Chapter 1

Superconnection

1.1 $\mathbb{Z}_2$-graded algebra

Definition 1.1.1. A is a $\mathbb{C}$-algebra, A is $\mathbb{Z}_2$-graded, if there is a splitting: $A = A_+ \oplus A_-$, such that $A_+A_+ \subset A_+, A_-A_- \subset A_+, A_+A_- \subset A-, A_-A_+ \subset A_-$.

$A_{even} = A_+, A_{odd} = A_-$ are called the even and odd part of A. The usual algebra is trivially $\mathbb{Z}_2$-graded with $A = A_+$.

Definition 1.1.2. for homogeneous elements $a, b \in A_\pm$, the supercommutator is

$$[a,b] = a\,b - (-1)^{deg\,a \cdot deg\,b} b\,a, \quad deg\ a = \begin{cases} 0, & a \in A_+ \\ 1. & a \in A_- \end{cases}$$

we extend linearly to A. $a, b \in A$ is said to be supercommutative if $[a,b] = 0$. Note that $[A_+, A_+] \subset A_+, [A_-, A_-] \subset A_+, [A_+, A_-] \subset A-, [A_-, A_+] \subset A_-$.

Proposition 1.1.1 (Generalized Jacobi Identity)**.**

$$[a,[b,c]] + (-1)^{deg\,a(deg\,b+deg\,c)}[b,[c,a]] + (-1)^{deg\,c(deg\,a+deg\,b)}[c,[a,b]] = 0, \ \textit{or equivalently}$$

$$[a,[b,c]] = [[a,b],c] + (-1)^{deg\,a\cdot deg\,b}[b,[a,c]] \ \textit{(derivation property)}$$

Example 1.1.1.
1. E is a finite dimensional vector space. The exterior algebra: $\wedge^* E^* = \wedge^{even} E^* \oplus \wedge^{odd} E^*$ is a commutative $\mathbb{Z}_2$-graded algebra, since $\omega \wedge \eta = (-1)^{deg\omega \cdot deg\eta} \eta \wedge \omega$.
2. $\tau : E \longrightarrow E$ is an endomorphism, s.t.$\tau^2 = id$, then E is $\mathbb{Z}_2$-graded: $E = E_+ \oplus E_-$, $\tau | E_\pm = \pm 1$. $End(E)$ is a $\mathbb{Z}_2$-graded algebra:

$$End(E)^{even} = \{A \in End(E) : A\tau = \tau A\}, End(E)^{odd} = \{A \in End(E) : A\tau = -\tau A\}$$

3. q is a nondegenerate quadratic form on E, the Clifford algebra is $\mathbb{Z}_2$-graded: $\alpha : Cl(E,q) \longrightarrow Cl(E,q)$ is the involution defined by $\alpha(v) = -v, \forall v \in E$.

Definition 1.1.3. A, B are $\mathbb{Z}_2$-graded algebra, $A \otimes B$ has a product:

$$a \otimes b \cdot a' \otimes b' = (-1)^{deg\,b \cdot deg\,a'} aa' \otimes bb'$$

for homogeneous elements $a, a' \in A,\ b, b' \in B$. It's well defined and $A \otimes B$ becomes a $\mathbb{Z}_2$-graded algebra, denoted by $A \hat{\otimes} B$:

$$(A \hat{\otimes} B)_+ = (A_+ \otimes B_+) \oplus (A_- \otimes B_-),\ (A \hat{\otimes} B)_- = (A_+ \otimes B_-) \oplus (A_- \otimes B_+)$$

Remark 1. If A, B are unitary($1_A \in A, 1_B \in B$), then A, B are embedded in $A\hat{\otimes}B$. We can identify a with $a \otimes 1_B$, and b with $1_A \otimes b$, then $a \cdot b = (-1)^{dega \cdot degb} b \cdot a$, so A and B are supercommutative in $A\hat{\otimes}B$.

Example 1.1.2. (E_1, q_1), (E_2, q_2) are two vector spaces with nondegenerate quadratic forms. Let $E = E_1 \oplus E_2, q = q_1 \oplus q_2 : q((v_1, v_2)) = q_1(v_1) + q_2(v_2)$, then $Cl(E, q) = Cl(E_1, q_1)\hat{\otimes}Cl(E_2, q_2)$.

Definition 1.1.4. assume $E = E_+ \oplus E_-$ is $\mathbb{Z}_2$-graded defined by involution τ. $\forall A \in End(E)$, the supertrace of A is $Tr_s(A) = Tr(\tau A)$. In blocked matrix, $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $Tr_s(A) = Tr(a) - Tr(d)$.

Proposition 1.1.2. $\forall A, B \in End(E), Tr_s[A, B] = 0$.

assume $(E, \tau), (F, \sigma)$ are two $\mathbb{Z}_2$-graded vector space, then $E\hat{\otimes}F = (E \otimes F, \tau \otimes \sigma)$ is a $\mathbb{Z}_2$-graded vector space. $\forall A \in End(E), B \in End(F)$, we have

$$Tr_s(A \otimes B) = Tr(\tau A \otimes \sigma B) = Tr(\tau A)Tr(\sigma B) = Tr_s(A)Tr_s(B).$$

Definition 1.1.5. E is a finit dimensional, we define the supertrace on the $\mathbb{Z}_2$-graded algebra $\wedge^* E^* \hat{\otimes} End(E)$(matrix of forms):

$$\begin{aligned} Tr_s : \wedge^* E^* \hat{\otimes} End(E) &\longrightarrow \wedge^* E^* \\ \omega \otimes A &\mapsto \omega Tr_s(A) \end{aligned}$$

By the universal property of tensor product, this is well defined. For simplicity, we write ωA for $\omega \otimes A$. We have

$$[\omega A, \omega' A] = (-1)^{degA \cdot deg\omega} \omega\omega' [A, A']$$

1.2 Superconnection

Definition 1.2.1. E is a finite dimensional $\mathbb{Z}_2$-graded complex vector bundle on the manifold M. A superconnection A is an odd differential operator acting on $\Gamma(\wedge^* T^* M \hat{\otimes} E)$, which satisfies the Leibniz rule:

$$A(\omega s) = d\omega \cdot S + (-1)^{deg\,\omega} \omega \wedge As$$

Remark 2. $\wedge^* T^* M \hat{\otimes} E$ is a $\mathbb{Z}_2$-graded vector bundle, $\Gamma(\wedge^* T^* M \hat{\otimes} E)$ is the set of E-valued forms which is naturally $\mathbb{Z}_2$-graded. Locally on an open set U, we choose a cotangent frame $\{e^i\}$ and a frame of $\{s_\alpha\} = \{s^+_\lambda\} \cup \{s^-_\mu\}$ of $E = E_+ \oplus E_-$, then locally $\tau \in \Gamma(\wedge^* T^* M \hat{\otimes} E)$ is:

$$\tau = f^\alpha_{i_1,\ldots,i_r} e^{i_1} \wedge \ldots \wedge e^{i_r} \otimes s_\alpha,\ \tau^\alpha_{i_1,\ldots,i_r} \in C^\infty(U)$$

$A\tau = g^\beta_{j_1,\ldots,j_s} e^{j_1} \wedge \ldots \wedge e^{j_s} \otimes s_\beta$, $g = D(f)$, D is a matrix of differential operators. τ is even iff locally it's the sum of $\omega^{even} \cdot s^+_\lambda$ and $\omega^{odd} \cdot s^-_\mu$.

$$\Gamma = \Gamma_{even} \oplus \Gamma_{odd} = \Gamma((\wedge^* T^* M \hat{\otimes} E)_{even}) \oplus \Gamma((\wedge^* T^* M \hat{\otimes} E)_{odd})$$

By definition, $A(\Gamma_{even}) \subset \Gamma_{odd}$, $A(\Gamma_{odd}) \subset \Gamma_{even}$.

Example 1.2.1. ∇^E is a connection preserving the splitting $E = E_+ \oplus E_-$, i.e. $\nabla^E(\Gamma(E_\pm)) \subset \Gamma(E_\pm)$.

Write $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$, or in blocked matrix: $\nabla^E = \begin{pmatrix} \nabla^{E_+} & 0 \\ 0 & \nabla^{E_-} \end{pmatrix}$

We extend ∇^E to act on $\Gamma(\wedge^* T^* M \hat{\otimes} E)$ satisfying the Leibniz rule: $\nabla^E(\omega s) = d\omega \cdot s + (-1)^{deg\,\omega} \omega \wedge \nabla^E s$, Then ∇^E is a superconnection, since it increases the degree of forms by 1, while keeps the degree of sections of the bundle E, thus odd.

let $D \in \Gamma(End(E)^{odd})$, i.e. $D_x \in End(E_x)^{odd}$, in matrix $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$, then $\nabla^E + D$ is a superconnection, since D preserves the degree of forms, and interchanges the even and odd part of E.

In particular, when $M = \{pt\}$, E is a $\mathbb{Z}_2$-graded vector space, $D \in End(E)^{odd}$ is a superconnection.

Proposition 1.2.1. *Any superconnection A can be written as $A = \nabla^E + B$, with $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$, and $B \in \Gamma(\mathcal{B})$, where $\mathcal{B} = \wedge^* T^*M \hat{\otimes} End(E)$ is a bundle of algebras.*

Proof. Let $B = A - \nabla^E$, then B is an odd operator on $\Gamma(\wedge^* T^*M \hat{\otimes} E)$. $\forall f \in C^\infty(M)$, $s \in \Gamma(\wedge^* T^*M \hat{\otimes} E)$,

$$B(fs) = A(fs) - \nabla^E(fs) = df \cdot s + fAs - (df \cdot s + f\nabla^E s) = f(A - \nabla^E)s = fBs$$

so B is $C^\infty(M)$ linear, i.e. tensorial.

Let $Bs_\alpha = \theta_\alpha^\beta s_\beta$, $\theta_\alpha^\beta \in \Gamma(\wedge^* T^*U)$, define $\tilde{B} = \theta_\alpha^\beta s_\alpha^* \otimes s_\beta \in \Gamma(\wedge^* T^*M \hat{\otimes} End(E))$, where $\{s_\alpha^*\}$ is the dual basis of $\{s_\beta\}$, then $\tilde{B}$ is independent of the chosen frame since B is tensorial.

conversely $\forall \tilde{B} \in \Gamma(\wedge^* T^*M \hat{\otimes} End(E))$, $\tilde{B}$ acts naturally on $\Gamma(\wedge^* T^*M \hat{\otimes} E) : (\omega \otimes D) \cdot (\eta \otimes s) = (-1)^{deg\, D \cdot deg\, \eta} \omega \wedge \eta \otimes Ds$, this action is clearly C^∞ linear. So we can identify B and $\tilde{B}$, and the proposition is proved. □

Remark 3. if we choose local frame $\{s_\lambda^+\} \cup \{s_\mu^-\}$ of E on U, then $E \cong \mathbb{C}^n \oplus \mathbb{C}^m$. The de Rham operator d is a connection on trivial bundles. By the proposition, every connection is locally of the form:

$$A = d + B, B \in \Gamma(\wedge^* T^*M \hat{\otimes} End(\mathbb{C}^n \oplus \mathbb{C}^m))_{odd}$$

1.3 Chern-Weil theory for superconnections

Proposition 1.3.1. *A is a superconnection, then $A^2 \in \Gamma(\mathcal{B}_{even})$.*

recall that $\mathcal{B} = \wedge^* T^*M \hat{\otimes} End(E)$.

Proof. Since A is odd, A^2 is an even operator. $\forall f \in C^\infty(M)$,

$$A^2(fs) = A(df \cdot s + f \cdot As) = d(df)s - df \wedge As + df \wedge As + fA^2 s = fA^2 s$$

so A^2 is a tensor. □

Definition 1.3.1. The curvature of a superconnection A is A^2. The Chern character of A is $ch(A) = \varphi Tr_s(e^{-A^2}) \in \Gamma(\wedge^* T^*M)$, where

$$\begin{aligned} \varphi : \wedge_{\mathbb{C}}^* T^*M &\longrightarrow \wedge_{\mathbb{C}}^* T^*M = \wedge^* T^*M \otimes \mathbb{C} \\ \alpha &\mapsto (2\pi i)^{-\frac{deg\,\alpha}{2}} \alpha \end{aligned}$$

is the normalization endormorphism, which makes the definition agree with the Chern character of a complex vector bundle.

Theorem 1.3.1 (Quillen)**.** *$ch(A)$ is a closed, even form. $[ch(A)] = ch(E_+) - ch(E_-)$, $[ch(A)] \in H_{dR}^{even}(M)$ is the cohomological class represented by $ch(A)$.*

Proof. Take a local trivialization $E = E_+ \oplus E_- \cong \mathbb{C}^n \oplus \mathbb{C}^m$, then $A = d + B$, where $B \in \Gamma(\wedge^* T^*M \hat{\otimes} End(\mathbb{C}^n \oplus \mathbb{C}^m))_{odd}$

$$\begin{aligned} dTr_s(\exp(-A^2)) &= Tr_s(d \exp(-A^2)) = Tr_s[d, \exp(-A^2)] \\ &= Tr_s[d + B, \exp(-A^2)] = Tr_s[A, \exp(-A^2)] = 0 \end{aligned}$$

In the third equality, we use $Tr_s[B, \exp(-A^2)] = 0$: since $\exp(-A^2)$ is a tensor, the supertrace of supercommutator vanishes. The last equality follows from the Bianchi identity: $[A, A^2] = 0$, which is trivial in this setting.

Remark 4. It's easy to see that, $\forall B \in \Gamma(\wedge^* T^*M \hat{\otimes} End(E))$, $[A, B] \in \Gamma(\wedge^* T^*M \hat{\otimes} End(E))$.

Since A^2 is even, $\exp(-A^2)$ is even. It is the sum of the form

$$\text{even form} \otimes \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \text{ odd form} \otimes \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

but $Tr_s(\text{odd form} \otimes \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}) = 0$, so $Tr_s(\exp(-A^2))$ contains only even forms.

Next we want to prove [ch(A)] is independent of the superconnection. Let $\mathcal{A} = \{superconnection\}$, $\mathcal{A}$ is an affine space. Given $A_0, A_1 \in \mathcal{A}$, let $A_t = (1-t)A_0 + tA_1$. Define $\mathbb{A} = dt\frac{\partial}{\partial t} + A_t$, then $\mathbb{A}$ is a superconnection on the vector bundle $\pi_M^* E \longrightarrow M \times [0,1]$, where $\pi_M : M \times [0,1] \longrightarrow M$ is the projection. $\mathbb{A}^2 = A_t^2 + dt\frac{\partial A_t}{\partial t}$,

$$ch(\mathbb{A}) = \varphi Tr_s \exp(-(A_t^2 + dt\frac{\partial A_t}{\partial t})) = ch(A_t) + dt\beta_t$$

β_t is an odd form on $M \times [0,1]$. We know that $ch(\mathbb{A})$ is closed, so

$$0 = d\,ch(\mathbb{A}) = d_M ch(A_t) + dt\frac{\partial}{\partial t}ch(A_t) - dt d_M \beta_t = dt\frac{\partial}{\partial t}ch(A_t) - dt\, d_M\beta_t$$

so $\frac{\partial}{\partial t}ch(A_t) = d\beta_t$ and

$$ch(A_1) - ch(A_0) = \int_0^1 \frac{\partial}{\partial t}ch(A_t)dt = d\int_0^1 \beta_t dt$$

so $[ch(A_0)] = [ch(A_1)]$.

