Topics on Kähler-Einstein Metrics

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Riemann surface: surface with a complex structure:

Topology	Metric	Curvature
$\mathbb{S}^2 = \mathbb{CP}^1$	spherical	1
$\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$	flat	0
$\Sigma_{\mathfrak{g}} = \mathbb{B}^1/\pi_1(\Sigma_{\mathfrak{g}})$	hyperbolic	-1

Riemannian metric: $g = E|dz|^2 = \frac{\partial^2\varphi}{\partial z \partial \bar{z}}|dz|^2$.

$$\begin{aligned} & \text{Constant Gauss/Ricci curvature equation} \\ = \ & \text{1-dimensional complex Monge-Ampère equation} \end{aligned}$$

$$Ric(g) = \lambda g \Longleftrightarrow \Delta \log E = -\lambda E \Longleftrightarrow \varphi_{z\bar{z}} = e^{-\lambda\varphi}.$$

Kähler manifolds and Kähler metrics

X: complex manifold; J: $TX \to TX$ integrable complex structure; g: Kähler metric, $g(J\cdot, J\cdot) = g(\cdot,\cdot)$ and $d\omega = 0$.

$$\omega = g(\cdot, J\cdot) = \sqrt{-1}\sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad (g_{i\bar{j}}) > 0.$$

Kähler class $[\omega] \in H^2(X, \mathbb{R})$.

Fact ($\partial\bar{\partial}$-Lemma): any $\omega' \in [\omega]$ is of the form

$$\omega_u := \omega + \sqrt{-1}\partial\bar{\partial}u = \sqrt{-1}\sum_{i,j}\left(g_{i\bar{j}} + u_{i\bar{j}}\right) dz^i \wedge d\bar{z}^j.$$

Kähler metric as curvature forms

$L \to X$: a $\mathbb{C}$-line bundle with holomorphic transition $\{f_{\alpha\beta}\}$.
$e^{-\varphi} := \{e^{-\varphi_\alpha}\}$ Hermitian metric on L:

$$e^{-\varphi_\alpha} = |f_{\alpha\beta}|^2 e^{-\varphi_\beta}. \tag{1}$$

Definition: L is positive (=ample) if $\exists$ $e^{-\varphi} = \{e^{-\varphi_\alpha}\}$ on L s.t.

$$\omega + \sqrt{-1}\partial\bar{\partial}u = \sqrt{-1}\partial\bar{\partial}\varphi := \sqrt{-1}\partial\bar{\partial}\varphi_\alpha > 0. \tag{2}$$

Anticanonical line bundle: $-K_X = \wedge^n T_{\mathrm{hol}}X$, $K_X = \wedge^n T^*_{\mathrm{hol}}X$.
Fact: {smooth volume forms}={Hermitian metrics on $-K_X$ }

$$\begin{aligned} 2\pi c_1(X) \ni Ric(\omega) &= -\sqrt{-1}\partial\bar{\partial}\log\omega^n \\ &= -\sum_{i,j} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log\det(g_{k\bar{l}}) dz^i \wedge d\bar{z}^j. \end{aligned}$$

Kähler-Einstein metric and Monge-Ampère equation

KE equation:

$$Ric(\omega_u) = \lambda\, \omega_u \iff (\omega + \sqrt{-1}\partial\bar{\partial}u)^n = Fe^{-\lambda u}\omega^n \tag{3}$$

$$\iff \det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) = Fe^{-\lambda u}\det(g_{i\bar{j}}).$$

$\lambda = -1$	Solvable (Aubin, Yau)	$c_1(X) < 0$
$\lambda = 0$	Solvable (Yau)	$c_1(X) = 0$
$\lambda = 1$	$\exists$ obstructions	$c_1(X) > 0$

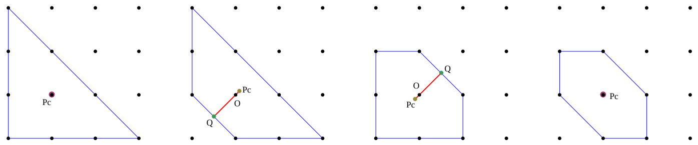
Fano manifolds

X Fano: $c_1(X) > 0 \iff \exists$ Kähler metric ω with $Ric(\omega) > 0$.

