YTD conjecture for generalized Kähler solitons and applications

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Moment map of torus actions on projective manifolds

- X a projective manifold. $L \to X$ an ample holomorphic line bundle.

  Kähler metric $\omega = \sqrt{-1}\sum_{i,j}\omega_{i\bar{j}}dz_i \wedge d\bar{z}_j > 0$ in $2\pi \cdot c_1(L)$.

- $T \cong (S^1)^r$, with Lie algebra $N_{\mathbb{R}}$, acts holomorphically on (X, L)

  Hamiltonian action: ω is T-invariant and there exists a moment map:

$$\mathbf{m} : X \to N_{\mathbb{R}}^{\vee} \cong \mathbb{R}^r, \quad \iota_\xi \omega = d\langle \mathbf{m}, \xi\rangle \quad \text{for any } \xi \in N_{\mathbb{R}}.$$

- Atiyah-Guillemin-Sternberg: the image $\mathbf{m}(X)$ is a convex polytope (which can be recovered by algebraic data using weight decompositions).

  Duistermaat-Heckman measure: $\mathrm{DH}_T := \mathbf{m}_*(\omega^n)$ does not depend on the Kähler form in the same Kähler class. $\Rightarrow$ For any smooth function g on $\mathbb{R}^r$,

$$\mathrm{V}_g := \int_X g(\mathbf{m})\omega^n = \int_P g(x)\mathrm{DH}_T \quad \text{is independent of } \omega \in [\omega].$$

Fano manifolds

- X Fano: $-K_X := \wedge^n T_{hol}X$ is ample, $c_1(X) = c_1(-K_X) > 0$.
  Local holomorphic frame: $\partial_z = \partial_{z_1} \wedge \cdots \wedge \partial_{z_n}, dz = dz_1 \wedge \cdots \wedge dz_n$.
- Hermitian metric on $-K_X \longleftrightarrow$ volume form

$$h_\varphi = h_0 e^{-\varphi} \quad \longleftrightarrow \quad \Omega_\varphi = |\partial_z|^2_{h_\varphi} (\sqrt{-1})^{n^2} dz \wedge d\bar{z}.$$

  Chern curvature:

$$Ric(\Omega_\varphi) = Ric(\Omega_0) + \sqrt{-1}\partial\bar{\partial}\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi = \omega_\varphi \in 2\pi c_1(X).$$

  Space of Kähler potentials:

$$\mathcal{H}(\omega_0) = \left\{\varphi \in C^\infty(X); \quad \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0\right\}.$$

- holomorphic vector field $v \longrightarrow$ a canonical Hamiltonian function:

$$\theta_v(\varphi) = -\frac{\mathfrak{L}_v \Omega_\varphi}{\Omega_\varphi} \quad \Longrightarrow \quad \iota_v \omega_\varphi = \sqrt{-1}\bar{\partial}\theta_v(\varphi).$$

  If $v = \frac{1}{2}(-J\xi - \sqrt{-1}\xi)$, $\theta_v(\varphi)$ is real $\Longleftrightarrow \xi$ is Killing w.r.t. ω_φ.

g-soliton equations

- Moment map of $T = (S^1)^r$ action with respect to ω_φ:

$$\mathbf{m}_\varphi : X \longrightarrow N_{\mathbb{R}}^\vee, \quad \langle \mathbf{m}_\varphi, \xi \rangle = \theta_{v_\xi}(\varphi).$$

$v_\kappa = \frac{1}{2}(-J\xi_\kappa - \sqrt{-1}\xi_\kappa)$: κ-th generator of $(\mathbb{C}^*)^r$

$$\mathbf{m}_\varphi = (\theta_{\kappa,0} + v_\kappa(\varphi))_{\kappa=1}^r : X \to \mathbb{R}^r.$$

- Let $g : P \to \mathbb{R} > 0$ be a smooth function. g-soliton equation:

$$g(\mathbf{m}_\varphi)(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-\varphi}\Omega_0$$
$$\iff \quad Ric(\omega_\varphi) - \omega_\varphi = \sqrt{-1}\partial\bar{\partial}\log g(\mathbf{m}_\varphi).$$

Examples: $g = \exp(\langle x, \xi \rangle)$: Käler-Ricci soliton; $g = 1$: Kähler-Einstein.