Thus we can choose any superconnection to compute $[ch(A)]$. We let $A = \nabla^E = \begin{pmatrix} \nabla^{E_+} & 0 \\ 0 & \nabla^{E_-} \end{pmatrix}$, then

$$\begin{aligned} ch(A) &= [\varphi Tr_s \exp(-\nabla^{E,2})] = [\varphi Tr\ \exp(-\nabla^{E_+,2})] - [\varphi Tr\ \exp(-\nabla^{E_-,2})] \\ &= [Tr\ \exp(\frac{\sqrt{-1}}{2\pi}\nabla^{E_+,2})] - [Tr\ \exp(\frac{\sqrt{-1}}{2\pi}\nabla^{E_-,2})] = ch(E_+) - ch(E_-) \end{aligned}$$

□

Example 1.3.1. $M = \mathbb{C}$, $E = E_+ \oplus E_- = \mathbb{C} \oplus \mathbb{C}$, $D_z = \begin{pmatrix} 0 & \bar{z} \\ z & 0 \end{pmatrix}$, $\nabla^E = d$. $A_t = \nabla^E + \sqrt{t}D = d + \sqrt{t}D$,

$$A_t^2 = d^2 + \sqrt{t}\,dD + tD^2 = \begin{pmatrix} t|z|^2 & 0 \\ 0 & t|z|^2 \end{pmatrix} + \sqrt{t}\begin{pmatrix} 0 & d\bar{z} \\ dz & 0 \end{pmatrix}$$

Since

$$\frac{1}{n!}Tr_s(t|z|^2 + \sqrt{t}\begin{pmatrix} 0 & d\bar{z} \\ dz & 0 \end{pmatrix})^n = \frac{1}{n!}\frac{n\cdot(n-1)}{2}(t|z|^2)^{n-2}t\,Tr_s\begin{pmatrix} d\bar{z}dz & 0 \\ 0 & dzd\bar{z} \end{pmatrix} = -t\frac{(t|z|^2)^{n-2}}{(n-2)!}dzd\bar{z}$$

$$Tr_s\ \exp(-A_t^2) = \sum_{n=0}^{\infty}(-1)^n Tr_s(A_t^2) = -t\exp(-t|z|^2)dzd\bar{z}$$

$$ch(A_t) = \varphi Tr_s \exp(-A_t^2) = \frac{1}{2\pi i}\cdot(-t)\exp(-t|z|^2)dzd\bar{z} = \frac{t}{\pi}\exp(-t|z|^2)dxdy = P_{\frac{1}{4t}}(z)dxdy$$

where $p_t(x,y) = \frac{1}{4\pi t}\exp(-\frac{|x-y|^2}{4t})$ is the heat kernel of Laplace $\Delta = -(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}) = -4\frac{\partial^2}{\partial z \partial \bar{z}}$ on $\mathbb{C}$. $\alpha_t = ch(A_t)$ has the following properties:

- $\int_{\mathbb{C}} \alpha_t = 1$

- $\alpha_t \xrightarrow{t\to 0} 0$, $\alpha_t \xrightarrow{t\to\infty} \delta_{\{0\}}$, in distributional sense.

We see that: $[\alpha_t] = 0 = ch(E_+) - ch(E_-)$(for trivial bundle E, $ch(E) = rank(E)$), this gives no information. But for $t > 0$, α_t represents the generator of $H_c^*(\mathbb{C})$, which is the compact supported cohomology(strictly, the forms are fast decaying). By Poincaré's duality,

$$H_c^*(\mathbb{R}^n) = H^{n-*}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & * = n \\ 0 & * \neq n \end{cases}$$

The isomorphism $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$ is given by integration over $\mathbb{R}^n$.

Example 1.3.2 (Heat kernel method)**.** H_1, H_2 are Hilbert spaces, a closed linear operator with dense domain $P : H_1 \longrightarrow H_2$ is Fredholm iff $dim\, kerP < \infty$, $dim\, cokerP = dimH_2/imP < \infty$. We can prove that imP is closed, so $cokerP \cong (imP)^{\perp} = kerP^*$, and

$$Ind(P) = dimkerP - dimcokerP = dimkerP - dimkerP^* = dimkerP^*P - dimkerPP^*$$

P^*P and PP^* have the same nonzero eigenvalues, and the corresponding eigenspaces are isomorphic:

$$\begin{array}{rcl} E_{P^*P}(\lambda) & \longrightarrow & E_{PP^*}(\lambda) \\ x & \mapsto & Px \\ \frac{1}{\lambda}P^*y & \leftarrow & y \end{array}$$

We can obtain the heat equation method:

$$\begin{aligned} Ind(P) &= dimkerP^*P - dimkerPP^* = \sum_{\lambda} e^{-t\lambda} dimE_{P^*P}(\lambda) - \sum_{\lambda} e^{-t\lambda} dimE_{PP^*}(\lambda) \\ &= Tr\, e^{-tP^*P} - Tr\, e^{-tPP^*} = Tr_s(e^{-tD^2}) = ch(D) \end{aligned}$$

where $D = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}$ is a superconnection on the infinite dimensional Hilbert bundle $H = H_1 \oplus H_2 \to pt$.

We can also use the transgression method to see this:

$$\frac{\partial}{\partial t} Tr_s e^{-tD^2} = -Tr_s(D^2 e^{-tD^2}) = -\frac{1}{2} Tr_s([D, D] e^{-tD^2}) = -\frac{1}{2} Tr_s[D, De^{-tD^2}] = 0$$

so $Tr_s e^{-tD^2}$ is a constant. Since $Tr_s e^{-tD^2} \xrightarrow{t\to\infty} dimkerP^*P - dimkerPP^* = IndP$, the formula follows.

1.4 Chern-Simon class

E is a complex vector bundle, ∇^E, ∇'^E are two connections, $R^E = \nabla^{E,2}$, $R'^E = \nabla'^{E,2}$ are their curvatures. P is an invariant polynomial of $GL(n, \mathbb{C})$, i.e. P satisfies:

$$P(TAT^{-1}) = P(A), \forall T \in GL(n, \mathbb{C}), A \in M_n$$

$P(R^E)$ is closed and $[P(R^E))]$ does not depend on ∇^E, so

$$P(R'^E) - P(R^E) = d\alpha, \text{ for some } \alpha \in \Omega^{odd}(M)/d\Omega^{even}(M)$$

We can have a canonical way to construct α, which is called the Chern-Simon form.

As in the proof of Theorem, choose a curve $C_t = \{\nabla_t^E : t \in [0, 1]\}$ connecting ∇^E and ∇'^E in $\mathcal{A}$ which is the affine space of connections. $\nabla^{\mathbb{E}} = \nabla_t^E + dt\frac{\partial}{\partial t}$ is a connection on the bundle $\pi_M^* E$ over $M \times [0, 1]$, where $\pi_M : M \times [0, 1] \to M$ is the projection. The curvature $R^{\mathbb{E}} = \nabla^{\mathbb{E},2} = \nabla_t^{E,2} + dt\frac{\partial \nabla_t^E}{\partial t}$.

$P : M_n \cong \mathbb{C}^{n^2} \to \mathbb{C}$ is a polynomial of n^2 variables, we have the Taylor expansion:

$$P(x+y) = \sum_{|\alpha| \leq deg\, P} \frac{\partial^\alpha P(x)}{\alpha!} y^\alpha, \ \forall\, x,\, y \in \mathbb{C}^{n^2}$$

Note that each term of $R^{\mathbb{E}}$ is a 2-form, which commutes with other forms. So we use the above formula to have:

$$P(R^{\mathbb{E}}) = P(R_t^E + dt \frac{\partial}{\partial t} \nabla_t^E) = P(R_t^E) + dt < P'(R_t^E), \frac{\partial}{\partial t} \nabla_t^E >$$

where $P'(x) : \mathbb{C}^{n^2} \to \mathbb{C}$ is the total derivative at $x \in \mathbb{C}^{n^2}$. We know that $P(R^{\mathbb{E}})$ is closed, so:

$$0 = d_{M \times [0,1]} P(R^{\mathbb{E}}) = d_M P(R_t^E) + dt \frac{\partial}{\partial t} P(R_t^E) - dt\, d_M < P'(R_t^E), \frac{\partial}{\partial t} \nabla_t^E >$$

so $\frac{\partial}{\partial t} P(R_t^E) = d < P'(R_t^E), \frac{\partial}{\partial t} \nabla_t^E >$. Define

$$\tilde{P} = \int_0^1 < P'(R_t^E), \frac{\partial}{\partial t} \nabla_t^E > dt$$

then $d\tilde{P} = P(R'^E) - P(R^E)$.

This is a case of integration along the fibre, see appendix. We write:

$$\tilde{P} = \int_{C_t} P(R_t^E) = \int_{[0,1]} P(R^{\mathbb{E}})$$

Proposition 1.4.1. *The class of $\tilde{P}$ in $\Omega^{odd}/d\Omega^{even}$ does not depend on the path connecting ∇^E and ∇'^E.*

Proof. If $C'_t = \{\nabla_t'^E\}$ is another path, let $\nabla_{s,t}^E = (1-s)\nabla_t^E + s\nabla_t'^E$, define $\nabla^{\mathcal{E}} = ds \frac{\partial}{\partial s} + dt \frac{\partial}{\partial t} + \nabla_{s,t}^E$. then $\nabla^{\mathcal{E}}$ is a connection on $\mathcal{E} = p_1^* E$, $p_1 : M \times [0,1]^2 \to M$ is the projection. The curvature:

$$R^{\mathcal{E}} = \nabla^{\mathcal{E},2} = \nabla_{s,t}^{E,2} + dt \frac{\partial}{\partial t} \nabla_{s,t}^E + ds \frac{\partial}{\partial s} \nabla_{s,t}^E = \nabla_{s,t}^{E,2} + dt((1-s)\frac{\partial}{\partial t}\nabla_t^E + s \frac{\partial}{\partial t} \nabla_t'^E) + ds(\nabla_t'^E - \nabla_t^E)$$

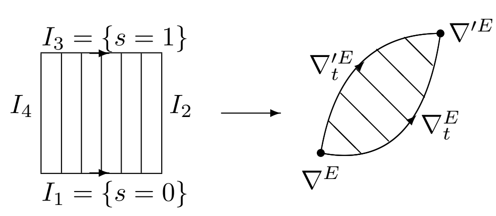
$$R^{\mathcal{E}}|_{s=0} = \nabla_t^{E,2} + dt \frac{\partial}{\partial t} \nabla_t^E = \nabla^{\mathbb{E},2},\ R^{\mathcal{E}}|_{s=1} = \nabla_t'^{E,2} + dt \frac{\partial}{\partial t} \nabla_t'^E = \nabla^{\mathbb{E}',2}$$

$$R^{\mathcal{E}}|_{t=0} = \nabla^{E,2},\ R^{\mathcal{E}}|_{t=1} = \nabla'^{E,2}$$

Note that $R^{\mathcal{E}}|_{t=0}$, $R^{\mathcal{E}}|_{t=1}$ have no ds term. Using the Stoke's Formula, we have:

$$\begin{aligned} d\int_{[0,1]^2} P(R^{\mathcal{E}}) &= \int_{[0,1]^2} d\,P(R^{\mathcal{E}}) - (\int_{I_1} P(R^{\mathcal{E}}|_{s=0}) - \int_{I_3} P(R^{\mathcal{E}}|_{s=1})) \\ &= -\int_{[0,1]} P(R^{\mathbb{E}}) + \int_{[0,1]} P(R^{\mathbb{E}'}) = \tilde{P}' - \tilde{P} \end{aligned}$$

□



Given ∇^E, ∇'^E, by the above proposition, we have constructed a class in $\Omega^{odd}/d\Omega^{even}$. We denote it by $\tilde{P}(\nabla^E, \nabla'^E)$, called Chern-Simon form. It satisfies 3 properties:

1. $d\,\tilde{P}(\nabla^E, \nabla'^E) = P(R'^E) - P(R^E)$
2. $\tilde{P}(\nabla^E, \nabla^E) = 0$ Just choose the constant path: $\nabla^E_t \equiv \nabla^E$.
3. $\tilde{P}(\nabla^E, \nabla'^E)$ is functorial: if $f : N \to M$ is a dfferentiable mapping, then

$$\tilde{P}(f^*\nabla^E, f^*\nabla'^E) = f^*\tilde{P}(\nabla^E, \nabla'^E)$$

where $f^*\nabla^E$, $f^*\nabla'^E$ is the induced connection on $F = f^*E$.
This is because: Choose a curve ∇^E_t connecting ∇^E and ∇'^E, $\nabla^F_t = f^*\nabla^E_t$ is a curve connecting $\nabla^F = f^*\nabla^E$ and $\nabla'^F = f^*\nabla'^E$. Let $\mathbb{E} = \pi^*_M E$, $\nabla^{\mathbb{E}} = \nabla^E_t + dt\frac{\partial}{\partial t}$; $\mathbb{F} = \pi^*_N F$, $\nabla^{\mathbb{F}} = \nabla^F_t + dt\frac{\partial}{\partial t} = (f \times id)^*\nabla^{\mathbb{E}}$.

$$\tilde{P}(f^*\nabla^E, f^*\nabla'^E) = \int_{[0,1]} P(R^{\mathbb{F}}) = \int_{[0,1]} P((f \times id)^* R^{\mathbb{E}}) = f^* \int_{[0,1]} P(R^{\mathbb{E}}) = f^*\tilde{P}(\nabla^E, \nabla'^E)$$

In fact, we have:

Proposition 1.4.2. *The class $\tilde{P}(\nabla^E, \nabla'^E)$ is uniquely defined by properties 1-3*

Proof. We need to check the uniqueness. Assume $\tilde{P}$ satisfies the 3 properties. Choose any path ∇^E_t connecting ∇^E, ∇'^E. Let $\nabla^{\mathbb{E}} = \nabla^E_t + dt\frac{\partial}{\partial t}$, $\nabla^{\mathbb{E}}_0 = \nabla^E + dt\frac{\partial}{\partial t}$ be connections on $\mathbb{E} = \pi^*_M E$. By property 1, we have $P(R^{\mathbb{E}}) - P(R'^{\mathbb{E}}) = d\tilde{P}(\nabla^{\mathbb{E}}_0, \nabla^{\mathbb{E}})$. Integrate this equality along the fibre, and use the Stoke's Formula:

$$\begin{aligned}\int_{[0,1]} P(R^{\mathbb{E}}) &= \int_{[0,1]} P(R^{\mathbb{E}}_0) + \int_{[0,1]} d\tilde{P}(\nabla^{\mathbb{E}_0}, \nabla^{\mathbb{E}}) \\ &= -d\int_{[0,1]} \tilde{P}(\nabla^{\mathbb{E}}_0, \nabla^{\mathbb{E}}) + \tilde{P}(\nabla^{\mathbb{E}}_0, \nabla^{\mathbb{E}})|_{t=1} - \tilde{P}(\nabla^{\mathbb{E}}_0, \nabla^{\mathbb{E}})|_{t=0}\end{aligned}$$

Since $\nabla^{\mathbb{E}}_0|_{t=1} = \nabla^E$, $\nabla^{\mathbb{E}}|_{t=1} = \nabla'^E$, by functoriality, $\tilde{P}(\nabla^{\mathbb{E}}_0, \nabla^{\mathbb{E}})|_{t=1} = \tilde{P}(\nabla^E, \nabla'^E)$. Also $\tilde{P}(\nabla^{\mathbb{E}}_0, \nabla^{\mathbb{E}})|_{t=0} = \tilde{P}(\nabla^E, \nabla^E) = 0$ by property 2. So $\tilde{P}(\nabla^E, \nabla'^E) - \int_{[0,1]} P(R^{\mathbb{E}})$ is exact, that is what we require. □

Example 1.4.1. λ is a line bundle, $c_1(\lambda)$ is the first Chern class, then $\tilde{c}_1(\nabla^\lambda, \nabla'^\lambda) = -\frac{\nabla'^\lambda - \nabla^\lambda}{2\pi i}$ is the Chern-Simon class. This is because $d(-\frac{\nabla'^\lambda - \nabla^\lambda}{2\pi i}) = \frac{i}{2\pi} r'^\lambda - \frac{i}{2\pi i} r^\lambda = c_1(\nabla'^\lambda) - c_1(\nabla^\lambda)$, and the other two properties are obvious.

