n!}$ is the Duistermaat-Heckman measure. As a consequence, the density function of $\mathrm{DH}(\xi)$ is the Fourier transform of $V(\theta)$.

The localization formula calculates the integral $V(\theta)$ via data on the fixed points sets (including the values of moment map and equivariant Euler curvature forms of normal bundles) which consists of three components in this case:

$$\begin{aligned} F_0 &= \mu^{-1}(0) \cong \mathbb{P}^1 : [*, 0, *, 0, \dots, 0], \\ F_1 &= \mu^{-1}(2) \cong \mathbb{P}^1 : [0, *, 0, *, 0, \dots, 0], \\ F_2 &= \mu^{-1}(1) \cong Q^{n-4} : [0, 0, 0, 0, Z_4, \dots, Z_{n+1}] \text{ with } Z_4^2 + \cdots Z_{n+1}^2 = 0. \end{aligned}$$

The contribution from F_0 is given by:

$$V_0(\theta) := \int_{\mathbb{P}^1} \frac{e^{\omega}}{((\omega - i\theta))^{n-2}(-i2\theta)} = \frac{(-1)^{n+1}2\pi}{2}((i\theta)^{1-n} + (n-2)(i\theta)^{-n}).$$

We have the following useful formula: for any $a \in \mathbb{R}$,

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ia\theta\xi}}{(i\theta)^k} = \frac{(a-\xi)^{k-1}}{2(k-1)!}\mathrm{Sign}(a-\xi).$$

So we can calculate the Fourier transform of $V_0(\theta)$:

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}} V_0(\theta)e^{-i\theta\xi}\mathrm{d}\theta &= \frac{1}{4(n-1)!}((n-2)\xi^{n-1} - (n-1)\xi^{n-2})\mathrm{Sign}(-\xi) \\ &=: \frac{f(\xi)}{2}\mathrm{Sign}(-\xi). \end{aligned}$$

The contribution from F_2 is given by:

$$V_2(\theta) := \int_{\mathbb{P}^1} \frac{e^{(\omega+2i\theta)}}{((\omega + i\theta))^{n-2}(i2\theta)} = \frac{2\pi}{2}e^{2i\theta}((i\theta)^{1-n} + (2-n)(i\theta)^{-n}).$$

Similar to above, the Fourier transform of $V_2(\theta)$ is equal to:

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}} V_2(\theta)e^{-i\theta\xi}\mathrm{d}\theta &= \frac{1}{4(n-1)!}(-(n-2)(2-\xi)^{n-1} + (n-1)(2-\xi)^{n-2})\mathrm{Sign}(2-\xi) \\ &=: \frac{g(\xi)}{2}\mathrm{Sign}(2-\xi). \end{aligned}$$

The contribution from F_1 is given by:

$$\begin{aligned} V_1(\theta) &:= \int_{Q^{n-4}} \frac{e^{(\omega+i\theta)}}{(\omega+i\theta)^2(\omega-i\theta)^2} = \int_{Q^{n-4}} \frac{e^{\omega}e^{i\theta}}{(\omega^2+\theta^2)^2} \\ &= \sum_{k=0}^{\lfloor \frac{n-4}{2} \rfloor} \frac{(k+1)}{(n-4-2k)!}\frac{e^{i\theta}}{(i\theta)^{2k+4}}. \end{aligned}$$

The Fourier transform of $V_1(\theta)$ is equal to:

$$\frac{1}{2\pi}\int_{\mathbb{R}} V_1(\theta)e^{-i\theta\xi}\mathrm{d}\theta = \sum_{k=0}^{\lfloor \frac{n-4}{2} \rfloor} \frac{(k+1)}{2(n-4-2k)!(2k+3)!}(1-\xi)^{2k+3}\mathrm{Sign}(1-\xi).$$

We claim that this is equal to $-\frac{f(\xi)+g(\xi)}{2}\mathrm{Sign}(1-\xi)$ so that the Fourier transform is $V(\theta)$ is equal to:

$$\begin{aligned}\rho(\xi) &= \frac{f(\xi)}{2}\mathrm{Sign}(-\xi) + \frac{g(\xi)}{2}\mathrm{Sign}(2-\xi) - \frac{f(\xi)+g(\xi)}{2}\mathrm{Sign}(1-\xi) \\ &= \begin{cases} -f(\xi) & 0 \le \xi \le 1 \\ g(\xi) & 1 \le \xi \le 2 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