$g(x) = 1 + \langle x - \bar{x}, \xi \rangle$: Mabuchi soliton

$g = (c + \langle x, \xi \rangle)^{-n-2}$: Ricci-flat Kähler cone metric (Apostolov et al.)

- Set $g_\varphi := g(\mathbf{m}_\varphi)$ and $f_\varphi = \log g_\varphi$. For any holomorphic vector field v

$$\mathrm{Fut_g(v)} \quad := \quad \int_X v(-\log \frac{\omega_\varphi^n}{\Omega_\varphi} - f_\varphi) e^{g_\varphi} \omega_\varphi^n = -\int_X \theta_v(\varphi) g_\varphi \omega_\varphi^n.$$

  Fact: $\mathrm{Fut_g}$ does not depend on the choice of $\omega \in 2\pi c_1(L)$.

  $\exists$ g-soliton in $\implies$ $\mathrm{Fut_g} \equiv 0$.

- There is a generalized Matsushima reductivity result:

Theorem

If $(X, \mathbb{T})$ admits a g-soliton, then the following group is reductive:

$$\mathrm{Aut}(X, \mathbb{T}) = \{\sigma \in \mathrm{Aut}(X); t \circ \sigma = \sigma \circ t \; \forall t \in \mathbb{T}\}. \tag{1}$$

This follows from the identity:

$$\mathfrak{aut}(X, \mathbb{T}) \cong \left\{\theta \in C^\infty(X, \mathbb{C})^T; (\Delta + \nabla f)\theta = -\theta\right\}. \tag{2}$$

Toric case

- X: toric manifold: $\mathbb{T} \cong (\mathbb{C}^*)^n$ has an open orbit $\longleftrightarrow$ moment polytope P
  X: toric Fano manifold $\longleftrightarrow$ reflexive polytope $P \ni 0$

$$P: \quad \ell_{n_i}(x) = \langle x, n_i \rangle \geq -1, \quad i = 1, \ldots, K.$$

- Holomorphic vector field $\longleftrightarrow \zeta \in \mathbb{R}^n$

$$\mathrm{Fut_g}(\zeta) = \int_{\mathrm{P}} \langle \mathrm{x}, \zeta \rangle \mathrm{gdx} = 0 \quad \leftrightarrow \quad g\text{-weighted barycenter} = 0.$$

If $g = f(\langle x, \xi \rangle)$ for a strictly convex f, this determines ξ uniquely.
Examples: $f = \exp$ (KR soliton); $f = \frac{1}{(c+t)^d}$ for any $c > 0$ and $d > 0$.

Theorem (Han-L. , a consequence of YTD theorem)

For toric manifolds, there exists a g-soliton iff the g-weighted barycenter=0.

- For Kähler-Ricci soliton this was proved by Wang-Zhu (2000).

Energy functionals and coercivity

- g-weighted functionals generalizing the unweighted case:

$$\begin{aligned}
\mathbf{E}_g(\varphi) &= \frac{1}{\mathrm{V}_g}\int_0^1 dt \int_X \varphi \omega_{t\varphi}^n, \quad \mathbf{\Lambda}_g(\varphi) = \frac{1}{\mathrm{V}_g}\int_X \varphi g_{\varphi_0}\omega_0^n \\
\mathbf{I}_g(\varphi) &= \int_X \varphi(g_{\varphi_0}\omega_{\varphi_0}^n - g_\varphi \omega_\varphi^n), \quad \mathbf{J}_g(\varphi) = \mathbf{\Lambda}_g(\varphi) - \mathbf{E}_g(\varphi) \\
\mathbf{L}(\varphi) &= -\log\left(\int_X e^{-\varphi}\Omega_0\right), \quad \mathbf{D}_g(\varphi) = -\mathbf{E}_g(\varphi) + \mathbf{L}(\varphi) \\
\mathbf{M}_g(\varphi) &= \frac{1}{\mathrm{V}_g}\int_X \log\frac{g_\varphi \omega_\varphi^n}{\Omega_0} g_\varphi \omega_\varphi^n - (\mathbf{I}_g - \mathbf{J}_g)(\varphi).
\end{aligned}$$