Since $\{superconnections\}$ is an affine space as $\{connections\}$, everything we've done can be carried out on superconnections, so we have:

Proposition 1.4.3. *The whole theory of Chern-Simon class extends to superconnections*

Proposition 1.4.4. *Let A_t be a curve of superconnections, then*

$$\frac{\partial}{\partial t} Tr_s \exp(-A_t^2) = -d\, Tr_s(\frac{\partial A_t}{\partial t} \exp(-A_t^2))$$

proof 1. Let $\mathbb{A} = A_t + dt\frac{\partial}{\partial t}$ be a superconnection on $\pi^*_M E$. By simple computation as example 1.3.1, we see that

$$Tr_s \exp(-\mathbb{A}^2) = Tr_s(\exp(-A_t^2)) - dtTr_s(\frac{\partial A_t}{\partial t} \exp(-A_t^2))$$

We know that $Tr_s \exp(-\mathbb{A}^2)$ is a closed even form on $M \times \mathbb{R}$, so

$$\frac{\partial}{\partial t} Tr_s \exp(-A_t^2) = -dTr_s(\frac{\partial A_t}{\partial t} \exp(-A_t^2))$$

□

proof 2.

$$\begin{aligned}\frac{\partial}{\partial t} Tr_s \exp(-A_t^2) &= -Tr_s(\frac{\partial A_t^2}{\partial t}\exp(-A_t^2)) = -Tr_s([A_t, \frac{\partial A_t}{\partial t}\exp(-A_t^2)]) \\ &= -Tr_s([A_t, \frac{\partial A_t}{\partial t}\exp(-A_t^2)]) = -d\,Tr_s(\frac{\partial A_t}{\partial t}\exp(-A_t^2))\end{aligned}$$

We used identities:

- $\frac{\partial A_t^2}{\partial t} = \frac{\partial A_t}{\partial t}A_t + A_t\frac{\partial A_t}{\partial t} = [A_t, \frac{\partial A_t}{\partial t}]$ (note both A_t and $\frac{\partial A_t}{\partial t}$ are odd)
- $d\,Tr_s(B) = Tr_s[A, B]$, for A a superconnection and $B \in \Gamma(\mathcal{B})$ (see the proof of theorem 1.3.1)

□

When $A_t = \nabla^E + \sqrt{t}V$, $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$, $V = \begin{pmatrix} 0 & V_- \\ V_+ & 0 \end{pmatrix} \in \Gamma(End(E)^{odd})$,

$$\frac{\partial}{\partial t} Tr_s \exp(-A_t^2) = -d\,Tr_s(\frac{V}{2\sqrt{t}}\exp(-A_t^2))$$

We want to integrate this equality on $[0, \infty)$.
Now assume $g^E = g^{E_+} \oplus g^{E_-}$ is an Hermitian metric on E, ∇^E is a unitary connection, and V is self-adjoint, so $V_- = V_+^*$.

Proposition 1.4.5. *$\alpha_t = ch(A_t)$ is a real form.*

Proof. We define an adjoint operator $*$ on $\mathcal{B} = \wedge^* T^*M \hat{\otimes} End(E)$:

1. $\forall \omega \in \wedge^1_{\mathbb{C}} T^*M$, $\omega^* = -\bar{\omega}$

2. $\forall A \in End(E)$, A^* is the adjoint of A

3. $\forall f, g \in \mathcal{B}$, $(f \cdot g)^* = g^* \cdot f^*$.

This is well defined. Now $A_t^2 = \nabla^{E,2} + \sqrt{t}\nabla^E V + tV^2$. $(tV^2)^* = tV^2$, since V is self-adjoint. $\forall X \in TM$, $\nabla_X V$ is also self-adjoint, so $(\nabla^E V)^* = (e^i \nabla^E_{e_i} V)^* = \nabla^E_{e_i} V \cdot (-e^i) = e^i \nabla^E_{e_i} V = \nabla^E V$. Since ∇^E is unitary, $R^E(X, Y) = \nabla^{E,2}(X, Y)$ is skew-Hermitian w.r.t. g^E, $\forall X, Y \in TM$, so

$$(\nabla^{E,2})^* = (e^i \wedge e^j R^E(e_i, e_j))^* = -R^E(e_i, e_j)(-e^j) \cdot (-e^i) = e^i \wedge e^j R^E(e_i, e_j) = \nabla^{E,2}$$

So we see $(A_t^2)^* = A_t^2$, also $(\exp(-A_t^2))^* = \exp(-A_t^2)$. Note that $\forall\, \omega A \in \mathcal{B}_{even}$,

$$(\omega A)^* = A^*\omega^* = A^*(-1)^{deg\,\omega} \cdot (-1)^{\frac{deg\,\omega\cdot(\deg\,\omega-1)}{2}}\bar{\omega} = (-1)^{\frac{deg\,\omega\cdot(deg\,\omega-1)}{2}}\bar{\omega}A^*$$

so

$$\overline{Tr_s(\omega A)} = Tr_s(\bar{\omega}A^*) = (-1)^{\frac{deg\,\omega\cdot(deg\,\omega-1)}{2}} Tr_s((\omega A)^*) = \begin{cases} Tr_s((\omega A)^*) & deg\,\omega \equiv 0 \mod 4 \\ -Tr_s((\omega A)^*) & deg\,\omega \equiv 2 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

So $(Tr_s \exp(-A_t^2))^{(4k)}$ is real, and $(Tr_s \exp(-A_t^2))^{(4k+2)}$ is purely imaginary, then it's clear $\varphi Tr_s \exp(-A_t^2) = ch(A_t)$ is real. □

Remark 5. Similarly, $\forall\, \omega A \in \mathcal{B}_{odd}$, $(\omega A)^* = (-1)^{\frac{deg\,\omega\cdot(deg\,\omega+1)}{2}}\bar{\omega}A^*$, so $\forall B \in \mathcal{B}_{odd}$ satisfying $B^* = B$, we have $(Tr_s(B))^{(4k+1)}$ is purely imaginary, and $Tr_s(B)^{(4k+3)}$ is real, then $\frac{1}{\sqrt{2\pi i}}\varphi Tr_s(B)$ is real.

Proposition 1.4.6. *Assume $kerV$ has locally constant dimensions, then $kerV$ is a smooth sub-bundle of E.*

Proof. Since $kerV$ has locally constant dimensions, $\forall x \in M$, $\exists \varepsilon > 0$, and a neighborhood $U \ni x$, s.t. $\forall y \in U$, $V_y : E_y \to E_y$ has no nonzero eigenvalues in the disk $B(0,\varepsilon) = \{z \in \mathbb{C} : |z| \leq \varepsilon\}$. Let

$$P^{kerV} = \frac{1}{2\pi i}\int_{\partial B(0,\varepsilon)} \frac{d\lambda}{\lambda - V}$$

then $P^{kerV} : E \to E$ is a smooth projection onto $kerV$. We have the direct sum:

$$E = P^{kerV}(E) \oplus (1 - P^{kerV})(E) = kerV \oplus (kerV)^{\perp} = kerV \oplus imV$$

□

Since $kerV = kerV_+ \oplus kerV_-$, and $dimkerV_- = dim(imV_+)^{\perp} = dimE_- - (dimE_+ - dimkerV_+)$,

$$dimkerV_+ + dimkerV_- = dimkerV,\ dimkerV_+ - dimkerV = dimE_+ - dimE_-$$

So $dimkerV_+$, $dimkerV_-$ is locally constant, by the proposition $kerV_{\pm}$ is subbundles of $E_{\pm}$. Let g^{kerV} be the metric on $kerV$ induced by g^E, and $\nabla^{kerV} = P^{kerV}\nabla^E$ be the induced connection on $kerV$ by orthogonal projection, then

$$\nabla^{kerV} = P^{kerV}(\nabla^{E_+} \oplus \nabla^{E_-}) = P^{kerV_+}\nabla^{E_+} \oplus P^{kerV_-}\nabla^{E_-} = \nabla^{kerV_+} \oplus \nabla^{kerV_-}$$

Proposition 1.4.7. *$ch(E) = ch(kerV)$ in $H^*(M)$, where $ch(E) = ch(E_+) - ch(E_-)$, $ch(kerV) = ch(kerV_+) - ch(kerV_-)$.*

Proof. We have an exact sequence of vector bundles:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & kerV_+ & \longrightarrow & E_+ & \xrightarrow{V_+} & E_- & \longrightarrow & kerV_- & \longrightarrow & 0 \\ & & & & \| & & \| & & & & \\ & & & & kerV_+ \oplus (kerV_+)^{\perp} & \xrightarrow{V_+} & (kerV_-)^{\perp} \oplus kerV_- & & & & \end{array}$$

since we can take any connection to compute $ch(E)$, we let

$$\nabla^{E_+} = \nabla^{kerV_+} \oplus \nabla^{(kerV_+)^{\perp}},\ \nabla^{E_-} = \nabla^{(kerV_-)^{\perp}} \oplus \nabla^{kerV_-}$$

where $\nabla^{(kerV_+)^{\perp}} = V_+^*\nabla^{(kerV_-)^{\perp}}$, then

$$\begin{aligned} ch(E) &= ch(E_+) - ch(E_-) = [ch(\nabla^{E_+})] - [ch(\nabla^{E_-})] \\ &= [ch(\nabla^{kerV_+})] - [ch(\nabla^{kerV_-})] \end{aligned}$$

□

Theorem 1.4.1. *When $t \to +\infty$, we have asymptotic formula:*

$$ch(A_t) = ch(\nabla^{kerV}) + O(\frac{1}{\sqrt{t}})$$

uniformly on compact sets.

Proof. We first compute $\nabla^{kerV,2}$. Consider the orthogonal splitting $E = kerV \oplus (kerV)^{\perp}$, $\nabla^E = \nabla^{split} + A$, where $\nabla^{split} = \nabla^{kerV} \oplus \nabla^{(kerV)^{\perp}}$. In blocked matrix with respect to the new splitting, because ∇^E is unitary,

$$\nabla^E = \begin{pmatrix} \nabla^{kerV} & 0 \\ 0 & \nabla^{(kerV)^{\perp}} \end{pmatrix} + \begin{pmatrix} 0 & -B^* \\ B & 0 \end{pmatrix}$$

Because $\forall s \in \Gamma(kerV)$, $X \in TM$,

$$(\nabla^E_X V)s = \nabla^E_X(Vs) - V(\nabla^E_X s) = -V(\nabla^E_X s) \in imV$$

$\nabla^E V$ is a 1-form with value in $End(E)$, which maps $kerV$ into $(kerV)^\perp = imV$, and $P^{(kerV)^\perp}\nabla^E_X s = -V^{-1}(\nabla^E_X V)s$, where $V : (kerV)^\perp \to imV = (kerV)^\perp$ is an isomorphism. So

$$B(X) = P^{(kerV)^\perp}\nabla^E_X \cdot P^{kerV} = -V^{-1}(\nabla^E_X V)P^{kerV},\ B(X)^* = -P^{kerV}\cdot\nabla^E_X V\cdot V^{-1}$$

or in our sign rule

$$B = V^{-1}\cdot(\nabla^E V)\cdot P^{kerV},\ B^* = -P^{kerV}\nabla^E V\cdot V^{-1}$$

so

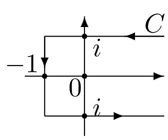
$$A \triangleq \begin{pmatrix} 0 & -B^* \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & P^{kerV}\nabla^E V\cdot V^{-1} \\ V^{-1}\nabla^E V & 0 \end{pmatrix}$$

$\nabla^{E,2} = \nabla^{split,2} + \nabla^{split}A + A^2$, since $\nabla^{split}A$ interchanges $kerV$ and $(kerV)^\perp$,

$$\begin{aligned}\nabla^{kerV,2} &= P^{kerV}\nabla^{E,2}P^{kerV} - P^{kerV}A^2P^{kerV}\\ &= P^{kerV}(\nabla^{E,2} - \nabla^E V\cdot V^{-2}\cdot\nabla^E V)P^{kerV}\end{aligned}$$

we next take the limit. $A_t^2 = tV^2 + \sqrt{t}\nabla^E V + \nabla^{E,2} \in \Gamma(\mathcal{B}_{even})$, $\sqrt{t}\nabla^E V + \nabla^{E,2}$ is nilpotent in $\mathcal{B}$ because they contain Grassmannian variables. So the spectrum of A_t^2 in $\mathcal{B}$ is the same as tV^2. Since tV^2 is positive, $Sp(A_t^2) = Sp(tV^2) \subset \mathbb{R}_+$. We choose a contour C as in the figure and use the Cauchy integral formula:

$$\exp(-A_t^2) = \frac{1}{2\pi i}\int_C \frac{\exp(-\lambda)d\lambda}{\lambda - A_t^2}$$



We can compute $(\lambda - A_t^2)^{-1}$ by the following

Lemma 1. $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a blocked matrix, then*

$$(\lambda I - M)^{-1} = \begin{pmatrix} \alpha & \alpha B(\lambda - D)^{-1} \\ (\lambda - D)^{-1}C\alpha & (\lambda - D)^{-1}(1 + C\alpha B(\lambda - D)^{-1}) \end{pmatrix}$$

where

$$\alpha = (\lambda - A - B(\lambda - D)^{-1}C)^{-1}$$

the proof is just computation.

$$\begin{aligned}A_t^2 &= \begin{pmatrix} 0 & 0 \\ 0 & tV^2 \end{pmatrix} + \sqrt{t}\begin{pmatrix} 0 & P^{kerV}\nabla^E V P^{(kerV)^\perp} \\ P^{(kerV)^\perp}\nabla^E V P^{kerV} & P^{(kerV)^\perp}\nabla^E V P^{(kerV)^\perp} \end{pmatrix} + \\ &\qquad \begin{pmatrix} P^{kerV}\nabla^{E,2}P^{kerV} & P^{kerV}\nabla^{E,2}P^{(kerV)^\perp} \\ P^{(kerV)^\perp}\nabla^{E,2}P^{kerV} & P^{(kerV)^\perp}\nabla^{E,2}P^{(kerV)^\perp} \end{pmatrix}\\ &= \begin{pmatrix} P^{kerV}\nabla^{E,2}P^{kerV} & \sqrt{t}P^{kerV}\nabla^E V P^{(kerV)^\perp} + O(1) \\ \sqrt{t}P^{(kerV)^\perp}\nabla^E V P^{kerV} + O(1) & tV^2 + O(\sqrt{t}) \end{pmatrix}\end{aligned}$$

Using the lemma, we have

$$\begin{aligned}\alpha_t &= [\lambda - P^{kerV}\nabla^{E,2}P^{kerV} - (\sqrt{t}P^{kerV}\nabla^E V P^{(kerV)^\perp} + O(1))(\lambda - tV^2 - O(\sqrt{t}))^{-1}\\ &\qquad (\sqrt{t}P^{(kerV)^\perp}\nabla^E V P^{kerV} + O(1))]^{-1}\\ &= (\lambda - P^{kerV}(\nabla^{E,2} - \nabla^E V V^{-2}\nabla^E V)P^{kerV})^{-1} + O(\frac{1}{\sqrt{t}})\\ &= (\lambda - \nabla^{kerV,2})^{-1} + O(\frac{1}{\sqrt{t}})\end{aligned}$$

and estimates for other blocks:

$$\underset{1}{\underset{\downarrow}{\alpha}}\ \underset{\sqrt{t}}{\underset{\downarrow}{B}}\ \underset{t^{-1}}{\underset{\downarrow}{(\lambda-D)^{-1}}} = O(\tfrac{1}{\sqrt{t}}),\quad \underset{t^{-1}}{\underset{\downarrow}{(\lambda-D)^{-1}}}\ (1+\underset{\sqrt{t}}{\underset{\downarrow}{C}}\ \underset{1}{\underset{\downarrow}{\alpha}}\ \underset{\sqrt{t}}{\underset{\downarrow}{B}}\ \underset{t^{-1}}{\underset{\downarrow}{(\lambda-D)^{-1}}}) = O(\tfrac{1}{t})$$

$(\lambda - D)^{-1}C\alpha = O(\frac{1}{\sqrt{t}})$

So $(\lambda - A_t^2)^{-1} = \begin{pmatrix} (\lambda - \nabla^{kerV,2})^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O(\frac{1}{\sqrt{t}})$, and

$$\exp(-A_t^2) = \frac{1}{2\pi i}\int_C \frac{\exp(-\lambda)}{\lambda - A_t^2}d\lambda = \frac{1}{2\pi i}\int_C \frac{\exp(-\lambda)}{\lambda - \nabla^{kerV,2}}P^{kerV} + O(\frac{1}{\sqrt{t}})$$

$$ch(A_t) = ch(\nabla^{kerV}) + O(\frac{1}{\sqrt{t}})$$

as $t \to +\infty$, uniformly on compact sets of M. □

Proposition 1.4.8.

$$Tr_s(\sqrt{t}V\exp(-A_t^2)) = O(\frac{1}{t}),\ \ as\ t \to +\infty$$

Proof. Consider the projection $\pi:\ M\times(0,\infty) \to M$, $\nabla^{\mathbb{E}} = \nabla^E + da\frac{\partial}{\partial a}$ is a connection on $\mathbb{E} = \pi^*E$. Let $\mathbb{A}_t = \nabla^{\mathbb{E}} + \sqrt{t}aV$ be a superconnection.

$$\mathbb{A}_t^2 = \nabla^{E,2} + ta^2V^2 + \sqrt{t}a\nabla^E V + da\sqrt{t}V$$

So

$$Tr_s(\exp(-\mathbb{A}_t^2)) = Tr_s(\exp(-A_t(a)^2)) - daTr_s(\sqrt{t}V\exp(-A_t(a)^2))$$

where $A_t(a) = \nabla^E + \sqrt{t}aV$.
Note $ker(aV) = \pi^*kerV$, by the above theorem, $Tr_s(\exp(-A_t(a)^2)){=}Tr_s(\exp(-\nabla^{kerV,2}))+O(\frac{1}{\sqrt{t}})$, as $t \to +\infty$, so

$$Tr_s(\exp(-\mathbb{A}_t^2)) = Tr_s(\exp(-\nabla^{kerV,2})) - daTr_s(\sqrt{t}V\exp(-A_t(a)^2)) + O(\frac{1}{\sqrt{t}})$$

On the other hand, by the theorem again,

$$Tr_s(\exp(-\mathbb{A}_t^2)) = Tr_s(\exp(-\nabla^{ker(aV),2})) + O(\frac{1}{\sqrt{t}}) = Tr_s(\exp(-\nabla^{kerV,2})) + O(\frac{1}{\sqrt{t}})$$

We used the fact: $\nabla^{ker(aV)} = \pi^*\nabla^{kerV} + da\frac{\partial}{\partial a}$, and so $\nabla^{ker(aV),2} = \pi^*\nabla^{kerV,2}$. Comparing the above two equation, and note that $A_t(1) = A_t$, we see that

$$Tr_s(\sqrt{t}V\exp(-A_t^2)) = O(\frac{1}{\sqrt{t}})$$

□

By the above proposition, we have

$$\frac{\partial}{\partial t}Tr_s\exp(-A_t^2) = -dTr_s(\frac{V}{2\sqrt{t}}\exp(-A_t^2)) = \begin{cases} O(\frac{1}{t^{\frac{3}{2}}}) & \text{as}\, t \to +\infty \\ O(\frac{1}{\sqrt{t}}) & \text{as}\, t \to 0 \end{cases}$$

So we can integrate it from 0 to $+\infty$.