1. $\dim_{\mathbb{C}} = 1$: $\mathbb{P}^1 = S^2$.
2. $\dim_{\mathbb{C}} = 2$: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2 \sharp k \overline{\mathbb{P}^2}$, $1 \leq k \leq 8$ (del Pezzo).
3. $\dim_{\mathbb{C}} = 3$: 105 deformation families (Iskovskikh, Mori-Mukai)
4. Hypersurface in $\mathbb{P}^{n+1}$ of degree $\leq n+1$;
5. Toric Fano manifolds

Fact: there are finitely many deformation family in each dimension (Campana, Kollár-Miyaoka-Mori, Nadel '90).

Toric manifolds $\leftrightarrow$ lattice polytopes. Fano $\leftrightarrow$ reflexive polytope.



(a) $\mathbb{P}^2$ (b) $\mathbb{P}^2\sharp\overline{\mathbb{P}^2}$ (c) $\mathbb{P}^2\sharp 2\overline{\mathbb{P}^2}$ (d) $\mathbb{P}^2\sharp 3\overline{\mathbb{P}^2}$

Set $\beta(X) = \sup\{t; \exists\, \omega \in 2\pi c_1(X) \text{ s.t. } Ric(\omega) > t\omega\} \in (0,1]$.

Fact: $\mathrm{KE} \Longrightarrow \beta(X) = 1$.

Theorem (Li '09)

If $P_c \neq O$, then $\beta(X_\triangle) = \left|\overline{OQ}\right| / \left|\overline{P_cQ}\right|$, where $Q = \overrightarrow{P_cO} \cap \partial\triangle$.

Example: $\beta(\mathbb{P}^2\sharp\overline{\mathbb{P}^2}) = 6/7$, $\beta(\mathbb{P}^2\sharp 2\overline{\mathbb{P}^2}) = 21/25$.

$\mathbb{Q}$-Fano varieties: building blocks of algebraic varieties

Definition

$\mathbb{Q}$-Fano variety X is a normal projective variety satisfying:

1. *Fano: $\mathbb{Q}$-line bundle $-K_X$ is ample.*
2. *klt (Kawamata log terminal): $\forall s_\alpha^* \sim dz^1 \wedge \dots dz^n \in \mathcal{O}_{K_X}(U_\alpha)$*

$$\int_{U^{\mathrm{reg}}} (\sqrt{-1}^{n^2} s_\alpha^* \wedge \bar{s}_\alpha^*) < +\infty. \quad (4)$$

Let $\mu : Y \to X$ by a resolution of singularities (Hironaka)

$$K_Y = \mu^* K_X + \sum_i (A(E_i) - 1) E_i. \quad (5)$$

The condition (4) $\Longleftrightarrow \mathrm{mld} := \min_i A(E_i) > 0$.

Fact: (Birkar '16) ϵ-klt (i.e. $\mathrm{mld} \geq \epsilon > 0$) Fanos are bounded.

KE equation on $\mathbb{Q}$-Fano varieties

Hermitian metric on the $\mathbb{Q}$-line bundle $-K_X$: $e^{-\varphi} = \{e^{-\varphi_\alpha}\}$ s.t.

$$e^{-\varphi_\alpha} = |s_\alpha|^2 e^{-\varphi}, \quad \{s_\alpha\} \text{ trivializing sections of } -K_X$$

Kähler-Einstein equation on Fano varieties:

$$\boxed{(\sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-\varphi}} := |s_\alpha|^2 e^{-\varphi} \left(\sqrt{-1}^{n^2} s_\alpha^* \wedge \bar{s}_\alpha^*\right). \tag{6}$$

$\omega = \sqrt{-1}\partial\bar{\partial}\psi \Rightarrow u = \varphi - \psi$ is globally defined. Then (6)$\Longleftrightarrow$ (3).

(weak) KE potential: generalized solutions in pluripotential sense.

Fact: KE potential is unique up to automorphism (Berndtsson) and is smooth on X^{reg} (Berman-Boucksom-Eddyssieux-Guedj-Zeriahi)

Obstructions to KEs on Fano varieties

1. KE $\Longrightarrow \mathrm{Aut}(X)$ is reductive: $\mathrm{Aut}(X)_0$ is the complexification of a compact Lie group (Matsushima, BBEGZ, CDS).
2. Futaki invariant: $\forall$ holomorphic vector field v , $\exists$ canonical Hamiltonian function θ_v,

$$\exists\,\mathrm{KE} \quad \Longrightarrow \quad \mathrm{Fut(v)} := \int_{\mathrm{X}} \theta_{\mathrm{v}} \omega^{\mathrm{n}} = 0. \qquad (7)$$

3. Energy coerciveness (Tian, Tian-Zhu, Phong-Song-Sturm-Weinkove, Darvas-Rubinstein, Hisamoto)
4. K-stability (Tian, Donaldson)
   Ding stability (Berman, Boucksom-Jonsson)
   all equivalent (**Li**-Xu '12, Berman-Boucksom-Jonsson, Fujita).