- Automorphisms: $\mathrm{Aut}(X, \mathbb{T}) = \{\sigma \in \mathrm{Aut}(X); \sigma \cdot t = t \cdot \sigma, \forall t \in \mathbb{R}\}$
  $\tilde{\mathbb{T}}$: maximal torus of $\mathrm{Aut}(X, \mathbb{T})$; T: maximal compact torus of $\tilde{\mathbb{T}}$.
- $\mathbf{F}_g$ is reduced coercive if $\exists \gamma, C > 0$ s.t. $\forall \tilde{T}$-invariant $\varphi \in \mathcal{H}(\omega_0)$

$$\mathbf{F}_g(\varphi) \geq \gamma \cdot \inf_{\sigma \in \tilde{\mathbb{T}}} \mathbf{J}_g(\sigma^* \omega_\varphi) - C.$$

Analytic criterion

Theorem (Han-L. , generalize Tian, Tian-Zhu, Phong-Song-Sturm-Weinkove, Darvas-Rubinstein)

The following are equivalent:

1. $(X, \mathbb{T})$ *admits a g-soliton.*
2. $\mathbf{D}_g$ *is reduced coercive.*
3. $\mathbf{M}_g$ *is reduced coercive.*

- $1 \to 2$: generalize the argument of Darvas-Rubinstein based on the convexity and uniqueness:

Theorem (Berndtsson, BBEGZ)

$\mathbf{L}$ *is convex along geodesic segment in* $\mathcal{E}^1_g$*. It is affine iff the geodesic is generated by a holomorphic vector field that commutes with* $\mathbb{T}$*.*

$2 \to 3$: $\mathbf{D}_g \le \mathbf{M}_g$.
$3 \to 2$: A duality argument.
$2 \to 1$: Variational argument

g-Monge-Ampère measure for singular potentials

- $\varphi \in L^1(\omega_0^n)$ is ω_0-psh if it is u.s.c. and $\psi + \varphi$ is psh ($\omega_0 = \sqrt{-1}\partial\bar{\partial}\psi$).
$$\mathrm{PSH}(\omega_0) = \{\varphi; \varphi \text{ is } \omega_0\text{-psh}\}.$$
Non-pluripolar product:
$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \lim_{j\to+\infty}(\omega_0 + \sqrt{-1}\partial\bar{\partial}\max\{\varphi, -j\})^n.$$
- $g = \prod_\kappa (x_\kappa + c)^{d_\kappa}$ such that $P + c(1, \ldots, 1) \in \mathbb{R}^r_{>0}$, then
$$\int_X fg_\varphi(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_{X^{[\vec{d}]}}(f^T)^{[\vec{d}]}\Big(\sum_\kappa \theta_\kappa(\varphi)\omega_{\mathrm{FS},\kappa} + \omega_\varphi\Big)^{n+d}.$$
- General g, find polynomials $g_j \to g$ uniformly and set:
$$\int_X fg_\varphi\omega_\varphi^n = \lim_{j\to+\infty}\int_X f(g_j)_\varphi\omega_\varphi^n.$$

Fibration construction

- $P \to B$: hol. line bundle with Hermitian metric h_P; $\bar{P}$ circle bundle.

  $L \to X$: hol. line bundle with $\mathbb{C}^*$-action, S^1-invariant Hermitian metric h_L

  $(Y, F) = (P^* \times (X, L))/\mathbb{C}^* \to B$: associated holomorphic fibre bundle.
- Induced horizontal distribution and splitting $TY = \pi^* TB \bigoplus TX$.

  Induced Hermitian metric h_F on F whose Chern curvature at $\bar{P}$:

$$\sqrt{-1}\bar{\partial}\partial \log h_F = \pi^*(\theta_v \sqrt{-1}\bar{\partial}\partial \log h_P) + \sqrt{-1}\bar{\partial}\partial \log h_L.$$

  The right-hand-side is considered as an equivariantly closed form on X.
- Example:
  - $(P \to B) = (\mathbb{P}^{d+1} \setminus \mathbb{P}^d_\infty \to \mathbb{P}^k)$, $\bar{P} \cong \mathbb{S}^{2d+1}$, $\mathbb{P}^* = \mathbb{C}^{d+1} \setminus \{0\}$.
    Curvature: $\theta_v(\varphi)\omega_{\rm FS} + \omega_\varphi$.
  - $\mathbb{S}^{[\vec{d}]} = S^{2d_1+1} \times \cdots \times S^{2d_r+1} \to \mathbb{P}^{[\vec{d}]} = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_r}$.

$$(X^{[\vec{d}]}, L^{[\vec{d}]}) = \mathbb{S}^{[\vec{d}]} \times_{(S^1)^r} (X, L) \to \mathbb{P}^{[\vec{d}]}.$$

    Curvature: $\sum_{\kappa=1}^r \theta_\kappa(\varphi)\omega_{{\rm FS}, \mathbb{P}^{d_\kappa}} + \omega_\varphi$.