Definition 1.4.1. $\beta = \frac{1}{\sqrt{2\pi i}}\int_0^{+\infty}\varphi Tr_s(\frac{V}{2\sqrt{t}}\exp(-A_t^2))dt$

Theorem 1.4.2. *β is an odd real form and*

$$d\beta = ch(\nabla^E) - ch(\nabla^{kerV})$$

Proof. By the Remark following proposition 1.4.5, β is an odd real form. By theorem 1.4.1 and proposition 1.4.4

$$\begin{aligned} Tr_s(\exp(-\nabla^{kerV,2})) - Tr_s(\exp(-\nabla^{E,2})) &= \int_0^\infty \frac{\partial}{\partial t} Tr_s(\exp(-A_t^2))dt \\ &= -d\int_0^\infty Tr_s(\frac{V}{2\sqrt{t}}\exp(-A_t^2))dt \end{aligned}$$

Since $\varphi \circ d = \frac{1}{\sqrt{2i\pi}} d \circ \varphi$,

$$\varphi Tr_s(\exp(-\nabla^{E,2}) - \varphi Tr_s(\exp(-\nabla^{kerV,2})) = \frac{1}{\sqrt{2i\pi}} d\int_0^\infty \varphi Tr_s(\frac{V}{2\sqrt{t}}\exp(-A_t^2))dt$$

i.e. $ch(\nabla^E) - ch(\nabla^{kerV}) = d\beta$. □

We will show β is a Chern-Simon class. First we generalize the definition.

Theorem 1.4.3. *Given an exact sequence of vector bundles:*

$$E:\ 0 \longrightarrow E_1 \xrightarrow{v_1} E_2 \xrightarrow{v_2} \cdots \xrightarrow{v_{n-1}} E_n \longrightarrow 0$$

and connections $\nabla^E = \{\nabla^{E_i}\}$ on $E = \{E_i, v_i\}$, there is a unique way to associate a class $\widetilde{ch}(\nabla^E)$ in $\Omega^{odd}(M)/d\Omega^{even}$, which satisfies 3 properties:

1. *$d\widetilde{ch}(E) = ch(\nabla^{E_1}) - ch(\nabla^{E_2}) + \ldots + (-1)^{n-1}ch(\nabla^{E_n}) =: ch(\nabla^E)$*

2. *If the sequence (E, ∇^E) splits, $\widetilde{ch}(\nabla^E) = 0$. "$(E, \nabla^E)$ splits" means:*

$$E:\ 0 \longrightarrow E_1 \xrightarrow{v_1} E_2' \oplus E_2'' \xrightarrow{v_2} \cdots \xrightarrow{v_{n-1}} E_n \longrightarrow 0$$

and $\nabla^{E_i} = \nabla^{E_i'} \oplus \nabla^{E_i''}$, $\nabla^{E_i''} = v_i^*\nabla^{E_{i+1}'}$, where $E_i' = imv_{i-1} = kerv_i$, $E_i'' \cong E_i/E_i'$.

3. *$\widetilde{ch}(\nabla^E)$ is functorial.*

Proof. Fix a splitting: $E_i = E_i' \oplus E_i''$, $\nabla^{E_i,split} = \nabla^{E_i'} \oplus \nabla^{E_i''}$, $\nabla^{E_i''} = v_i^*\nabla^{E_{i+1}'}$. Let

$$\alpha(\nabla^E) = \sum_{i=1}^n (-1)^{i-1}\widetilde{ch}(\nabla^{E_i,split}, \nabla^{E_i})$$

where $\widetilde{ch}(\nabla^E, \nabla'^E)$ is the Chern-Simon class defined before(see propositin 1.4.2). We'll show $\alpha(\nabla^E)$ satisfies the 3 properties.

1. Since $\sum_{i=1}^n (-1)^{i-1}ch(\nabla^{E_i,split}) = 0$,

$$\begin{aligned} d\alpha(\nabla^E) &= \sum_{i=1}^n (-1)^{i-1} d\widetilde{ch}(\nabla^{E_i,split}, \nabla^{E_i}) = \sum_{i=1}^n (-1)^{i-1}(ch(\nabla^{E_i}) - ch(\nabla^{E_i,split})) \\ &= \sum_{i=1}^n (-1)^{i-1}ch(\nabla^{E_i}) = ch(\nabla^E) \end{aligned}$$

2. First consider another splitting of the kind: $E_i = E_i'\tilde{\oplus}\tilde{E}_i''$, $\nabla^{E_i} = \nabla^{E_i'}\tilde{\oplus}\nabla^{\tilde{E}_i''}$, $\nabla^{\tilde{E}_i''} = v_i^*\nabla^{E_i'}$, i.e. the bundles have a new splitting $E_i = E_i'\tilde{\oplus}E_i''$, but we use the fixed $\nabla^{E_i'}$. They together determine the new splitting connections $\nabla^{E_i} = \nabla^{E_i'}\tilde{\oplus}\nabla^{\tilde{E}_i''}$.

Claim 1. $\widetilde{ch}(\nabla^{E_i,split}, \nabla^{E_i}) = 0$.

proof of the Claim. Let $\tilde{P}^{E_i'}$ and $\tilde{P}^{\tilde{E}_i''}$ be the projection w.r.t this new splitting, we have isomorphisms

$$(E_i'', \nabla^{E_i''}) \underset{P^{E_i''}|_{\tilde{E}_i''}\,:\,\tilde{E}_i'' \overset{\cong}{\to} E_i''}{\overset{\tilde{P}^{\tilde{E}_i''}|_{E_i''}\,:\,E_i'' \overset{\cong}{\to} \tilde{E}_i''}{\rightleftarrows}} (\tilde{E}_i'', \nabla^{\tilde{E}_i''})$$

So $\nabla^{E_i} = \nabla^{E_i'}\tilde{\oplus}(P^{E_i''})^*(\nabla^{E_i''})$. $\forall s \in E$, $s = u + v = \tilde{u} + \tilde{v}$ is decompositions w.r.t the two splittings, we calculate

$$\begin{aligned} P^{E_i'}(\nabla^{E_i'}\tilde{\oplus}(P^{E_i''})^*(\nabla^{E_i''}))P^{E_i'}s &= P^{E_i'}\nabla^{E_i'}\tilde{P}^{E_i'}P^{E_i'}s + P^{E_i'}(P^{E_i''})^{-1}\nabla^{E_i''}P^{E_i''}\tilde{P}^{\tilde{E}_i''}P^{E_i'}s \\ &= \nabla^{E_i'}P^{E_i'}s \end{aligned}$$

$$\begin{aligned} P^{E_i''}(\nabla^{E_i'}\tilde{\oplus}(P^{E_i''})^*(\nabla^{E_i''}))P^{E_i''}s &= P^{E_i''}\nabla^{E_i'}\tilde{P}^{E_i'}P^{E_i''}s + P^{E_i''}((P^{E_i''})^{-1}\nabla^{E_i''}P^{E_i''})\tilde{P}^{\tilde{E}_i''}P^{E_i''}s \\ &= \nabla^{E_i''}P^{E_i''}s \end{aligned}$$

$$P^{E_i''}(\nabla^{E_i'}\tilde{\oplus}(P^{E_i''})^*(\nabla^{E_i''}))P^{E_i'}s = P^{E_i''}\nabla^{E_i'}\tilde{P}^{E_i'}P^{E_i'}s + P^{E_i''}(P^{E_i''})^{-1}\nabla^{E_i''}P^{E_i''}\tilde{P}^{\tilde{E}_i''}P^{E_i'}s = 0$$

$$\begin{aligned} P^{E_i'}(\nabla^{E_i'}\tilde{\oplus}(P^{E_i''})^*(\nabla^{E_i''}))P^{E_i''}s &= P^{E_i'}\nabla^{E_i'}\tilde{P}^{E_i'}P^{E_i''}s + P^{E_i'}(P^{E_i''})^{-1}\nabla^{E_i''}P^{E_i''}\tilde{P}^{\tilde{E}_i''}P^{E_i''}s \\ &= \nabla^{E_i'}\tilde{P}^{E_i'}P^{E_i''}s =: B \end{aligned}$$

In the above calculation, we use some identities, like

$$\tilde{P}^{E_i'}P^{E_i'} = P^{E_i'},\ \tilde{P}^{E_i''}P^{E_i'} = 0,\ \tilde{P}^{\tilde{E}_i''}P^{E_i''} = P^{E_i''},\ P^{E_i'}(P^{E_i''})^{-1} = P^{E_i'}\tilde{P}^{\tilde{E}_i''} = 0$$

So with respect to the initial splitting $E_i = E_i' \oplus E_i''$, we have

$$\nabla^{E_i} = \begin{pmatrix} \nabla^{E_i'} & 0 \\ 0 & \nabla^{E_i''} \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} := \nabla^{E_i,split} + D$$

Let $\nabla_t^{E_i} = \nabla^{E_i,split} + tD$ be a curve connecting $\nabla^{E_i,split}$ and ∇^{E_i}. It's easy to see that

$$ch(\nabla_t^{E_i}) \equiv ch(\nabla^{E_i,split})$$

So $\widetilde{ch}(\nabla^{E_i,split}, \nabla^{E_i}) = 0$. □

Now we consider the general splitting $E_i = E_i'\tilde{\oplus}\tilde{E}_i''$, $\tilde{\nabla}^{E_i} = \tilde{\nabla}^{E_i'}\tilde{\oplus}\tilde{\nabla}^{\tilde{E}_i''}$, $\tilde{\nabla}^{\tilde{E}_i''} = v_i^*\tilde{\nabla}^{E_{i+1}'}$. From the definition, we can see that

$$\widetilde{ch}(\nabla^{E_i,split}, \tilde{\nabla}^{E_i}) = \widetilde{ch}(\nabla^{E_i,split}, \nabla^{E_i}) + \widetilde{ch}(\nabla^{E_i}, \tilde{\nabla}^{E_i})$$

By the above construction, we have

$$\begin{aligned} \widetilde{ch}(\nabla^{E_i}, \tilde{\nabla}^{E_i}) &= \widetilde{ch}(\nabla^{E_i'}\tilde{\oplus}\nabla^{\tilde{E}_i''}, \tilde{\nabla}^{E_i'}\tilde{\oplus}\tilde{\nabla}^{\tilde{E}_i''}) \\ &= \widetilde{ch}(\nabla^{E_i'}, \tilde{\nabla}^{E_i'}) + \widetilde{ch}(v_i^*\nabla^{E_{i+1}'}, v_i^*\tilde{\nabla}^{E_{i+1}'}) \\ &= \widetilde{ch}(\nabla^{E_i'}, \tilde{\nabla}^{E_i'}) + \widetilde{ch}(\nabla^{E_{i+1}'}, \tilde{\nabla}^{E_{i+1}'}) \end{aligned}$$

Using the Claim,

$$\begin{aligned}
\alpha(\tilde{\nabla}^E) &= \sum_{i=1}^{n}(-1)^{i-1}\widetilde{ch}(\nabla^{E_i,split},\tilde{\nabla}^{E_i}) \\
&= \sum_{i=1}^{n}(-1)^{i-1}(\widetilde{ch}(\nabla^{E_i,split},\nabla^{E_i})+\widetilde{ch}(\nabla^{E_i},\tilde{\nabla}^{E_i})) \\
&= \sum_{i=1}^{n}(-1)^{i-1}\widetilde{ch}(\nabla^{E_i},\tilde{\nabla}^{E_i}) \\
&= \sum_{i=1}^{n}(-1)^{i-1}(\widetilde{ch}(\nabla^{E'_i},\tilde{\nabla}^{E'_i})+\widetilde{ch}(\nabla^{E'_{i+1}},\tilde{\nabla}^{E'_{i+1}})) \\
&= 0
\end{aligned}$$

3. By the functoriality of $\widetilde{ch}(\nabla^E,\nabla'^E)$, this is clear.

So we have the existence. We prove the uniqueness. The proof is same as the proof of proposition 1.4.2.
Assume $\alpha(\nabla^E)$ is any class satisfying the 3 properties. Let ∇^E_t be any path connecting $\nabla^{E,split}=\{\nabla^{E_i,split}\}$ and $\nabla^E=\{\nabla^{E_i}\}$. Let $\nabla^{\mathbb{E}}=\nabla^E_t+dt\frac{\partial}{\partial t}$, then $\nabla^{\mathbb{E}}|_{t=0}=\nabla^{E,split}$, $\nabla^{\mathbb{E}}|_{t=1}=\nabla^E$, so

$$\begin{aligned}
\int_{[0,1]}ch(\nabla^{\mathbb{E}}) &= \int_{[0,1]}d\alpha(\nabla^{\mathbb{E}}) \\
&= -d\int_{[0,1]}\alpha(\nabla^{\mathbb{E}})+\alpha(\nabla^{\mathbb{E}})|_{t=1}-\alpha(\nabla^{\mathbb{E}})|_{t=0} \\
&= -d\int_{[0,1]}\alpha(\nabla^{\mathbb{E}})+\alpha(\nabla^E)
\end{aligned}$$

Note that $\int_{[0,1]}ch(\nabla^{\mathbb{E}})$ is what we've constructed. □

We can now show $\beta=\frac{1}{\sqrt{2i\pi}}\int_0^\infty\varphi Tr_s(\frac{V}{2\sqrt{t}}\exp(-A_t^2))dt$ is a Chern-Simon class.

Theorem 1.4.4. *We have an exact sequence of vector bundles*

$$0\longrightarrow kerV_+\longrightarrow E_+\xrightarrow{V_+}E_-\longrightarrow cokerV_+\cong kerV_-\longrightarrow 0$$

and corresponding connections ∇^{kerV_+}, ∇^{E_+}, ∇^{E_-}, ∇^{kerV_-}.
$\beta=-\widetilde{ch}$ *in* $\Omega^{odd}/d\Omega^{even}$.