It is immediate to check that $\rho(\xi) = -\frac{1}{n!}\frac{d}{d\xi}\mathrm{vol}(H-\xi E)$ by using the expression from (21), which verifies the formula (21) after integration. To verify the claim, we calculate:

$$\begin{aligned}\xi^{n-1} - (2-\xi)^{n-1} &= (1-(1-\xi))^{n-1} - (1+(1-\xi))^{n-1} = \sum_{k=0}^{n-1}\binom{n-1}{k}((-1)^k - 1)(1-\xi)^k \\ &= -\sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{2j+1}(1-\xi)^{2j+1},\end{aligned}$$

and finally verify the claimed identity:

$$\begin{aligned}-2(n-1)!(f(\xi)+g(\xi)) &= (n-2)(\xi^{n-1} - (2-\xi)^{n-1}) - (n-1)(\xi^{n-2} - (2-\xi)^{n-2}), \\ &= \sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}(n-2)\binom{n-1}{2j+1}(1-\xi)^{2j+1} - \sum_{j=0}^{\lfloor\frac{n-3}{2}\rfloor}(n-1)\binom{n-2}{2j+1}(1-\xi)^{2j+1} \\ &= \sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\frac{(n-1)!2j}{(2j+1)!(n-2-2j)!}(1-\xi)^{2j+1} \\ &= 2(n-1)!\sum_{k=0}^{\lfloor\frac{n-4}{2}\rfloor}\frac{k+1}{(n-4-2k)!(2k+3)!}(1-\xi)^{2k+3}.\end{aligned}$$

REFERENCES

[AB24] Rolf Andreasson and Robert J. Berman. Sharp bounds on the height of K-semistable Fano varieties I, the toric case. *Compos. Math.*, 160(10):2366–2406, 2024.

[ADL23] Kenneth Ascher, Kristin DeVleming, and Yuchen Liu. K-stability and birational models of moduli of quartic K3 surfaces. *Invent. Math.*, 232(2):471–552, 2023.

[ADL24] Kenneth Ascher, Kristin DeVleming, and Yuchen Liu. Wall crossing for K-moduli spaces of plane curves. *Proc. Lond. Math. Soc. (3)*, 128(6):Paper No. e12615, 113, 2024.

[AIM20] AIM. AIM problem list: K-stability and related topics, 2020. Available at http://aimpl.org/kstability.

[Ara06] Carolina Araujo. Rational curves of minimal degree and characterizations of projective spaces. *Math. Ann.*, (335):937–951, 2006.

[AZ22] Hamid Abban and Ziquan Zhuang. K-stability of Fano varieties via admissible flags. *Forum Math. Pi*, 10:Paper No. e15, 43, 2022.

[BB13] Robert J. Berman and Bo Berndtsson. Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. *Ann. Fac. Sci. Toulouse Math. (6)*, 22(4):649–711, 2013.

[BB17] Robert J. Berman and Bo Berndtsson. The volume of Kähler-Einstein Fano varieties and convex bodies. *J. Reine Angew. Math.*, 723:127–152, 2017.

[BGV92] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren Math. Wiss.* Springer-Verlag, Berlin, 1992.

[BHJ17] Sébastien Boucksom, Tomoyuki Hisamoto, and Mattias Jonsson. Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs. *Ann. Inst. Fourier (Grenoble)*, 67(2):743–841, 2017.

[BJ20] Harold Blum and Mattias Jonsson. Thresholds, valuations, and K-stability. *Adv. Math.*, 365:107062, 57, 2020.

[CD15] Cinzia Casagrande and Stéphane Druel. Locally unsplit families of rational curves of large anticanonical degree on Fano manifolds. *Int. Math. Res. Not. IMRN*, (21):10756–10800, 2015.

[CFH14] Yifei Chen, Baohua Fu, and Jun-Muk Hwang. Minimal rational curves on complete toric manifolds and applications. *Proc. Edinb. Math. Soc. (2)*, 57(1):111–123, 2014.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[CMSB02] Koji Cho, Yoichi Miyaoka, and N.I. Shepherd-Barron. Characterizations of projective space and applications to complex symplectic manifolds. In *Heigher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math., 25*, pages 1–88. Mathematical Society of Japan, 2002.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.

[dFKL07] Tommaso de Fernex, Alex Küronya, and Robert Lazarsfeld. Higher cohomology of divisors on a projective variety. *Math. Ann.*, 337(2):443–455, 2007.

[DH17] Thomas Dedieu and Andreas Höring. Numerical characterisation of quadrics. *Algebraic Geometry*, 4(1):120–135, 2017.

[Don02] S. K. Donaldson. Scalar curvature and stability of toric varieties. *J. Differential Geom.*, 62(2):289–349, 2002.

[FO18] Kento Fujita and Yuji Odaka. On the K-stability of Fano varieties and anticanonical divisors. *Tohoku Math. J. (2)*, 70(4):511–521, 2018.