Main result: Generalized Yau-Tian-Donaldson conjecture

Theorem (Li-Tian-Wang, Li '19)

A $\mathbb{Q}$-Fano variety X has a KE potential if (and only if) X is $\mathrm{Aut}(X)_0$-uniformly K/Ding-stable.

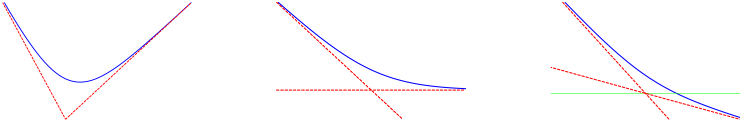
1. X Smooth (Chen-Donaldson-Sun, Tian, Datar-Székelyhidi).
2. $\mathbb{Q}$-Gorenstein smoothable (Spotti-Sun-Yao, **Li**-Wang-Xu);
3. *Good* (e.g. crepant) resolution of singularities (**Li**-Tian-Wang).

Proofs in above special cases depend on compactness/regularity theory in metric geometry and do NOT generalize to singular case.

4. X smooth & $\mathrm{Aut}(X)$ discrete: Berman-Boucksom-Jonsson (BBJ) in 2015 proposed an approach using pluripotential theory/non-Archimedean analysis. Our work greatly extends their work and removes the two assumptions.

Variational point of view

Consider energy functionals on a pluripotential version of Sobolev space, denoted by $\mathcal{E}^1(X, -K_X)$ (Cegrell, Guedj-Zeriahi)



(e) Proper (f) Bounded (g) Unbounded

There is a distance-like energy:

$$\mathbf{J}(\varphi) = \mathbf{\Lambda}(\varphi) - \mathbf{E}(\varphi) \quad \sim \quad \sup(\varphi - \psi) - \mathbf{E}(\varphi) > 0. \tag{8}$$

E is the primitive of complex Monge-Ampère operator:

$$\delta\mathbf{E}(\delta\varphi) = \int_X (\delta\varphi)(\sqrt{-1}\partial\bar{\partial}\varphi)^n \tag{9}$$

Analytic criterion for KE potentials

Energy functional with KE as critical points:

$$\mathbf{D} = -\mathbf{E} + \mathbf{L} = -\mathbf{E} - \log\left(\int_X e^{-\varphi}\right). \tag{10}$$

The Euler-Lagrangian equation is just the KE equation:

$$\delta\mathbf{D}(\delta\varphi) = \int_X (\delta\varphi)\left(-(\sqrt{-1}\partial\bar{\partial}\varphi)^n + C\cdot e^{-\varphi}\right). \tag{11}$$

Theorem (Darvas-Rubinstein, Darvas, Di-Nezza-Guedj, Hisamoto)

A $\mathbb{Q}$-Fano variety X admits a KE potential if and only if

1. $\mathrm{Aut}(X)_0$ *is reductive* (with center $\mathbb{T} \cong (\mathbb{C}^*)^r$)
2. *there exist $\gamma > 0$ and $C > 0$ s.t.* $\forall \varphi \in \mathcal{E}^1(X, -K_X)^{\mathbb{K}}$,

$$\mathbf{D}(\varphi) \geq \gamma \cdot \inf_{\sigma\in\mathbb{T}} \mathbf{J}(\sigma^*\varphi) - C. \quad \textit{(coercive)}$$

Test of coerciveness along algebraic rays (Tian)

For $k \gg 1$, fix an $\mathrm{Aut}(X)_0$-equivariant Kodaira embedding

$$\iota_k : X \hookrightarrow \mathbb{P}^{N_k - 1} = \mathbb{P}(H^0(X, -kK_X)^*).$$

Pick $\eta \in \mathrm{Mat}_{N_k \times N_k}(\mathbb{C})$, hermitian and commutes with $\mathrm{Aut}(X)_0$.
Set 1-psg: $\sigma_\eta(t) = \exp(-(\log t)\eta)$ and a path:

$$\varphi(t) := \frac{1}{k} \sigma_\eta(t)^* \varphi_{\mathrm{FS}}|_X \in \mathcal{E}^1(-K_X), \quad \Phi = \{\varphi(t); t \in \mathbb{C}^*\}.$$