Proof. By proposition 1.4.2, β is a real odd form, and

$$d\beta=ch(\nabla^E)-ch(\nabla^{kerV})=-(ch(\nabla^{kerV_+})-ch(\nabla^{E_+})+ch(\nabla^{E_-})-ch(\nabla^{kerV_-}))$$

The functorial property is easy. We need to prove β satisfies the 2nd property. If the sequence splits, we assume:

$$\nabla^{E_+}=\nabla^{kerV_+}\oplus\nabla^{(kerV_+)^\perp},\ \nabla^{E_-}=\nabla^{kerV_-}\oplus\nabla^{(kerV_-)^\perp},\ \nabla^{(kerV_+)^\perp}=V_+^*\nabla^{(kerV_-)^\perp}$$

Under the splitting $E=E_+\oplus E_-=kerV_+\oplus(kerV_+)^\perp\oplus kerV_-\oplus(kerV_-)^\perp$, we have

$$\nabla^E=\begin{pmatrix}\begin{pmatrix}\nabla^{kerV_+} & 0\\ 0 & \nabla^{(kerV_+)^\perp}\end{pmatrix} & 0\\ 0 & \begin{pmatrix}\nabla^{kerV_-} & 0\\ 0 & \nabla^{(kerV_-)^\perp}\end{pmatrix}\end{pmatrix}$$

$$V = \begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 \\ 0 & V_- \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & V_+ \end{pmatrix} & 0 \end{pmatrix}, \nabla^E V = \begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 \\ 0 & \nabla^E V_- \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & \nabla^E V_+ \end{pmatrix} & 0 \end{pmatrix} = 0$$

$$\begin{aligned} A_t^2 &= \nabla^{E,2} + \sqrt{t}\nabla^E V + tV^2 \\ &= \begin{pmatrix} \begin{pmatrix} \nabla^{kerV_+,2} & 0 \\ 0 & \nabla^{(kerV_+)^\perp,2} + tV_-V_+ \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \nabla^{kerV_-,2} & 0 \\ 0 & \nabla^{(kerV_-)^\perp,2} + tV_+V_- \end{pmatrix} \end{pmatrix} \end{aligned}$$

So $Tr_s(V\exp(-A_t^2)) = Tr_s \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 0$, so $\beta = 0$.

So β is indeed the Chern-Simon class of the sequence. □

Chapter 2

Atiyah-Singer Family Index Theorem

2.1 Family Index Theorem

$X \longrightarrow M \xrightarrow{\pi} S$ is a fibration with compact fibre X of dim $2l$. TX (subbundle of TM) is even dimensional, orientable and spin. This is a global condition over M. g^{TX} is a metric on TX. E is a complex vector bundle on M with Hermitian metric g^E and unitary connection ∇^E. $S^{TX} = S_+^{TX} \oplus S_-^{TX}$ is spinor bundle of TX, rank$(S^{TX}) = 2^{\frac{dim X}{2}} = 2^l$. $\forall s \in S$, D_s^X is the Dirac operator along the fibre X_s with twisting bundle $E|_{X_s}$, which acts on $\Gamma(X_s, S^{TX} \otimes E|_{X_x})$. $D_s^X = \begin{pmatrix} 0 & D_{-,s}^X \\ D_{+,s}^X & 0 \end{pmatrix}$ is self-adjoint, it's Fredholm. $Ind(D_{+,s}^X) = dimker D_{+,s}^X - dimker D_{-,s}^X \in \mathbb{Z}$ does not depend on s by the homotopy invariance of the index. By Atiyah-Singer index theorem,

$$Ind D_{+,s}^X = \int_{X_s} \hat{A}(TX) ch(E)$$

The idea of family index is to see $Ind D_+^X$ as "vector bundle" on S, s.t. "rank"$(Ind\, D_+^X)$=classical index.

Suppose that $ker D_+^X$ and $ker D_-^X$ have constant dimensions, then one can prove easily that $ker D_{\pm,s}^X$ are the fibres of smooth vector bundles. Let $\mathcal{V}(M)$ denote the class of vector bundles on M, then we have

$$\begin{array}{rcl} \mathcal{V}(M) & \longrightarrow & \mathcal{V}(S) \\ E & \mapsto & ker D_-^X \ominus ker D_-^X \end{array}$$

One of the purposes of topological K-theory is to make sense such a $\ominus$ sign, the idea is analogous to the idea of extending natural numbers $\mathbb{N}$ to integers $\mathbb{Z}$. But for vector bundles we must consider stable equivalent relations rather than equality.

Definition 2.1.1. $E, E' \in \mathcal{V}(M)$, $E \overset{s}{\sim} E' \Leftrightarrow \exists F \in \mathcal{V}(M), s.t. E \oplus F \cong E' \oplus F$. E and E' are called stable equivalent. Let $\mathcal{W}(M) = \mathcal{V}(M)/ \overset{s}{\sim}$. Define a equivalent relation on $\mathcal{V}(M) \times \mathcal{V}(M)$:

$$(E, E') \sim (F, F') \Leftrightarrow E \oplus F' = E' \oplus F \,\text{in}\, \mathcal{W}(M) \Leftrightarrow \exists H \in \mathcal{V}(M), \text{s.t.}\, E \oplus F' \oplus H = E' \oplus F \oplus H$$

The K-group of M is defined to be $K(M) = \{(E, E') : E, E' \in \mathcal{V}(M)\}/ \sim$.

Remark 6. There are several equivalent definitions of $K(M)$. See ref.

If $E \overset{s}{\sim} E'$, then $ch(E) = ch(E')$. It's clear the map $ch : \mathcal{V}(M) \longrightarrow H^*(M)$ induces

$$ch : K(M) \longrightarrow H^*(M)$$

Assume again that $kerD^X_\pm$ are smooth vector bundles, (it's sufficient to assume constant dimensions) then

$$Ind(D^X_+) = kerD^X_+ - kerD^X_- \in K(S)$$

If the assumption is not verified, we can add a finite dimensional space, and find a surjective homomorphism $\tilde{D}_+ : S^{TX}_+ \oplus \mathbb{C}^n \to S^{TX}_-$, for which the assumption is verified, then

$$Ind(D^X_+) = Ind(\tilde{D}_+) - \mathbb{C}^n \in K(S)$$

Theorem 2.1.1 (Family Index Theorem)**.**

$$ch(Ind(D^X_+)) = \pi_*[\hat{A}(TX)ch(E)]$$

in $H^{even}(S, \mathbb{Q})$*, i.e. the diagram commutes:*

$$\begin{array}{ccc} K(M) & \xrightarrow[\pi_!]{Ind(D^X_+)} & K(S) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H^{even}(M,\mathbb{Q}) & \xrightarrow{\pi_*[\hat{A}(TX)\cdot]} & H^{even}(S,\mathbb{Q}) \end{array}$$

Example 2.1.1. If S is a single point, $\mathcal{V}(S)$ =vector spaces, $K(S) = \mathbb{Z}$, $ch(E \ominus F) = dimE - dimF$. We get the index theorem

$$Ind(D^X_+) = \int_X \hat{A}(TX)ch(E)$$

2.2 Consequences of the Family Index Theorem

Assume S is even dimensional, compact and spin and we have a given horizontal space $T^H M$ which is a subbundle of TM. So $T^H M \cong \pi^*TS$, $TM = T^H M \oplus TX \cong \pi^*TS \oplus TX$. Let $g^{TM} = g^{TX} \oplus \pi^* g^{TS}$. Then TM is spin, and $S^{TM} = S^{TX} \hat{\otimes} \pi^* S^{TS}$. Since $\hat{A}$ is multiplicative, we have $\hat{A}(TM) = \hat{A}(TX)\pi^*\hat{A}(TS)$. So

$$\begin{aligned} Ind(D^{M,E}_+) &= \int_M \hat{A}(TM)ch(E) = \int_M \pi^*(\hat{A}(TS))\hat{A}(TX)ch(E) \\ &= \int_S \hat{A}(TS)\pi_*[\hat{A}(TX)ch(E)] \overset{F.I.T}{=} \int_S \hat{A}(TS)ch(Ind\, D^X_+) \\ &= Ind(D^{S,Ind\, D^X_+}_+) \end{aligned} \tag{2.1}$$

Claim 2. (2.1) is equivalent to family index theorem:

Proof. Let F be a vector bundle on S, (2.1) is valid when $E \rightsquigarrow E \otimes \pi^* F$. By the same computation of (2.1)

$$Ind(D^{M,E\otimes\pi^*F}_+) = \int_S \hat{A}(TS)ch(F)\pi_*[\hat{A}(TX)ch(E)]$$

Note that $Ind(D^{X,E\otimes\pi^*F}_+) = F \otimes Ind(D^{X,E}_+)$, so by A-S index theorem,

$$Ind\, D^{S,Ind(D^{X,E\otimes\pi^*F}_+)}_+ = \int_S \hat{A}(TS)ch(F)ch(Ind\, D^{X,E}_+)$$

Since $ch(F)$ generate the full $H^{even}(S, \mathbb{Q})$ as F varies, compare the above two equation, we have

$$\hat{A}(TS)\pi_*[\hat{A}(TX)ch(E)] = \hat{A}(TS)ch(Ind\, D^{X,E}_+)$$

Since $\hat{A} = 1 + \cdots$ is invertible in $H^{even}(S, \mathbb{Q})$, so

$$\pi_*[\hat{A}(TX)ch(E)] = ch(Ind\, D^{X,E}_+)$$

□

Remark 7. Atiyah-Singer proved the family index theorem by proving (2.1).

2.3 Adiabatic Limit

From now on, we assume $g_\varepsilon = g_\varepsilon^{TM} = g^{TX} \oplus \frac{1}{\varepsilon}\pi^* g^{TS}$, then $T^H M = (TX)^\perp \cong \pi^* TS$, and $\pi : M \to S$ is a Riemannian submersion. Let $\nabla^{TM,\varepsilon}$ denote the Levi-Civita connection of g_ε^{TM}, $\nabla^{TM,L} = \nabla^{TM,1}$ denote the Levi-Civita connection of $g^{TM} = g^{TX} \oplus \pi^* g^{TS} = g_1^{TM}$.

We want to calculate the Dirac operator D_ε^M using the metric g_ε^{TM}.

Theorem 2.3.1. *As $\varepsilon \to 0$, $\nabla^{TM,\varepsilon} = \begin{pmatrix} \nabla^{TX} & * \\ 0 & \pi^*\nabla^{TS} \end{pmatrix} + O(\varepsilon) =: \nabla^{TM,0} + O(\varepsilon)$.*

Proof. $\forall U \in TS$, U^H denotes the lift of U to $T^H M$. If U is a smooth vector field on S, then U^H is a smooth vector field on M. Let φ_t, ψ_t be one-parameter transformation group generated by U and U^H, then $\varphi_t \circ \pi = \pi \circ \psi_t$, and $\psi_t : X_s \to X_{\varphi_t(s)}$ is a diffeomorphism. So if V is a smooth section of TX, then $[U^H, V] \in TX$.
By properties of connection, we can assume X is either vertical or $X = U^H \in T^H M$ in the following proof. We have defining equation of the Levi-Civita connection:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle Z, [X,Y]\rangle - \langle Y, [X,Z]\rangle - \langle X, [Y,Z]\rangle \qquad (2.2)$$

Using this formula, we calculate all possible cases:

1. Y, Z are vertical, X is vertical
$$\langle \nabla_X^{TM,\varepsilon} Y, Z\rangle_{g^{TX}} = \langle \nabla_X^{TX} Y, Z\rangle_{g^{TX}}$$

2. Y, Z are vertical, $X = U^H$ is horizontal
$$\begin{aligned} 2\langle \nabla_{U^H}^{TM,\varepsilon} Y, Z\rangle_{g^{TX}} &= U^H\langle Y, Z\rangle_{g^{TX}} + \langle Z, [U^H, Y]\rangle_{g^{TX}} - \langle Y, [U^H, Z]\rangle_{g^{TX}} \\ &= 2\langle [U^H, Y], Z\rangle + (L_{U^H} g^{TX})(Y, Z) \\ &= 2\langle [U^H, Y], Z\rangle + \langle (g^{TX})^{-1} L_{U^H} g^{TX}(Y), Z\rangle \end{aligned}$$
So $P^{TX}\nabla_{U^H}^{TM,\varepsilon} Y = [U^H, Y] + \frac{1}{2}((g^{TX})^{-1} L_{U^H} g^{TX})Y$.

3. $Y = V^H$ is horizontal, Z is vertical, X is vertical
$$2\langle \nabla_X^{TM,\varepsilon} V^H, Z\rangle_{g^{TX}} = V^H\langle X, Z\rangle_{g^{TX}} - \langle [V^H, X], Z\rangle_{g^{TX}} - \langle X, [V^H, Z]\rangle_{g^{TX}} = (L_{V^H} g^{TX})(X, Z)$$
So $P^{TX}\nabla_X^{TM,\varepsilon} V^H = \frac{1}{2}(g^{TX})^{-1}(L_{V^H} g^{TX})(X)$.

4. $Y = V^H$ is horizontal, Z is vertical, $X = U^H$ is horizontal
$$2\langle \nabla_{U^H}^{TM,\varepsilon} V^H, Z\rangle_{g^{TX}} = \langle P^{TX}[U^H, V^H], Z\rangle_{g^{TX}}$$
So $P^{TX}\nabla_{U^H} V^H = \frac{1}{2} P^{TX}[U^H, V^H]$.

5. Y is vertical, $Z = W^H$ is horizontal, X is vertical
$$2\langle \nabla_X^{TM,\varepsilon} Y, W^H\rangle_{\pi^* g^{TS}} = -\varepsilon (L_{W^H} g^{TX})(X, Y)$$
So $P^{T^H M}\nabla_X^{TM,\varepsilon} Y = O(\varepsilon)$.

6. Y is vertical, $Z = W^H$ is horizontal, $X = U^H$ is horizontal
$$2\langle \nabla_{U^H}^{TM,\varepsilon} Y, W^H\rangle_{\pi^* g^{TS}} = -\varepsilon\langle Y, P^{TX}[U^H, W^H]\rangle_{g^{TX}}$$
So $P^{T^H M}\nabla_{U^H}^{TM,\varepsilon} Y = O(\varepsilon)$

7. $Y = V^H$, $Z = W^H$ are horizontal, X is vertical

$$2\langle \nabla_X^{TM,\varepsilon} V^H, W^H \rangle_{\pi^* g^{TS}} = -\varepsilon \langle X, P^{TX}[V^H, W^H] \rangle_{g^{TX}}$$

So $P^{T^H M} \nabla_X^{TM,\varepsilon} V^H = O(\varepsilon)$.

8. $Y = V^H$, $Z = W^H$ are horizontal, $X = U^H$ is horizontal

$$\langle \nabla_{U^H}^{TM,\varepsilon} V^H, W^H \rangle_{\pi^* g^{TS}} = \langle \nabla_U^{TS} V, W \rangle_{g^{TS}}$$

So $P^{T^H M} \nabla_{U^H}^{TM,\varepsilon} V^H = (\nabla_U^{TS} V)^H$.

Write in matrix,

$$\nabla^{TM,\varepsilon} = \begin{pmatrix} P^{TX} \nabla^{TM,\varepsilon} P^{TX} & P^{TX} \nabla^{TM,\varepsilon} P^{T^H M} \\ P^{T^H M} \nabla^{TM,\varepsilon} P^{TX} & P^{T^H M} \nabla^{TM,\varepsilon} P^{T^H M} \end{pmatrix} = \begin{pmatrix} \nabla^{TX} & * \\ O(\varepsilon) & \pi^* \nabla^{TS} + O(\varepsilon) \end{pmatrix}$$

where $* = P^{TX} \nabla^{TM,\varepsilon} P^{T^H M}$ is determined by above calculations in cases 3 and 4. The theorem is proved. □

From above calculations in cases 1 and 2, we see that

Theorem 2.3.2. *The connection $\nabla^{TX} = P^{TX} \nabla^{TM,\varepsilon} P^{TX}$ is characterized by the following two properties:*

1. *∇^{TX} restricts to the Levi-Civita connection of (X, g^{TX}) along the fibre X.*

2. *$\forall U \in TS$, $Y \in TX$, $\nabla_{U^H}^{TX} Y = [U^H, Y] + \frac{1}{2}((g^{TX})^{-1} L_{U^H} g^{TX}) Y$.*

Remark 8. ∇^{TX} preserves the metric g^{TX}. In general, if ∇^E is any connection on an Hermitian vector bundle E, we can modify it to be metric preserving: $\nabla'^E = \nabla^E + \frac{1}{2}(g^E)^{-1}(\nabla^E g^E)$.

$$\begin{aligned} X\langle s, s' \rangle &= (\nabla_X^E g^E)(s, s') + \langle \nabla_X^E s, s' \rangle + \langle s, \nabla_X^E s' \rangle \\ &= \langle \frac{1}{2}(g^E)^{-1}(\nabla_X^E g^E) s, s' \rangle + \langle s, \frac{1}{2}(g^E)^{-1}(\nabla_X^E g^E) s' \rangle + \langle \nabla_X^E s, s' \rangle + \langle s, \nabla_X^E s' \rangle \\ &= \langle \nabla_X'^E s, s' \rangle + \langle s, \nabla_X'^E s' \rangle \end{aligned}$$

Let $\nabla^\oplus = \nabla^{TX} \oplus \pi^* \nabla^{TS}$ denote the splitting connection, we write

$$S^\varepsilon = \nabla^{TM,\varepsilon} - \nabla^\oplus, \; S = \nabla^{TM,L} - \nabla^\oplus$$

Proposition 2.3.1. *Assume $\nabla^{TM,L}$ is the Levi-Civita connection of g^{TM}, and ∇ is a metric preserving connection with torsion T. Let $\nabla^{TM,L} = \nabla + S$, then*

$$2\langle S(X)Y, Z \rangle = \langle T(X,Z), Y \rangle + \langle T(Y,Z), X \rangle - \langle T(X,Y), Z \rangle \tag{2.3}$$

We first evaluate the torsion T of $\nabla^\oplus$.