$\mathcal{E}^1_g$ space

- Define:

$$\mathcal{E}^1 = \left\{\varphi \in \mathrm{PSH}(\omega_0); \int_X |\varphi|\omega_\varphi^n < +\infty\right\}, \quad \text{(Guedj-Zeriahi)}$$

$$\mathcal{E}^1_g = \left\{\varphi \in \mathrm{PSH}(\omega_0)^T; \int_X |\varphi| g_\varphi \omega_\varphi^n < +\infty\right\} = (\mathcal{E}^1)^T.$$

Strong topology: $\varphi_j \to \varphi$ in strong topology iff it converges in L^1 and $\mathbf{E}_g(\varphi_j) \to \mathbf{E}_g(\varphi)$ iff

$$\varphi_j \to \varphi \text{ in } L^1, \quad \sup \varphi_j \to \sup \varphi, \quad \mathbf{I}_g(\varphi, \varphi_j) \to 0.$$

Theorem (generalized version of Berman-Boucksom-Eyssidieux-Guedj-Zeriahi)

For any constant $C > 0$, the following subset is compact in strong topology: $\left\{\varphi \in \mathcal{E}^1_g; \int_X \log \frac{g_\varphi \omega_\varphi^n}{\Omega_0} g_\varphi \omega_\varphi^n < C, \sup \varphi = 0\right\}.$

Test configurations

- Let $X \to \mathbb{P} = \mathbb{P}^{N_m - 1}$ be the $\mathbb{T}$-equivariant Kodaira embedding via $|mL|$.

  $\sigma(s) = \exp(s\zeta), s \in [0, +\infty)$: one parameter subgroup of $GL(N_m, \mathbb{C})$.

  Limit scheme: $[\mathcal{X}_0] = \lim_{s \to +\infty} \sigma(s) \circ [X]$ and induced test configuration:

$$\begin{aligned} \mathcal{X} &= \{(z, t) \in \mathbb{P}^{N_m - 1} \times \mathbb{C}; z \in \sigma(-\log |t|^2) \circ X\} \\ \mathcal{L} &= (p_1^* \mathcal{O}_{\mathbb{P}}(1))^{1/m}. \end{aligned}$$

  Path in $\mathcal{H}(\omega_0)$: $\Phi = \{\varphi(s)\}$ with $\varphi(s) = \frac{1}{m} \log \frac{|\sigma(s) \cdot Z|^2}{|Z|^2}$.

- For $\xi \in N_{\mathbb{R}}$, twist $(\mathcal{X}, \mathcal{L}) \to (\mathcal{X}_\xi, \mathcal{L}_\xi) \longleftrightarrow \sigma \cdot \exp(s\xi)$.
- σ commutes with $\mathbb{T} \Longrightarrow \mathbb{T} \times \mathbb{C}^*$ acts on $(\mathcal{X}_0, \mathcal{L}_0)$ and hence on $V_m = H^0(\mathcal{X}_0, m\mathcal{L}_0)$ with weight decomposition:

$$V_m = \bigoplus_{\alpha} V_{m,\alpha} = \bigoplus_{\alpha \in \mathbb{Z}^r} \bigoplus_{i \in \mathbb{Z}} V_{m,\alpha}(\lambda_i^{(m,\alpha)}).$$