Slope at infinity:

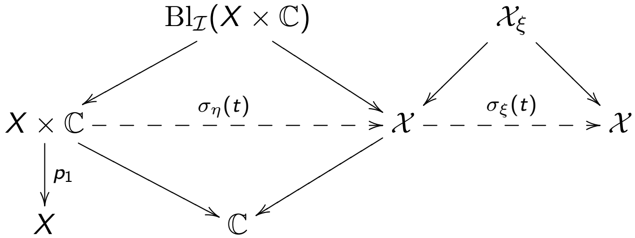
$$\mathbf{D}'^\infty(\Phi) := \lim_{t \to 0} \frac{\mathbf{D}(\varphi(t))}{-\log|t|^2} = -\mathbf{E}'^\infty(\Phi) + \mathbf{L}'^\infty(\Phi),$$

$$\mathbf{J}'^\infty(\Phi) := \lim_{t \to 0} \frac{\mathbf{J}(\varphi(t))}{-\log|t|^2} = \mathbf{\Lambda}'^\infty(\Phi) - \mathbf{E}'^\infty(\Phi).$$

coerciveness (2) $\Longrightarrow$ $\mathbf{D}'^\infty(\Phi) \geq \gamma \cdot \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}'^\infty(\sigma_\xi(t)^* \Phi).$

Algebraic rays=Test configurations=smooth non-Archimedean metrics

Set $\mathcal{X} = \overline{\{(\sigma_\eta(t)(X)), t\}} \subseteq \mathbb{P}^{N_k-1} \times \mathbb{C}, \quad \mathcal{L} = k^{-1}\mathcal{O}_{\mathbb{P}}(1)|_{\mathcal{X}}$.



$$\begin{aligned} \mathbf{E}'^{\infty}(\Phi) &= \frac{\bar{\mathcal{L}}^{\cdot(n+1)}}{n+1} = \mathbf{E}^{\mathrm{NA}}, \quad \mathbf{\Lambda}'^{\infty}(\Phi) = \bar{\mathcal{L}} \cdot p_1^*(-K_X)^{\cdot n} = \mathbf{\Lambda}^{\mathrm{NA}}. \\ \mathbf{J}'^{\infty}(\Phi) &= \mathbf{\Lambda}^{\mathrm{NA}} - \mathbf{E}^{\mathrm{NA}} =: \mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \\ \mathbf{L}'^{\infty}(\Phi) &= \inf_{v \in X^{\mathrm{div}}_{\mathbb{Q}}} (A_X(v) - G(v)(\mathcal{I})) =: \mathbf{L}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}). \end{aligned}$$

$X^{\mathrm{div}}_{\mathbb{Q}}$: space of divisorial valuations; $G(v)$: Gauss extension.

K-stability and Ding-stability

Definition-Theorem (Berman, Hisamoto, Boucksom-Hisamoto-Jonsson)

X KE implies that it is $\mathrm{Aut}(X)_0$-uniformly Ding-stable: $\exists \gamma > 0$ (slope) such that for all $\mathrm{Aut}(X)_0$-equivariant test configurations

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi). \tag{12}$$

Using Minimal Model Program as (**Li**-Xu '12), one can derive:

1. valuative criterions (**Li**, Fujita, Boucksom-Jonsson).
2. algebraically checkable for (singular) Fano surfaces, and Fano varieties with large symmetries (e.g. all toric Fano varieties) **Example:** toric Fano case (12) $\Longleftrightarrow$ $p_c = O \Longleftrightarrow \mathrm{KE}$.

BBJ's proof in case X smooth and $\mathrm{Aut}(X)$ discrete

Assume **D** (and **M**) not coercive.