Proposition 2.3.2.

1. *T takes value in TX;*

2. *T vanishes on $TX \times TX$;*

3. *$T(U^H, V^H) = -P^{TX}[U^H, V^H]$;*

4. *$T(U^H, A) = \frac{1}{2}(g^{TX})^{-1}(L_{U^H} g^{TX}) A$, $U \in TS$, $A \in TX$.*

The proof is straightforward computation. By above two proposition, we have the properties of S and S^ε:

Proposition 2.3.3. *1.* $P^{TX}S^\varepsilon = P^{TX}S$

2. $P^{T^HM}S^\varepsilon = \varepsilon P^{T^HM}S$

3. $S(\cdot)$ *maps* TX *into* T^HM

4. $\langle S(U^H)V^H, W^H\rangle = 0$, *for* $U, V, W \in TS$

Proof. 1. by (2.3)

2. by (2.3) and properties 3,4 in proposition 2.3.2

3. because $\nabla^{TX} = P^{TX}\nabla^{TM,L}P^{TX} = P^{TX}\nabla^{\oplus}P^{TX}$

4. by (2.3)

□

Remark 9. The proposition also follows from the proof of theorem (2.3.1), and the formula

$$S^\varepsilon = \begin{pmatrix} 0 & * \\ O(\varepsilon) & O(\varepsilon) \end{pmatrix}$$

Theorem 2.3.3.

$$D_\varepsilon^M = D^X + \sqrt{\varepsilon}D^H - \frac{\varepsilon c(T^H)}{4} \tag{2.4}$$

- D^X *is the Dirac operator along the fibre:* $D^X = \sum_i c(e_i)\nabla_{e_i}^{S^{TX}\otimes E}$, $\{e_i\}$ *is a orthonormal basis of* TX.
- T^H *is the* HH *component of* T*:* $T^H(U^H, V^H) = -P^{TX}[U^H, V^H]$, *and*

  $$c(T^H) = \frac{1}{2}\langle T^H(f_\alpha^H, f_\beta^H), e_i\rangle c(f_\alpha)c(f_\beta)c(e_i)$$

  $\{f_\alpha\}$ *is a orthonormal basis of* TS.
- $D^H = c(f_\alpha)(\nabla_{f_\alpha^H}^{S^{TX}\otimes S^{TS}\otimes E} + k(f_\alpha))$, $k(U) \triangleq \frac{1}{2}div^X(U^H) = \frac{1}{4}Tr[(g^{TX})^{-1}L_{U^H}g^{TX}]$

Proof.

$$D_\varepsilon^M = c(e_i)\nabla_{e_i}^\varepsilon + c(f_\alpha)\nabla_{\sqrt{\varepsilon}f_\alpha^H}^\varepsilon \tag{2.5}$$

∇^ε is the induced connection on the spinor bundle $S^{TM} = S^{TX}\hat{\otimes}\pi^*S^{TS}$ by the Levi-Civita connection $\nabla^{TM,\varepsilon}$. Note that the Clifford multiplication doesn't change when rescaling the horizontal metric. We have $\nabla^{TM,\varepsilon} = \nabla^{TM} + S^\varepsilon$, and Lie algebra homomorphism:

$$\begin{array}{ccc} so(n) & \longrightarrow & spin(n) \subset Cl_n \\ A & \mapsto & c(A) = \frac{1}{4}\langle Ae_i, e_j\rangle c(e_i)c(e_j) \end{array}$$

so we get

$$\nabla^\varepsilon = \nabla^{S^{TX}} + \nabla^{S^{TS}} + \frac{1}{4}\langle S^\varepsilon(\cdot)e_i, e_j\rangle_{g_\varepsilon} c(e_i)c(e_j) + \frac{1}{4}\langle S^\varepsilon(\cdot)\sqrt{\varepsilon}f_\alpha^H, \sqrt{\varepsilon}f_\beta^H\rangle_{g_\varepsilon} + \frac{1}{2}\langle S^\varepsilon(\cdot)\sqrt{\varepsilon}f_\alpha^H, e_i\rangle_{g_\varepsilon} c(f_\alpha)c(e_i) \tag{2.6}$$

$\nabla^{S^{TX}}$ and $\nabla^{S^{TS}}$ is the connection on spinors induced by ∇^{TX} and ∇^{TS}. By (2.3), proposition 2.3.3, and propostion 2.3.2

- $\langle S^\varepsilon(\cdot)e_i, e_j\rangle_{g_\varepsilon} = 0$
- $\langle S^\varepsilon(\cdot)\sqrt{\varepsilon}f_\alpha^H, \sqrt{\varepsilon}f_\beta^H\rangle_{g_\varepsilon} = \varepsilon\langle S(\cdot)f_\alpha^H, f_\beta^H\rangle$
  $c(e_i)\cdot\frac{1}{4}\langle S(e_i)f_\alpha^H, f_\beta^H\rangle c(f_\alpha)c(f_\beta) = \frac{1}{8}\langle T(f_\alpha^H, f_\beta^H), e_i\rangle c(f_\alpha)c(f_\beta)c(e_i)$
  $\langle S(f_\gamma^H)f_\alpha^H, f_\beta^H\rangle = 0.$
- $\langle S^\varepsilon(\cdot)\sqrt{\varepsilon}f_\alpha^H, e_i\rangle_{g_\varepsilon} = \sqrt{\varepsilon}\langle S(\cdot)f_\alpha^H, e_i\rangle$
  $c(f_\alpha)\cdot\frac{1}{2}\langle S(f_\alpha^H)f_\beta^H, e_i\rangle c(f_\beta)c(e_i) = -\frac{1}{4}\langle T(f_\alpha^H, f_\beta^H), e_i\rangle c(f_\alpha)c(f_\beta)c(e_i)$

$$\begin{aligned}
c(e_i)\cdot\frac{1}{2}\langle S(e_i)f_\alpha^H, e_j\rangle c(f_\alpha)c(e_j) &= \frac{1}{4}(\langle T(f_\alpha^H, e_j), e_i\rangle - \langle T(e_i, f_\alpha^H), e_j\rangle)c(e_i)c(f_\alpha)c(e_j)\\
&= \sum_i \frac{1}{2}\langle T(f_\alpha^H, e_i), e_i\rangle c(f_\alpha) + \sum_{i\neq j}\cdots\\
&= \frac{1}{2}\langle T(f_\alpha^H, e_i), e_i\rangle c(f_\alpha)
\end{aligned}$$

$\langle T(f_\alpha^H, e_i), e_i\rangle = \langle \frac{1}{2}(g^{TX})^{-1}(L_{f_\alpha^H}g^{TX})e_i, e_i\rangle = \frac{1}{2}Tr[(g^{TX})^{-1}L_{f_\alpha^H}g^{TX}]$

Lemma 2. *Let dV_X be the volume element along the fibre, then*

$$div^X(U^H) = \frac{L_{U^H}dV_X}{dV_X} = \frac{1}{2}Tr[(g^{TX})^{-1}L_{U^H}g^{TX}]$$

proof of the lemma. $\{e_i\}$ is an orthonormal basis of TX, $\{e^i\}$ is the dual basis. We calculate

$$\begin{aligned}
L_{U^H}dV_X &= L_{U^H}(e^1\wedge\cdots\wedge e^n) = \sum_i e^1\wedge\cdots\wedge L_{U^H}e^i\wedge\cdots\wedge e^n\\
&= \sum_i e^1\wedge\cdots\wedge(\sum_j -\langle L_{U^H}e_j, e_i\rangle e^j)\wedge\cdots\wedge e^n\\
&= -(\sum_i\langle L_{U^H}e_i, e_i\rangle)dV_X
\end{aligned}$$

$$\begin{aligned}
Tr[(g^{TX})^{-1}L_{U^H}g^{TX}] &= Tr_{g^{TX}}(L_{U^H}g^{TX}) = \sum_i(L_{U^H}g^{TX})(e_i, e_i)\\
&= \sum_i(U^H\langle e_i, e_i\rangle - \langle L_{U^H}e_i, e_i\rangle - \langle e_i, L_{U^H}e_i\rangle)\\
&= -2\sum_i\langle L_{U^H}e_i, e_i\rangle
\end{aligned}$$

□

Put the results together, finally we have

$$\begin{aligned}
D_\varepsilon^M &= c(e_i)\nabla_{e_i}^{TX\otimes E} + \sqrt{\varepsilon}c(f_\alpha)(\nabla_{f_\alpha^H}^{S^{TX}\otimes S^{TS}\otimes E} + k(U)) - \frac{\varepsilon}{8}\langle T(f_\alpha^H, f_\beta^H), e_i\rangle c(f_\alpha)c(f_\beta)c(e_i)\\
&= D^X + \sqrt{\varepsilon}D^H - \frac{\varepsilon c(T^H)}{4}
\end{aligned}$$

□

2.4 Levi-Civita Superconnection

Definition 2.4.1. $\forall s\in S$, $H_s = \Gamma(X_s, (S^{TX}\otimes E)|_{X_s})$. $H_s = H_{s,+}\oplus H_{s,-}$.

H is an infinite dimensional $\mathbb{Z}_2$-graded vector bundle with L^2 Hermitian metric .

Definition 2.4.2. $\forall U \in TS$, f is a smooth section of H,

$$\nabla^H_U f = \nabla^{S^{TX}\otimes E}_{U^H} f + \frac{1}{2} div^X(U^H) f =: \nabla'^H_U f + k(U) f$$

Note that $\Gamma(S, H) = \Gamma(S, \Gamma(X, S^{TX} \otimes E|_X)) = \Gamma(M, S^{TX} \otimes E)$

Proposition 2.4.1. *∇^H is a unitary connection H, whose curvature is a two form on the base with values in 1st order differential operator along the fibre.*

Proof. Let $U, U^H, \varphi_t, \psi_t$ be the same as in the beginning of proof of theorem (2.3.1). $\forall f,\ f' \in \Gamma(H)$,

$$\begin{aligned} U\langle f, f'\rangle_H &= U\int_t \langle f, f'\rangle dV_X = \frac{d}{dt}\int_{\varphi_t(X)} \langle f, f'\rangle dV_{\varphi_t(X)} \\ &= \int_X \frac{d}{dt}\psi_t^*(\langle f, f'\rangle dV_{\varphi_t(X)}) = \int_X (U^H\langle f, f'\rangle + \frac{L_{U^H} dV_X}{dV_X}) dV_X \\ &= \int_X (\langle \nabla_{U^H} f, f'\rangle + \langle f, \nabla_{U^H} f'\rangle + div^X(U^H)\langle f, f'\rangle) dV_X \\ &= \langle \nabla^H_U f, f'\rangle_H + \langle f, \nabla^H_U f'\rangle \end{aligned}$$

So ∇^H is unitary. Since $\nabla^H = \nabla'^H + k$, $\nabla^{H,2} = \nabla'^{H,2} + dk$. So $\forall U, V \in \Gamma(TS)$,

$$\begin{aligned} R^H(U, V) &= \nabla^{S^{TX}\otimes E}_{U^H}\nabla^{S^{TX}\otimes E}_{V^H} - \nabla^{S^{TX}\otimes E}_{V^H}\nabla^{S^{TX}\otimes E}_{U^H} - \nabla^{S^{TX}\otimes E}_{[U,V]^H} + dk(U, V) \\ &= R^{S^{TX}}(U^H, V^H) + R^E(U^H, V^H) - \nabla^{S^{TX}\otimes E}_{P^{TX}[U^H, V^H]} + dk(U, V) \end{aligned}$$

So R^H is a two form on the base S with value in 1st order differential operator along the fibre. ☐

By this proposition and $\Gamma(M, S^{TX}\hat{\otimes} S^{TS} \otimes E) = \Gamma(S, S^{TS}\hat{\otimes} H)$, we see D^H is just the Dirac operator on S with twisting bundle $(H, \langle\cdot,\cdot\rangle_{L^2}, \nabla^H)$.

Definition 2.4.3. The Levi-Civita superconnection on H is given by

$$A = D^X + \nabla^H - \frac{c(T^H)}{4}$$

$c(T^H) = \frac{1}{2}\langle T^H(f^H_\alpha, f^H_\beta), e_i\rangle f^\alpha f^\beta c(e_i)$.

Remark 10. Formally, this is obtained by changing Clifford variable $c(f_\alpha)$ in D^M to Grassman variable f^α. In fact, it's obtained by Getzler rescaling: in the expression of D^M_ε, we substitute:

$$c(f_\alpha) \rightsquigarrow \frac{f^\alpha\wedge}{\varepsilon} - \varepsilon i_{f_\alpha}$$

and let $\varepsilon \to 0$, then we get the Levi-Civita superconnection A.

We want to calculate the curvature of A. We use the Lichnerowicz formula for D^M_ε:

$$D^{M,2}_\varepsilon = -\Delta^M_\varepsilon + \frac{K^M_\varepsilon}{4} + \mathcal{R}^E \tag{2.7}$$

where $\Delta^M_\varepsilon = (\nabla^\varepsilon_{e_i})^2 + \varepsilon(\nabla^\varepsilon_{f_\alpha})^2$ is the Bochner Laplacian, ∇^ε is given by equation (2.6). K^M_ε is the scalar curvature of g^{TM}_ε, and

$$\mathcal{R}^E = \frac{1}{2}c(e_i)c(e_j)R^E(e_i, e_j) + \frac{\sqrt{\varepsilon}}{2}c(e_i)c(f_\alpha)R^E(e_i, f^H_\alpha) + \frac{\varepsilon}{2}c(f_\alpha)c(f_\beta)R^E(f^H_\alpha, f^H_\beta)$$

The idea is to change Clifford variable $c(f_\alpha)$ to f^α, note that

- $(\sqrt{\varepsilon}c(f_\alpha))^2 = -\varepsilon \rightsquigarrow (f^\alpha)^2 = f^\alpha \wedge f^\alpha = 0$

- $\alpha \neq \beta$, $\sqrt{\varepsilon}c(f_\alpha)\sqrt{\varepsilon}c(f_\beta) = \varepsilon c(f_\alpha)c(f_\beta) \rightsquigarrow f^\alpha \wedge f^\beta$

So if the power of $\sqrt{\varepsilon} > (=)$ the length of Clifford variables, then the term is killed(survives). $\nabla^{TM,\varepsilon} = \nabla^{TM} + S^\varepsilon$, $\nabla^{TM,\varepsilon,2} = \nabla^{TM,2} + \nabla^{TM} S^\varepsilon + [S^\varepsilon, S^\varepsilon]$.

Definition 2.4.4.

$$\alpha_t = \varphi Tr_s(\exp(-A_t^2))$$

Theorem 2.4.1. *1. The α_t are real, even, closed forms on S.*

$$[\alpha_t] = ch(Ind(D_+^X))$$

2. As $t \to 0$, $\alpha_t = \pi_*[\hat{A}(\nabla^{TX})ch(\nabla^E)] + O(\sqrt{t})$

3. If $dimkerD_\pm^X$ is locally constant, as $t \to \infty$,

$$\alpha_t = ch(\nabla^{kerD^X}) + O(\frac{1}{\sqrt{t}})$$

$\nabla^{kerD^X} = P^{kerD^X}\nabla^H$.

Remark 11. 1. The theorem implies the family index theorem:

$$ch(Ind(D_+^X)) = [\alpha_t] = \pi_*[\hat{A}(TX)ch(E)]$$

2. the theorem asserts the following diagram commutes.