[Fuj18] Kento Fujita. Optimal bounds for the volumes of Kähler-Einstein Fano manifolds. *Amer. J. Math.*, 140(2):391–414, 2018.

[Fuj19] Kento Fujita. A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties. *J. Reine Angew. Math.*, 751:309–338, 2019.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HM03] Jun-Muk Hwang and Ngaiming Mok. Finite morphisms onto Fano manifolds of Picard number 1 which have rational curves with trivial normal bundles. *J. Algebraic Geom.*, 12(4):627–651, 2003.

[IF00] A. R. Iano-Fletcher. Working with weighted complete intersections. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 101–173. Cambridge Univ. Press, Cambridge, 2000.

[Jia20] Chen Jiang. Boundedness of $\mathbb{Q}$-Fano varieties with degrees and alpha-invariants bounded from below. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(5):1235–1248, 2020.

[JM12] Mattias Jonsson and Mircea Mustata. Valuations and asymptotic invariants for sequences of ideals. *Ann. Inst. Fourier (Grenoble)*, 62(6):2145–2209, 2012.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998.

[KO73] Shoshichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1996.

[Kol13] János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

[KSC04] János Kollár, Karen E. Smith, and Alessio Corti. *Rational and nearly rational varieties*, volume 92 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Li17] Chi Li. K-semistability is equivariant volume minimization. *Duke Math. J.*, 166(16):3147–3218, 2017.

[Li18] Chi Li. Minimizing normalized volumes of valuations. *Math. Z.*, 289(1-2):491–513, 2018.

[Liu18] Yuchen Liu. The volume of singular Kähler-Einstein Fano varieties. *Compos. Math.*, 154(6):1131–1158, 2018.

[LL19] Chi Li and Yuchen Liu. Kähler-Einstein metrics and volume minimization. *Adv. Math.*, 341:440–492, 2019.

[LX19] Yuchen Liu and Chenyang Xu. K-stability of cubic threefolds. *Duke Math. J.*, 168(11):2029–2073, 2019.

[LZ18] Yuchen Liu and Ziquan Zhuang. Characterization of projective spaces by Seshadri constants. *Math. Z.*, 289(1-2):25–38, 2018.

[LZ22] Yuchen Liu and Ziwen Zhu. Equivariant K-stability under finite group action. *Internat. J. Math.*, 33(1):Paper No. 2250007, 21, 2022.

[Miy04] Y. Miyaoka. Numerical characterisations of hyperquadrics. In *Complex Analysis in Several Variables-Memorial Conference of Kiyoshi Oka's Centennial Birthday*, volume 42 of *Adv. Stud. Pure Math.*, pages 209–235. Math. Soc. Japan, Tokyo, 2004.

[Mor79] Shigefumi Mori. Projective manifolds with ample tangent bundles. *Ann. of Math. (2)*, 110(3):593–606, 1979.

[MS24] Joaquín Moraga and Hendrik Süß. Bounding toric singularities with normalized volume. *Bull. Lond. Math. Soc.*, 56(6):2212–2229, 2024.

[Obr07] Mikkel Obro. An algorithm for the classification of smooth Fano polytopes. arXiv:0704.0049, 2007. Database at http://www.grdb.co.uk/forms/toricsmooth.

[QR12] M.H. Quek and D. Rydh. Weighted blow-ups. 2012. in preparation.

[SS17] Cristiano Spotti and Song Sun. Explicit Gromov-Hausdorff compactifications of moduli spaces of Kähler-Einstein Fano manifolds. *Pure Appl. Math. Q.*, 13(3):477–515, 2017.

[ST24] Taro Sano and Luca Tasin. Delta invariants of weighted hypersurfaces. *arXiv preprint arXiv:2408.03057*, 2024.

[SZ12] Yalong Shi and Xiaohua Zhu. Kähler-Ricci solitons on toric Fano orbifolds. *Math. Z.*, 271(3-4):1241–1251, 2012.

[Tia97] Gang Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*, 130(1):1–37, 1997.

[WZ04] Xu-Jia Wang and Xiaohua Zhu. Kähler-Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.*, 188(1):87–103, 2004.

[Xu24] Chenyang Xu. K-stability of Fano varieties, 2024. Available at https://web.math.princeton.edu/∼chenyang/Kstabilitybook.pdf.

[Zhu21] Ziquan Zhuang. Optimal destabilizing centers and equivariant K-stability. *Invent. Math.*, 226(1):195–223, 2021.

[Zhu25] Ziquan Zhuang. Stability of klt singularities. In *Handbook of geometry and topology of singularities VII*, pages 501–551. Springer, Cham, [2025] ©2025.

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