Step 1: construct a ray Φ in $\mathcal{E}^1(-K_X)$ such that

$$0 \geq \mathbf{D}'^{\infty}(\Phi) = -\mathbf{E}'^{\infty}(\Phi) + \mathbf{L}'^{\infty}(\Phi), \quad \mathbf{E}'^{\infty}(\Phi) = -1.$$

Step 2: $\Phi_m := (\mathrm{Bl}_{\mathcal{J}(m\Phi)}(X \times \mathbb{C}), \mathcal{L}_m = \pi_m^* p_1^*(-K_X) - \frac{1}{m+m_0} E_m)$.
Just need to show that the TC $\Phi_m, m \gg 1$ is destabilising:

Step 3+: Comparison of slopes:

$$\Phi_m \geq \Phi \quad (\Rightarrow \mathbf{E}^{\mathrm{NA}}(\Phi_m) \geq \mathbf{E}'^{\infty}(\Phi), \text{FAILS when } X \text{ is singular!})$$

Contradiction to uniform stability:

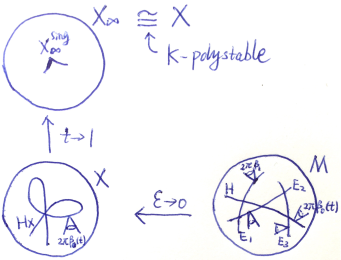
$$\begin{aligned} -1 = \mathbf{E}'^{\infty}(\Phi) &\geq \mathbf{L}'^{\infty}(\Phi) \leftarrow \mathbf{L}^{\mathrm{NA}}(\Phi_m) \\ \geq_{\mathrm{Stability}} \quad (1-\gamma)\mathbf{E}^{\mathrm{NA}}(\Phi_m) &\geq (1-\gamma)\mathbf{E}'^{\infty}(\Phi) = \gamma - 1. \end{aligned}$$

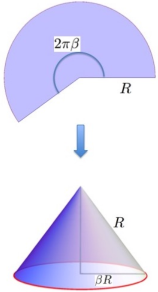
New techniques for singularities I (**Li**-Tian-Wang '17)

Take a resolution $\mu : Y \to X$:

$$-K_Y = \left(\mu^*(-K_X) - \epsilon \sum_i \theta_i E_i\right) + \overbrace{\sum_i (1 - A(E_i) + \epsilon\theta_i) E_i}^{B_\epsilon} . \quad (13)$$

If the cone angle $0 < 2\pi A(E_i) \leq 2\pi$, then we can construct KE metrics on Y with edge cone singularities along E_i and take potential=metric=algebraic limit as $\epsilon \to 0$ (**Li**-Tian-Wang '17).





New techniques for singularities II (**Li**-Tian-Wang, **Li**'19)

1. uniform stability of *in-effective* pair (Y, B_ϵ) with slope $\gamma_\epsilon \to \gamma > 0$ (proved by valuative criterion).
2. perturbed destabilizing geodesic ray Φ_ϵ.
3. blow-up $\mathcal{J}(m\Phi_\epsilon)$ to get: $\phi_{\epsilon,m} := (\mathcal{Y}_{\epsilon,m}, \mathcal{B}_{\epsilon,m}, \mathcal{L}_{\epsilon,m})$ of (Y, B_ϵ).
4. Comparison of slopes

$$\mathbf{E}^{\mathrm{NA}}(\phi_{\epsilon,m}) \geq \mathbf{E}'^{\infty}(\Phi_\epsilon) \quad \text{(true since } Y \text{ is smooth)}$$
$$\lim_{\epsilon\to 0} \mathbf{E}'^{\infty}(\Phi_\epsilon) = \mathbf{E}'^{\infty}(\Phi) \quad \text{(key convergence)}$$
$$\lim_{\epsilon\to 0} \mathbf{L}'^{\infty}(\Phi_\epsilon) = \mathbf{L}'^{\infty}(\Phi) \quad \text{(key convergence)}$$

2-parameters chain of contradiction:

$$\begin{aligned} & -1 = \mathbf{E}'^{\infty}(\Phi) \geq \mathbf{L}'^{\infty}(\Phi) \leftarrow \mathbf{L}'^{\infty}(\Phi_\epsilon) \leftarrow \mathbf{L}^{\mathrm{NA}}(\phi_{\epsilon,m}) \\ \geq_{\mathrm{Stab.}} \quad & (1-\gamma_\epsilon)\mathbf{E}^{\mathrm{NA}}(\phi_{\epsilon,m}) \geq (1-\gamma_\epsilon)\mathbf{E}'^{\infty}(\Phi_\epsilon) \to (1-\gamma)\mathbf{E}'^{\infty}(\Phi) \end{aligned}$$

New ingredients for continuous $\mathrm{Aut}(X)$ (**Li**'19)