Chapter 3

Determinant Bundle

3.1 Finite Dimensional Case

Let $\mathcal{C}$ denote the category of complex line(complex vector space of dim 1) up to canonical isomorphism. For example

- $\mathbb{C}$ denotes the canonical line. If λ has a canonical nonzero element $s \in \lambda$, then λ can be canonically identified with $\mathbb{C}$: $a \in \mathbb{C} \mapsto as \in \lambda$.
- If λ, μ are complex lines, then canonically $\lambda \otimes \mu \cong \mu \otimes \lambda$, $a \otimes b \mapsto b \otimes a$. The operator $\otimes$ is commutative, associative and has a neutral element:
  - $\lambda \otimes \mathbb{C} \cong \lambda$: $s \otimes 1 \mapsto s$.
  - $\lambda^* \otimes \lambda \cong \mathbb{C}$: choose any $s \neq 0 \in \lambda$, there is $s^{-1} \in \lambda^*$, s.t. $\langle s^{-1}, s\rangle = 1$. $s^{-1} \otimes s$ does not depend on s, so it's a canonical nonzero element in $\lambda^* \otimes \lambda$. We define $\lambda^{-1} = \lambda^*$.

Definition 3.1.1. E is a finite dimensional complex vector space

$$det\, E \stackrel{\triangle}{=} \wedge^{max} E$$

$\wedge^{max} E$ means the elements of max degree in $\wedge^* E$.

Remark 12. More correctly, $\mathcal{C} = \{(\lambda, \pm)\}$, $\mathcal{V}$={finite dimensional complex vector space}.

$$\widehat{det} : \mathcal{V} \longrightarrow \mathcal{C}, \quad \widehat{det} E = (det\, E, (-1)^{dim E})$$

then $\widehat{det}(E \oplus F) = \widehat{det}\, E \hat{\otimes} \widehat{det}\, F$, where $\lambda \hat{\otimes} \mu \cong \mu \hat{\otimes} \lambda$ is given by $a \hat{\otimes} b \mapsto (-1)^{\varepsilon_\lambda \varepsilon_\mu} b \hat{\otimes} a$.

Let $E : 0 \to E_0 \xrightarrow{V} E_1 \to 0$, $\lambda = det\, E \stackrel{\triangle}{=} (det\, E_0)^{-1} \otimes det\, E_1$. V induces $det\, V : det\, E_0 \to det\, E_1$, so $det\, V \in \lambda$.
Let E be a $\mathbb{Z}_2$-graded vector bundle with Hermitian metric $g^E = g^{E_+} \oplus g^{E_-}$, unitary connection $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$. We have the induced metric g^λ and unitary connection ∇^λ. Assume $V = \begin{pmatrix} 0 & V_- \\ V_+ & 0 \end{pmatrix}$ is self-adjoint and V_+ is invertible. Let $\{e_i\}$ be an orthonormal basis of E_+

$$\begin{aligned}
(\nabla^\lambda_X (det\, V_+))(e_1 \wedge \cdots \wedge e_n) &= \nabla_X (V_+(e_1) \wedge \cdots \wedge (e_n)) - det\, V_+(\sum_i e_1 \wedge \cdots \wedge \nabla^E_X e_i \wedge \cdots \wedge e_n) \\
&= \sum_i V_+(e_1) \wedge \cdots \wedge (\nabla^{E_-}_X (V_+(e_i)) - V_+(\nabla^{E_+}_X e_i)) \wedge \cdots \wedge V_+(e_n) \\
&= \sum_i V_+(e_1) \wedge \cdots \wedge V_+(V_+^{-1} \nabla_X V_+)(e_i) \wedge \cdots \wedge V_+(e_n) \\
&= Tr(V_+^{-1} \nabla_X V_+) \cdot det\, V_+(e_1 \wedge \cdots \wedge e_n)
\end{aligned}$$

So in anticommutative sign rule

$$\nabla^\lambda(det\,V_+) = -(det\,V_+)\cdot Tr(V_+^{-1}\nabla^E V_+) \tag{3.1}$$

Note that

$$V^{-1}\nabla^E V = \begin{pmatrix} 0 & V_+^{-1} \\ V_-^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \nabla^E V_- \\ \nabla^E V_+ & 0 \end{pmatrix} = \begin{pmatrix} V_+^{-1}\nabla^E V_+ & 0 \\ 0 & V_-^{-1}\nabla^E V_- \end{pmatrix}$$

So

$$Tr(V_+^{-1}\nabla^E V_+) = \frac{1}{2}Tr(V^{-1}\nabla^E V) + \frac{1}{2}Tr_s(V_-^{-1}\nabla^E V) \tag{3.2}$$

Since $(V_+^{-1}\nabla_X^E V_+)^* = (\nabla_X^E V_-)\cdot V_-^{-1}$, $\frac{1}{2}Tr(V^{-1}\nabla_X^E V)$ is real, $\frac{1}{2}Tr_s(V^{-1}\nabla_X^E V)$ is purely imaginary. Moreover, $Tr(V^{-1}\nabla^E V)$ is exact: $det\,V \in (det\,E)^* \otimes det\,E \cong \underline{\mathbb{C}}$ is a well defined function, so it's easy to see

$$Tr(V^{-1}\nabla^E V) = \frac{d(det\,V)}{det\,V} = d\,\log|detV| = 2d\,\log|detV_+| \tag{3.3}$$

Let r^λ be the curvature of ∇^λ, by formula (3.1),(3.2),(3.3)

$$r^\lambda = d\frac{\nabla^\lambda det\,V_+}{det\,V_+} = -d\,Tr(V_+^{-1}\nabla^E V_+) = -\frac{1}{2}d\,Tr_s(V^{-1}\nabla^E V) \tag{3.4}$$

Lemma 3. $c_1(det\,E) = ch(E)^{(2)} = c_1(E)$

in definition 1.4.1, we defined

$$\beta = \frac{1}{\sqrt{2\pi i}}\int_0^\infty \varphi Tr_s(\frac{V}{2\sqrt{t}}\exp(-A_t^2))dt$$

where $A_t^2 = \nabla^{E,2} + \sqrt{t}\nabla^E V + tV^2$.
Note that $\frac{(-1)^n}{n!}Tr_s(V(\sqrt{t}\nabla^E V + tV^2)^n)^{(1)} = \frac{(-1)^n}{(n-1)!}\sqrt{t}Tr_s(V\nabla^E V(tV^2)^{n-1})$,

$$Tr_s(\frac{V}{2\sqrt{t}}\exp(-A_t^2))^{(1)} = -\frac{1}{2}Tr_s(V(\nabla^E V)\exp(-tV^2))$$

$$\beta^{(1)} = -\frac{1}{2\pi i}\cdot\frac{1}{2}\int_0^\infty Tr_s(V\nabla^E V\exp(-tV^2))dt = \frac{\sqrt{-1}}{2\pi}\cdot\frac{1}{2}Tr_s(V^{-1}\nabla^E V)$$

By formula (3.4)

$$d\beta^{(1)} = -\frac{\sqrt{-1}}{2\pi}r^\lambda = -c_1(detE) = -ch(E)^{(2)} \tag{3.5}$$

3.2 Determinant Bundle

We want to construct the determinant bundle λ of $\{H = H_+ \oplus H_-, D_+^X\}$. If $ker\,D_\pm^X$ has constant dimension, $\lambda \triangleq (det\,ker\,D_+^X)^{-1}\otimes(det\,ker\,D_-^X)$. In general cases, $\forall a > 0$,

$$U_a \triangleq \{s \in S | a \notin Sp(D_s^{X,2})\}$$

U_a is an open set on S. $\bigcup\limits_{a>0} U_a$ is an open covering of S.

Definition 3.2.1. On U_a, $H^{<a} = \bigoplus\limits_{\substack{\lambda\in Sp(D^{X,2})\\ \lambda<a}}$ (bundle of eigenvalue λ of $D^{X,2}$).

We use other similar notations, like $H^{(0,a)}, D^{(0,a)} = D^X|_{H^{(0,a)}}$, e.t..

Proposition 3.2.1. *$H^{<a}$ is a finite dimensional smooth vector bundle on U_a. $H^{<a} = H_+^{<a} \oplus H_-^{<a}$.*

Definition 3.2.2. On U_a, $\lambda^{<a} = detH^{<a} = (detH_+^{<a})^{-1} \otimes detH_-^{<a}$. $\lambda^{<a}$ is a smooth line bundle on U_a.

The idea is that we can't define $detH = (detH_+)^{-1} \otimes detH_-$ directly, but we can patch $\lambda^{<a}$ together to get a well defined line bundle.

Proposition 3.2.2. *$\forall s \in U_a$, $\lambda_s^{<a} \cong (det\, kerD_+^{X_s})^{-1} \otimes (det\, kerD_-^{X_s}) =: det\, kerD^{X_s}$. This is a canonical isomorphism.*

Proof. We have an exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & kerD_+^{X_s} & \longrightarrow & H_+^{<a} & \xrightarrow{D_+} & H_-^{<a} & \longrightarrow & kerD_-^{X_s} & \longrightarrow & 0 \\ & & & & \| & & \| & & & & \\ & & & & kerD_+^{X_s} \oplus H_+^{(0,a)} & \xrightarrow{D_+} & H_-^{(0,a)} \oplus kerD_-^{X_s} & & & & \end{array}$$

Let $\sigma_\pm \neq 0 \in det\, kerD_\pm^{X_s}$, $\tau \neq 0 \in detH_+^{(0,a)}$, define

$$\begin{array}{ccc} det\, kerD^X & \xrightarrow{\cong} & detH^{<a} \\ \sigma_+^{-1} \otimes \sigma_- & \mapsto & (\sigma_+ \wedge \tau)^{-1} \otimes (detD_+^X \tau \wedge \sigma_-) \end{array}$$

This isomorphism is independent of τ. In fact, $detD_+^{(0,a)} = \tau^{-1} \otimes (detD_+^X \tau)$ is a nonzero canonical section of $detH^{(0,a)}$. So $\sigma_+^{-1} \otimes \sigma_- \mapsto (\sigma_+^{-1} \otimes \sigma_-) \cdot (detD_+^{(0,a)})$ is a canonical isomorphism from $det\, kerD^X$ to $detH^{<a} = (det\, kerD^X) \otimes (detH^{(0,a)})$. □

$\forall 0 < a < b$, $a, b \notin Sp(D^{X,2})$, $H^{<b} = H^{<a} \oplus H^{(a,b)}$, $detH^{<b} = detH^{<a} \otimes detH^{(a,b)}$. $detD_+^{(a,b)}$ is a canonical nonzero section of $detH^{(a,b)}$, so we have a canonical isomorphism

$$\begin{array}{rccc} \varphi_b^a : & \lambda^{<a} & \longrightarrow & \lambda^{<b} \\ & \sigma & \mapsto & \sigma \otimes (detD_+^{(a,b)}) \end{array}$$

Since $D_+^{(a,c)} = D_+^{(a,b)} \oplus D_+^{(b,c)}$, $detD_+^{(a,c)} = detD_+^{(a,b)} \otimes detD_+^{(b,c)} \in detH^{(a,b)} \otimes detH^{(b,c)} = detH^{(a,c)}$. So $\varphi_c^a = \varphi_c^b \circ \varphi_b^a : \lambda^{<a} \to \lambda^{<b}$. We can define the determinant line bundle:

Definition 3.2.3. The determinant line bundle λ is the complex line bundle obtained by pasting $(\lambda^{<a}, U_a)$ together via the canonical isomorphisms $\{\varphi_b^a : \lambda^{<a} \to \lambda^{<b}\}$

Remark 13. If $kerD_\pm^X$ have constant dimension, then

$$\lambda \cong det\, kerD^X$$

Proposition 3.2.3. *$c_1(\lambda) = \pi_*[\hat{A}(TX)ch(E)]^{(2)}$ in $H^2(S)$.*

Proof. By the family index theorem, $ch(IndD_+^X) = \pi_*[\hat{A}(TX)ch(E)]$ in $H^2(S)$. So $c_1(\lambda) = c_1(det\, kerD^X) = c_1(det\, IndD_+^X) = ch(IndD_+^X)^{(2)} = \pi_*[\hat{A}(TX)ch(E)]^{(2)}$. □

We want to construct a metric on λ. First we have a induced metric $|\cdot|_{\lambda^{<a}}$ on $\lambda^{<a}$ by the metric of H. $\forall \sigma \in \lambda^{<a}$,

$$|\varphi_b^a(\sigma)|_{\lambda^{<b}} = |\sigma \otimes (detD_+^{(a,b)})|_{\lambda^{<b}} = |\sigma|_{\lambda^{<a}} |det(D_-^{(a,b)} D_+^{(a,b)})|^{\frac{1}{2}}_{H_+^{(a,b)}}$$

We want to define $\|\cdot\|_{\lambda^{<a}} = |\cdot|_{\lambda^{<a}} |detD_+^{(a,\infty)}|$, then

$$\|\varphi_b^a(\sigma)\|_{\lambda^{<b}} = |\varphi_b^a(\sigma)|_{\lambda^{<b}} |detD_+^{(b,\infty)}| = |\sigma|_{\lambda^{<a}} |detD_+^{(a,b)}||detD_+^{(b,\infty)}| = |\sigma|_{\lambda^{<a}} |detD^{(a,\infty)}| = \|\sigma\|_{\lambda^{<a}}$$

so the metric patch together. We have to make sense the meaning of $|detD_+^{(a,\infty)}|$ in this infinite dimensional case.
$L = D_-^X D_+^X$ is a second order, elliptic, positive differential operator, we define

$$\zeta(s) = Tr^*(L^{-s}) = \sum_{\lambda \in Sp^*(D_-^X D_+^X)} \frac{1}{\lambda^s} \quad \text{(eigenvalue 0 excluded)} \tag{3.6}$$

We have the result of Seeley:

Proposition 3.2.4. *$\zeta(s)$ is holomorphic in $\{s \in \mathbb{C} | Re\, s > \frac{n}{2}\}$. It extends to a meromorphic function of $s \in \mathbb{C}$ with simple poles, which is holomorphic at $s = 0$.*

Proof. Use the Melin transform:

$$L^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tL} dt$$

$$Tr^*(L^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr^*(e^{-tL}) dt = \frac{1}{\Gamma(s)} (\int_0^1 + \int_1^\infty)$$

$\int_1^{+\infty} t^{s-1} Tr^*(e^{-tL}) dt$ is uniformly convergent w.r.t s, so is holomorphic in $s \in \mathbb{C}$. When $t \to 0$, we have asymptotic expansion:

$$Tr(e^{-tD^2}) = \frac{A_{-\frac{n}{2}}}{t^{\frac{n}{2}}} + \cdots + A_0 + A_1 t \cdots + O(t^k)$$

So

$$\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} Tr^*(e^{-tL}) dt = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} Tr(e^{-tL}) dt - \frac{1}{\Gamma(s+1)} (dim\, kerL)$$

is holomorphic when $Re\, s > \frac{n}{2}$. □

We also define

$$\zeta^a(s) = Tr(L^{-s} P^{(a,+\infty)}) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} Tr(e^{-tL} P^{(a,+\infty)}) dt$$

and $\zeta^{(a,b)}(s) = Tr(L^{-s} P^{(a,b)})$. Then $\forall 0 < a < b$, $\zeta^a(s) = \zeta^{(a,b)}(s) + \zeta^b(s)$. Note that

$$\frac{\partial \zeta^{(a,b)}}{\partial s}(0) = -\log \prod_{\substack{\lambda \in Sp(L) \\ a < \lambda < b}} \lambda = -\log\, detL^{(a,b)} = -2 \log |detD_+^{(a,b)}|$$

So $|detD_+^{(a,b)}| = \exp(-\frac{1}{2} \frac{\partial \zeta^{(a,b)}}{\partial s}(0))$. We define

$$|detD^{(a,\infty)}| = \exp(-\frac{1}{2} \frac{\partial \zeta^a}{\partial s}(0)), \ |det^* D_+| = \exp(-\frac{1}{2} \frac{\partial \zeta}{\partial s}(0))$$

then

$$|detD_+^{(a,\infty)}| = |detD^{(a,b)}||detD^{(b,\infty)}|, \ |det^* D_+| = |detD_+^{(0,a)}||detD^{(a,+\infty)}|$$

Definition 3.2.4. $\| \cdot \|_{\lambda^{<a}} = | \cdot |_{\lambda^{<a}} |detD_+^{(a,\infty)}| = | \cdot |_{\lambda^{<a}} \exp(-\frac{1}{2} \frac{\partial \zeta^a}{\partial s}(0))$.