Non-Archimedean functionals

$$
\begin{aligned}
\mathbf{E}_g^{\mathrm{NA}} &= \frac{1}{\mathrm{V}_g} \lim_{m\to+\infty} \frac{n!}{m^n} \sum_{\alpha,i} \frac{\lambda_i^{(m,\alpha)}}{m} g\left(\frac{\alpha}{m}\right) = \frac{1}{\mathrm{V}_g} \int_{\mathcal{X}_0} \theta_\zeta g_\varphi \omega_\varphi^n \\
\mathbf{\Lambda}_g^{\mathrm{NA}} &= \lim_{m\to+\infty} \max_{\alpha,i} \frac{\lambda_i^{(m,\alpha)}}{m} = \sup_{\mathcal{X}_0}(\theta_\zeta), \quad \theta_\zeta = \frac{1}{m} \frac{\sum_{\alpha,i} \lambda_i^{(m,\alpha)} |s_i^{(m,\alpha)}|^2}{\sum_{\alpha,i} |s_i^{(m,\alpha)}|^2} \\
\mathbf{J}_g^{\mathrm{NA}} &= \mathbf{\Lambda}_g^{\mathrm{NA}} - \mathbf{E}_g^{\mathrm{NA}} \\
\mathbf{L}^{\mathrm{NA}} &= \mathrm{lct}\,(\mathcal{X}, -(K_{\mathcal{X}} + \mathcal{L}); \mathcal{X}_0) - 1 \\
&= \sup\{c; (\mathcal{X}, -K_{\mathcal{X}} - \mathcal{L} + (c+1)\mathcal{X}_0) \text{ is sub-log-canonical}\} \\
\mathbf{D}_g^{\mathrm{NA}} &= -\mathbf{E}_g^{\mathrm{NA}} + \mathbf{L}^{\mathrm{NA}}.
\end{aligned}
$$

Theorem (Generalized slope formula, proof uses fibration construction)

For each $\mathbf{F} \in \{\mathbf{E}_g, \mathbf{J}_g, \mathbf{D}_g\}$*, we have the identity:*

$$
\mathbf{F}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{s\to+\infty} \frac{\mathbf{F}(\varphi(s))}{s}.
$$

Stability (generalization of Tian, Donaldson, . . .)

- $(X, \mathbb{T})$ is reduced uniformly g-Ding stable if there exists $\gamma > 0$ such that for any $\tilde{\mathbb{T}}$-equivariant test configuration $(\mathcal{X}, \mathcal{L})$:

$$\mathbf{D}_g^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \mathbf{J}_{g,\tilde{\mathbb{T}}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$$

  where

$$\mathbf{J}_{g,\tilde{\mathbb{T}}}(\mathcal{X}, \mathcal{L}) = \inf_{\xi \in \tilde{N}_{\mathbb{R}}} \mathbf{J}_g^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi).$$

- It is g-(Ding)-polystable if $\mathbf{D}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq 0$ and equality holds iff it is induced by a holomorphic vector field.

Theorem (Liu-Xu-Zhuang, Blum-Liu-Xu-Zhuang)

$(X, \mathbb{T})$ is reduced uniformly g-Ding-stable if and only if it is polystable with respect to $\tilde{\mathbb{T}}$.

MMP and Valuative criterion

Theorem (Generalizing K. Fujita and L. , based on L. -Xu)

$(X, \mathbb{T})$ *is reduced uniformly g-Ding stable if and only if it is so for special test configurations (the central fibre is a normal Fano variety).*

The proof uses Minimal Model Program and the fibration construction.

- $X^{\mathrm{div}}_{\mathbb{Q}}$: set of divisorial valuations. $v = \mathrm{ord}_E \in X^{\mathrm{div}}_{\mathbb{Q}}$.

  $A_X(v)$: log discrepancy of valuation; $S_g(v) = \frac{1}{\mathrm{V}_g} \int_0^{+\infty} \mathrm{vol}_g(\mathcal{F}_v^{(t)}) dt$.

Theorem (Han-L. , generalizing Fujita and L.)

$(X, \mathbb{T})$ *is reduced uniformly stable if and only if there exists* $\delta > 1$ *such that for any* $\tilde{\mathbb{T}}$*-invariant valuation* $v \in X^{\mathrm{div}}_{\mathbb{Q}}$*, there exists a* $\xi \in \tilde{N}_{\mathbb{R}}$ *such that* $A_X(v_\xi) - \delta \cdot S_g(v_\xi) \geq 0$.

Proved by studying filtration associated to the valuation $v = r(\mathrm{ord}_{\mathcal{X}_0})$.

Yau-Tian-Donaldson conjecture

Theorem (Han-L., generalization of Berman-Boucksom-Jonsson, L.)

$(X, \mathbb{T})$ *admits a* g*-soliton metric if and only if* $(X, \mathbb{T})$ *is reduced uniformly* g*-Ding stable over special test configurations.*

$\Longrightarrow$ Analytic criterion + slope formula

$\Longleftarrow$ Proof by contradiction. Suppose that $\mathbf{M}_g$ is not reduced coercive.

- Take a destabilizing sequence and construct a destabilizing geodesic ray, based on convexity and compactness
- Blow up multiplier ideal sheaf to construct test configurations
- Approximate slopes of energy functional by non-Archimedean functionals of test configurations and get a contradiction.