1. Valuative criterion for $\mathbb{G}$-uniform stability: $\exists \delta > 1$, s.t.

$$\inf_{v \in X_{\mathbb{Q}}^{\mathrm{div}}} \sup_{\xi \in N_{\mathbb{R}}} \left(A_X(v_\xi) - \delta S_L(v_\xi) \right) \geq 0. \tag{14}$$

2. Non-Archimedean metrics $\longleftrightarrow$ functions on $X_{\mathbb{Q}}^{\mathrm{div}}$.

$$\phi_\xi(v) = \phi(v_\xi) + \theta(\xi), \quad \theta(\xi) = A_X(v_\xi) - A_X(v). \tag{15}$$

3. Reduce the infimum (resp. supremum) to "bounded" subsets of $X_{\mathbb{Q}}^{\mathrm{div}}$ (resp. $N_{\mathbb{R}}$) (use *Strong Openness Conjecture*)
4. Delicate interplay between convexity and coerciveness of Archimedean and non-Archimedean energy.

Synthesis: Chain of Contradiction in general **Li**'19

3-parameters approximation argument: Don't follow!

$$
\begin{aligned}
\mathbf{E}'^{\infty}(\Phi) &\geq \mathbf{L}'^{\infty}(\Phi) + O(k^{-1}) \\
&\leftarrow \mathbf{L}'^{\infty}(\Phi_\epsilon) + O(\epsilon, k^{-1}) \\
&\leftarrow \mathbf{L}^{\mathrm{NA}}(\phi_{\epsilon,m}) + O(\epsilon, m^{-1}, k^{-1}) \\
&= A(v_k) + \phi_{\epsilon,m}(v_k) \\
&= A(v_{k,-\xi_k}) + \phi_{\epsilon,m,-\xi_k}(v_{k,\xi_k}) \\
&\geq \delta S_{L_\epsilon}(v_{k,-\xi_k}) + \phi_{\epsilon,m,-\xi_k}(v_{k,\xi_k}) \\
&\geq \delta \mathbf{E}^{\mathrm{NA}}(\delta^{-1}\phi_{\epsilon,m,-\xi}) \\
&\geq (1-\delta^{-1/n})\mathbf{J}^{\mathrm{NA}}(\phi_{\epsilon,m,-\xi_k}) + \mathbf{E}^{\mathrm{NA}}(\phi_{\epsilon,m,-\xi_k}) \\
&= (1-\delta^{-1/n})\mathbf{\Lambda}^{\mathrm{NA}}(\phi_{\epsilon,m,-\xi_k}) + \delta^{-1/n}\mathbf{E}^{\mathrm{NA}}(\phi_{\epsilon,m,-\xi_k}) \\
&\geq (1-\delta^{-1/n})\mathbf{\Lambda}'^{\infty}(\Phi_{\epsilon,-\xi_k}) + \delta^{-1/n}\mathbf{E}'^{\infty}(\Phi_{\epsilon,-\xi_k}) \\
&= (1-\delta^{-1/n})\mathbf{J}'^{\infty}(\Phi_{\epsilon,-\xi_k}) + \mathbf{E}'^{\infty}(\Phi_{\epsilon,-\xi_k}) \\
&\geq (1-\delta^{-1/n})\chi + \mathbf{E}'^{\infty}(\Phi).
\end{aligned}
$$

Metric structures from KE potentials

Question: Do (weak) KEs induce "good" metric structures?
Partial answer: Yes for orbifolds (Song-Tian). In general they have orbifold singularities away from $\mathrm{codim}_{\mathbb{C}}$ 3 subvariety (**Li**-Tian).

- Related to [Rong-Zhang, Song, Tian-Zhang, Tian-Wang, ...]

Theorem (Li-Tian-Wang '17)

If X admits a good (e.g. crepant) resolution and is K-(poly)stable, then X admits a KE metric. Moreover, the metric completion of $(X^{\mathrm{reg}}, \omega_{\mathrm{KE}})$ is homeomorphic to X.

- KE Fano varieties from Gromov-Hausdorff (GH) limits:

$$(M_i, \omega_{i,\mathrm{KE}}) \longrightarrow (X, d_X).$$

X is a $\mathbb{Q}$-Fano variety with a weak KE (Donaldson-Sun, Tian).

Metric tangent cones (MTC) on GH Fano limits

Question: What does the metric look like near the singularities?
1st order approximation of the metric structure (Cheeger-Colding):

$$C_x X := \overset{p-GH}{\lim_{r_k \to 0+}} \left(X, x, \frac{d_X}{r_k} \right)$$

Donaldson-Sun: $C_x X$ is homeomorphic to an affine variety and uniquely determined by the (unknown) *metric* structure.