Theorem 3.2.1. *Under the canonical identification of λ with $\lambda^{<a}$ over U_a, the metric $\| \cdot \|_{\lambda^{<a}}$ patch into a smooth metric $\| \cdot \|_\lambda$ on λ.*

the next step is to construct a natural unitary connection ∇^λ on $(\lambda, \| \cdot \|_\lambda)$. First we assume $kerD_\pm^X$ have constant dimensions. In theorem **??** we see that

$$d\tilde{\eta} = \pi_* [\hat{A}(\nabla^{TX}) ch(\nabla^E)] - ch(\nabla^{kerD^X})$$

so $d\tilde{\eta}^{(1)} = \pi_* [\hat{A}(\nabla^{TX}) ch(\nabla^E)]^{(2)} - c_1(\nabla^{det\, kerD^X})$.

Definition 3.2.5. $\nabla'^{\lambda} = \nabla^{det\, ker D^X} - 2\pi i\tilde{\eta}^{(1)}$.

then
$$c_1(\nabla'^{\lambda}) = c_1(\nabla^{det\, ker D^X}) + d\tilde{\eta}^{(1)} = \pi_*[\hat{A}(\nabla^{TX})ch(\nabla^E)]^{(2)}$$
Since $2\pi i\tilde{\eta}^{(1)}$ is purely imaginary, ∇'^{λ} is unitary with respect to $|\cdot|_{\lambda}$. Let g, g' be the metric associated with $\|\cdot\|_{\lambda}$, $|\cdot|_{\lambda}$, and θ the connection form of ∇'^{λ}, then
$$\begin{aligned} dg &= d(\frac{g}{g'}g') = d(\frac{g}{g'})\cdot g' + \frac{g}{g'}dg' = g\cdot d\log\frac{g}{g'} + \frac{g}{g'}(g'\theta + g'\overline{\theta}) \\ &= g(d\log\frac{g}{g'} + \theta + \overline{\theta}) \end{aligned}$$
So if we let
$$\nabla''^{\lambda} = \nabla'^{\lambda} + \frac{1}{2}d\log\frac{g}{g'} \tag{3.7}$$
then ∇''^{λ} is unitary with respect to $\|\cdot\|_{\lambda}$.
In general case, we want to patch $\nabla^{\lambda^{<a}}$ together. Let $\nabla^{H^{<a}}$ be the orthogonal projection of ∇^H to $H^{<a}$, which is unitary, then $\lambda^{<a}$ has an induced connection $\nabla^{<a}$ which is unitary w.r.t. $|\cdot|_{\lambda^{<a}}$. $\forall\sigma\in\lambda^{<a}$
$$\begin{aligned} \nabla^{\lambda^{<b}}(\varphi_b^a(\sigma)) &= \nabla^{\lambda^{<b}}(\sigma\otimes detD_+^{(a,b)}) = \nabla^{\lambda^{<a}}\sigma\otimes(detD_+^{(a,b)}) + \sigma\otimes(\nabla^{\lambda^{(a,b)}}detD_+^{(a,b)}) \\ &= (\nabla^{\lambda^{<a}}\sigma + \frac{\nabla^{\lambda^{(a,b)}}detD_+^{(a,b)}}{detD_+^{(a,b)}}\sigma)\otimes(detD_+^{(a,b)}) \end{aligned}$$
From formula (3.1), we know that
$$\begin{aligned} \frac{\nabla^{\lambda^{(a,b)}}(detD_+^{(a,b)})}{detD_+^{(a,b)}} &= -Tr[(D_+^{(a,b)})^{-1}\nabla^{H^{(a,b)}}D_+^{(a,b)}] \\ &= -\frac{1}{2}Tr[(D^{(a,b)})^{-1}\nabla^{H^{(a,b)}}D^{(a,b)}] - \frac{1}{2}Tr_s[(D^{(a,b)})^{-1}\nabla^{H^{(a,b)}}D^{(a,b)}] \end{aligned}$$
We want to define $\nabla'^{\lambda^{<a}} = \nabla^{\lambda^{<a}} - Tr[(D_+^{(a,\infty)})^{-1}\nabla^{H^{(a,\infty)}}D_+^{(a,\infty)}]$, then
$$\begin{aligned} \nabla'^{\lambda^{<b}}(\varphi_b^a(\sigma)) &= (\nabla^{\lambda^{<a}} - Tr[(D_+^{(a,b)})^{-1}\nabla^{H^{(a,b)}}D_+^{(a,b)}] - Tr[(D_+^{(b,\infty)})^{-1}\nabla^{H^{(b,\infty)}}D_+^{(b,\infty)}])\sigma\otimes detD_+^{(a,b)} \\ &= (\nabla^{\lambda^{<a}}\sigma - Tr[(D_+^{(a,\infty)})^{-1}\nabla^{H^{(a,\infty)}}D_+^{(a,\infty)}]\sigma)\otimes detD_+^{(a,b)} = \varphi_b^a(\nabla'^{\lambda^{<a}}\sigma) \end{aligned}$$
So the connection $\{\nabla'^{<a}\}$ patch together. First, we introduce

Definition 3.2.6.
$$\gamma_t^a = \int_t^\infty Tr[e^{-sD^2}\nabla^H D\cdot D\cdot P^{(a,\infty)}]ds = -Tr[e^{-tD^2}D^{-1}\nabla^H D\cdot P^{(a,\infty)}]$$
$$\delta_t^a = \int_t^\infty Tr_s[e^{-sD^2}\nabla^H D\cdot D\cdot P^{(a,\infty)}]ds = Tr_s[e^{-tD^2}D^{-1}\nabla^H D\cdot P^{(a,\infty)}]$$
similarly we can define $\gamma_t^{(a,b)}$ and $\delta_t^{(a,b)}$.

As $t\to 0$, we have asymptotic expansions:
$$\begin{aligned} \gamma_t^a &= \int_t^1 (Tr[e^{-sD^2}\nabla^H D\cdot D\cdot P^{(a,\infty)}] + \sum_{-\frac{n}{2}}^{0} dA_j\cdot s^{j-1})ds \\ &\qquad + \int_1^\infty Tr[e^{-sD^2}\nabla^H D\cdot D\cdot P^{(a,\infty)}]ds - \sum_{-\frac{n}{2}}^{-1}\frac{dA_j}{j} + \sum_{-\frac{n}{2}}^{-1}\frac{dA_j}{j}t^j + dA_0\cdot\log t \\ &= \sum_{-\frac{n}{2}}^{-1} dA_j\cdot\frac{t^j}{j} + dA_0\cdot\log t + \gamma_0^a + O(t) \end{aligned} \tag{3.8}$$

$$\delta_t^a = \delta_0^a + O(t)$$

we have $\gamma_t^b = \gamma_t^a + \gamma_t^{(a,b)}$, $\delta_t^b = \delta_t^a + \delta_t^{(a,b)}$ and

$$\gamma_0^{(a,b)} = -Tr[D^{-1}\nabla^H D \cdot P^{(a,b)}] = -Tr[(D^{(a,b)})^{-1}\nabla^{H^{(a,b)}} D^{(a,b)}]$$

$$\delta_0^{(a,b)} = Tr_s[D^{-1}\nabla^H D \cdot P^{(a,b)}] = Tr_s[(D^{(a,b)})^{-1}\nabla^{H^{(a,b)}} D^{(a,b)}]$$

So $\frac{\nabla^{\lambda^{(a,b)}}(detD_+^{(a,b)})}{detD_+^{(a,b)}} = \frac{1}{2}(\gamma_0^{(a,b)} - \delta_0^{(a,b)})$. If we let

$$\nabla'^{\lambda^{<a}} = \nabla^{\lambda^{<a}} + \frac{1}{2}(\gamma_0^a - \delta_0^a)$$

then $\nabla'^{\lambda^{<a}}$ will patch together. But it may not be unitary. By equation (3.7), we compute $d\log\frac{g}{g'} = d(-\frac{\partial\zeta^a}{\partial s}(0))$. Recall that

$$\zeta^a(s) = Tr(L^{-s}P^{(a,\infty)}) = \frac{1}{2}Tr[(D^2)^{-s}P^{(a,\infty)}] = \frac{1}{2\Gamma(s)}\int_0^\infty t^{s-1}Tr[e^{-tD^2}P^{(a,\infty)}]dt$$

so

$$\begin{aligned}
d\zeta^a(s) &= -\frac{1}{\Gamma(s)}\int_0^\infty t^s Tr[e^{-tD^2}\nabla^H D\cdot D\cdot P^{(a,\infty)}]dt \\
&= -\frac{1}{\Gamma(s)}\int_0^1 t^s(Tr[e^{-tD^2}\nabla^H D\cdot D\cdot P^{(a,\infty)}] + \sum_{-\frac{n}{2}}^{0} dA_j\cdot t^{j-1})dt \\
&\quad -\frac{1}{\Gamma(s)}\int_1^\infty t^s Tr[e^{-tD^2}\nabla^H D\cdot D\cdot P^{(a,\infty)}]dt + \frac{1}{\Gamma(s)}\sum_{j=-\frac{n}{2}}^{-1}\frac{dA_j}{s+j} + \frac{dA_0}{\Gamma(s+1)}
\end{aligned}$$

Note that $\Gamma(s)$ has simple pole at $s=0$, $Res(\Gamma(s),0)=1$, and $(\frac{1}{\Gamma(s)})'(0)=1$. By equation (3.8), we get

$$\begin{aligned}
d[\frac{\partial\zeta^a}{\partial s}(0)] &= -\int_0^1(Tr[e^{-tD^2}\nabla^H D\cdot D\cdot P^{(a,\infty)}] + \sum_{-\frac{n}{2}}^{0} dA_j\cdot t^{j-1})dt \\
&\quad -\int_1^\infty Tr[e^{-tD^2}\nabla^H D\cdot D\cdot P^{(a,\infty)}]dt + \sum_{j=-\frac{n}{2}}^{-1}\frac{dA_j}{j} - \Gamma'(1)dA_0 \\
&= -\gamma_0^a - \Gamma'(1)dA_0
\end{aligned}$$

δ_0^a is purely imaginary, so $\nabla''^{\lambda^{<a}} = \nabla^{\lambda^{<a}} + \frac{1}{2}(\gamma_0^a - \delta_0^a) + \frac{1}{2}\Gamma'(1)dA_0$ is unitary and can be patched together.

We will see whether it satisfies $c_1(\nabla^\lambda) = \pi_*[\hat{A}(\nabla^{TX})ch(\nabla^E)]^{(2)}$, we know that as $t\to 0$, $ch(A_t) = \varphi Tr_s\exp(-A_t^2) = \pi_*[\hat{A}(\nabla^{TX})ch(\nabla^E)] + O(\sqrt{t})$. So

$$[\varphi Tr_s\exp(-A_t^2)]^{(2)} = \pi_*[\hat{A}(\nabla^{TX})ch(\nabla^E)]^{(2)} + O(\sqrt{t}) \tag{3.9}$$

Proposition 3.2.5.

$$[Tr_s\exp(-A_t^2)]^{(2)} = [Tr_s\exp(-(\sqrt{t}D^X + \nabla^H)^2)]^{(2)} \tag{3.10}$$

Proof. Let $A_t^l = \sqrt{t}D^X + \nabla^H - \frac{lc(T^H)}{4\sqrt{t}}$, then

$$\frac{\partial}{\partial l}[Tr_s\exp(-A_t^{l,2})]^{(2)} = -dTr_s[\frac{\partial A_t^l}{\partial l}\exp(-A_t^{l,2})]^{(1)} = dTr_s[\frac{c(T^H)}{4\sqrt{t}}\exp(-A_t^{l,2})]^{(1)} = 0$$

Since $A_t^1 = A_t$, $A_t^0 = \sqrt{t}D^X + \nabla^H$, the proposition follows. □

Proposition 3.2.6.

$$[Tr_s\exp(-(\sqrt{t}D^X+\nabla^H)^2)]^{(2)} = [Tr_s\exp(-(\sqrt{t}D^{<a}+\nabla^{H^{<a}})^2)]^{(2)}+[Tr_s\exp(-(\sqrt{t}D^{(a,\infty)}+\nabla^{H^{(a,\infty)}})^2)]^{(2)} \tag{3.11}$$

Proof. We use the same transgression trick: Let $M^a = \nabla^H - \nabla^{H,split}$, where $\nabla^{H,split} = \nabla^{H^{<a}} \oplus \nabla^{H^{(a,\infty)}}$, and $\nabla^H_l = \nabla^{H,split} + lM^a$. Then

$$\begin{aligned}\frac{\partial}{\partial l}[Tr_s \exp(-(\sqrt{t}D^X + \nabla^H_l)^2)]^{(2)} &= -dTr_s[M^a \exp(-(\sqrt{t}D^X + \nabla^H_l)^2)]^{(1)} \\ &= -dTr_s[M^a \exp(-tD^{X,2})]\end{aligned}$$

Clearly M^a interchanges $H^{<a}_\pm$ and $H^{(a,\infty)}_\pm$, while $\exp(-tD^{X,2})$ preserves the splitting $H_\pm = H^{<a}_\pm \oplus H^{(a,\infty)}_\pm$, so $Tr_s[M^a \exp(-tD^{X,2})] = 0$. Since $\nabla^{H,split}$ verifies the proposition, so does ∇^H. □

Now

$$\begin{aligned}\frac{\partial}{\partial t}[Tr_s \exp(-(\sqrt{t}D^{(a,\infty)} + \nabla^{H^{(a,\infty)}})^2)]^{(2)} &= -dTr_s[\frac{D^{(a,\infty)}}{2\sqrt{t}} \exp(-(\sqrt{t}D^{(a,\infty)} + \nabla^{H^{(a,\infty)}})^2)]^{(1)} \\ &= \frac{1}{2}dTr_s[D^{(a,\infty)}\nabla^{H^{(a,\infty)}} D^{(a,\infty)} \exp(-tD^{(a,\infty),2})] \\ &= \frac{1}{2}dTr_s[\exp(-tD^2)\nabla^H D \cdot D \cdot P^{(a,\infty)}]\end{aligned}$$

Since $a > 0$, as $t \to +\infty$, $[Tr_s \exp(-(\sqrt{t}D^{(a,\infty)} + \nabla^{H^{(a,\infty)}})^2)]^{(2)}$ decays exponentially. So we have

$$\begin{aligned}[Tr_s \exp(-(\sqrt{t}D^{(a,\infty)} + \nabla^{H^{(a,\infty)}})^2)]^{(2)} &= -\frac{1}{2}d\int_t^\infty Tr_s[\exp(-sD^2)\nabla^H D \cdot D \cdot P^{(a,\infty)}]ds \\ &= -\frac{1}{2}d\delta^a_t \end{aligned} \tag{3.12}$$

By formula (3.9),(3.10),(3.11),(3.12), we have

$$[Tr_s \exp(-A_t^2)]^{(2)} = [Tr_s \exp(-(\sqrt{t}D^{<a} + \nabla^{H^{<a}})^2)]^{(2)} - \frac{1}{2}d\delta^a_t$$

Let $t \to 0$, we get

$$\pi_*[\hat{A}(\nabla^{TX})ch(\nabla^E)]^{(2)} = c_1(\nabla^{\lambda^{<a}}) - \frac{1}{2\pi i}\frac{1}{2}d\delta^a_0 \tag{3.13}$$

Now we have $\nabla''^{\lambda^{<a}} = \nabla^{\lambda^{<a}} + \frac{1}{2}(\gamma^a_0 - \delta^a_0) + \frac{1}{2}\Gamma'(1)dA_0$. Note that $\gamma^a_0 = -d[\frac{\partial\zeta^a}{\partial s}(0)] - \Gamma'(1)dA_0$ is exact, so

$$c_1(\nabla''^{\lambda^{<a}}) = c_1(\nabla^{\lambda^{<a}}) - \frac{1}{2\pi i}\cdot\frac{1}{2}d\delta^a_0 \tag{3.14}$$

So $\nabla''^{\lambda^{<a}}$ is the connection we want:

Theorem 3.2.2. *Let* ${}^1\nabla^{\lambda^{<a}} = \nabla^{\lambda^{<a}} + \frac{1}{2}(\gamma^a_0 - \delta^a_0) + \frac{1}{2}\Gamma'(1)dA_0$*, then identifying* λ *with* $\lambda^{<a}$ *over* U_a*, the connection* ${}^1\nabla^{\lambda^{<a}}$ *patch together into a connection* ${}^1\nabla^\lambda$ *on* λ*, which is unitary w.r.t. the metric* $\|\cdot\|$*, and*

$$c_1({}^1\nabla^\lambda) = \pi_*[\hat{A}(\nabla^{TX})ch(\nabla^E)]^{(2)}$$