Conjecture (Donaldson-Sun)

$C_x X$ depends only on the algebraic structure of the germ $x \in X$.

Normalized volume on any klt singularity

$\mathrm{Val}_{X,x}$: space of real valuations centered at $x \in X$.

Definition (Li'15, the normalized volume)

$$\widehat{\mathrm{vol}} := \widehat{\mathrm{vol}}_{X,x} : \mathrm{Val}_{X,x} \longrightarrow \mathbb{R}_{>0} \cup \{+\infty\}$$
$$v \mapsto A_X(v)^n \cdot \mathrm{vol}(v).$$

Properties/Remarks:

1. $\widehat{\mathrm{vol}}(x, X) := \inf \widehat{\mathrm{vol}}(v) > 0$ coincides with volume density (GMT) on GH limits. (Hein-Sun, **Li**-Xu)
2. This is an "anti-derivative" of Futaki invariant, motivated by Martelli-Sparks-Yau's study of Sasaki-Einstein.
3. Related to previous works of de-Fernex-Ein-Mustaţă.

Minimizing normalized volumes

Conjecture (Proposed by Li, Li-Xu)

(i) *$\forall$ klt germ $x \in X$, $\exists$ a minimizer v_* unique up to scaling.*
(ii) *v_* is regular: quasi-monomial, finitely generated associated graded ring s.t. $\mathrm{Spec}(\mathrm{gr}_{v_*}(\mathcal{O}_{X,x}))$ is a K-semistable Fano cone.*

- Existence: Blum (uses coerciveness estimate from **Li**'15)
- Uniqueness:
  - Divisorival minimizers are plt blow-ups and unique (**Li**-Xu '16)
  - f.g. quasi-monomial minimizers are unique (**Li**-Xu '17).
- Regularity of minimizer:
  - True for valuations from GH limits (by Donaldson-Sun)
  - Quasi-monomial (Xu '19 by using Birkar's boundedness)

Examples

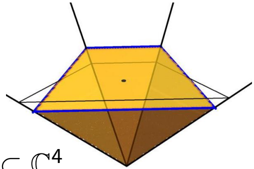
- $\widehat{\mathrm{vol}}(0, \mathbb{C}^n/G) = \frac{n^n}{|G|}$.
- Toric case: minimizing the volume of convex bodies.

  Example:

  $X = \mathrm{AffCone}(S = Bl_p\mathbb{P}^2, -K_S)$

  $v = (\frac{4-\sqrt{13}}{3}, \frac{4-\sqrt{13}}{3}, 1)$

  $\widehat{\mathrm{vol}}(x, X) = \frac{46+13\sqrt{13}}{12}$



- 3-dim A_{k-1}: $X = \{z_1^2 + z_2^2 + z_3^2 + z_4^k = 0\} \subset \mathbb{C}^4$.

k	$0 \le k \le 3$	$k = 4$ (**Li**-Sun'12)	$k \ge 5$
minimizer	$(k, k, k, 2)$	$(2, 2, 2, 1)$	$(2, 2, 2, 1)$
degeneration	X (stable)	X(semistable)	$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ (stable)
MTC	X	$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$	$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$

Application of normalized volumes

Theorem (Donaldson-Sun's conjecture, Li-Xu '17, Li-Wang-Xu '18)

For any x on GH limit, $\exists$ a unique valuation $v_* \in \mathrm{Val}_{X,x}$ satisfying:

1. *v_* minimizes $\widehat{\mathrm{vol}}$ and v_* is "regular".*
2. *$\mathrm{gr}_{v_*}\mathcal{O}_{X,x}$ uniquely degenerates to a K-polystable Fano cone which coincides with the MTC C_xX.*

This allows to determine metric tangent cones a priori without knowing metric structures. Other connections/applications:

1. Torus-equivariant criterions for the K-semistability and K-polystability (**Li**, **Li**-Liu, **Li**-Wang-Xu)
2. Bound the singularities of K-semistable Fano varieties (Liu) and application to moduli problem (Liu-Xu, Spotti-Sun)
3. 2-dimensional logarithmic normalized volume is equal to Langer's local orbifold Euler number (Borbon-Spotti, **Li**'18)

Thanks for your attention!