Problem: Prove partial C^0-estimates for the Bakry-Emery Ricci curvature tensor $Ric(\omega_\varphi) + \sqrt{-1}\partial\bar{\partial}f_\varphi$ under appropriate assumptions.

log Fano case

- X: projective normal variety; D: an effective Weil divisor satisfying: $-(K_X + D)$ is $\mathbb{Q}$-Cartier and ample.

  (X, D) has klt singularities if for any $v = \mathrm{ord}_E \in X_{\mathbb{Q}}^{\mathrm{div}}$, $A_{(X,D)}(v) = \mathrm{ord}_D(K_{X'/X}) + 1$ where E is an ordinary divisor on $X' \to X$.
- Example 1: Orbifold (X, D), for any $p \in X$, $\exists$ a neighborhood U s.t.

$$U \cong D/\Gamma, \quad D = \sum_i (1 - d_i^{-1}) D_i$$

  where Γ is a finite group that acts linearly on U and D_i is the set of points with non-trivial stabilizers isomorphic to $\mathbb{Z}_{d_i}$.
- Example 2: (S, Δ) is log Fano variety, $-(K_S + \Delta) = \gamma L$ with $\gamma > 0$, then the cone singularity $(X = \mathrm{Spec}(\bigoplus_m H^0(S, mL)), D = \mathrm{Spec}(\bigoplus_m H^0(\Delta, mL)))$ is klt.

Theorem (Han-L.)

$(X, D, \mathbb{T})$ admits a g-soliton iff it is reduced uniformly g-Ding stable.

Perturbation approach (developed by **L.** -Tian-Wang)

Choose resolution of singularities $\mu : Y \to X$ with SNC exceptional divisors $\{E_i\}$. There exist $\beta_i = A_{(X,D)}(E_i) > 0$ and a positive perturbation $P = \mu^* L - \sum_i \theta_i E_i$:

$$\begin{aligned} -K_{X'} - \sum_i (1 - \beta_i + \frac{\epsilon}{1+\epsilon}\theta_i) E_i &= \mu^*(-K_X - D) - \frac{\epsilon}{1+\epsilon} \sum_i \theta_i E_i \\ -(K_{X'} + B_\epsilon) &= L_\epsilon = \frac{1}{1+\epsilon}(\mu^* L + \epsilon P) > 0. \end{aligned}$$

Idea to overcome the difficulty caused by singularities: carry out the construction on (X', B_ϵ) and let $\epsilon \to 0$.

- Construct a destabilizing geodesic ray.
- Use multiplier ideals of perturbed sub-geodesic rays to construct destabilizing test configurations of (X', B_ϵ).
- Use valuative criterion to prove the uniform stability of (X', B_ϵ).
- Prove uniform convergence estimates as $\epsilon \to 0$ to get contradiction.

Kähler cone metric

- $Y = \mathrm{Spec}(R)$: affine variety isolated singularity $o \in Y$.

  $\hat{\mathbb{T}} \cong (\mathbb{C}^*)^{r+1}$-action $\Rightarrow$ Weight decomposition: $R = \sum_{\hat{\alpha} \in \mathbb{Z}^{r+1}} R_{\hat{\alpha}}$.

- $\hat{T}$-invariant Radius function: $r : Y \to \mathbb{R}_{\geq 0}$, $S = \{r = 1\}$.

  Kähler form: $\hat{\omega} = \sqrt{-1}\partial\bar{\partial}r^2$, $\hat{\omega}(\cdot, J\cdot) = dr^2 + r^2 g_S$.

- Reeb vector field: $J(r\partial_r) \in \hat{N}_{\mathbb{R}}^+ = \left\{\xi \in \hat{N}_{\mathbb{R}}; \langle \hat{\alpha}, \hat{\xi} \rangle > 0\right\}$ (Reeb cone).

  Quasi-regular: $\hat{\xi} \in N_{\mathbb{Q}}^+$. $\langle v_{\hat{\xi}} \cong \mathbb{C}^*$ and $Y/\langle v_{\hat{\xi}} \rangle = (X, D)$ is an orbifold.

  Regular: if $\langle v_\xi \rangle$ acts freely on Y^*. $Y/\langle v_{\hat{\xi}} \rangle$ is a projective manifold.

  Irregular: $\hat{\xi} \notin N_{\mathbb{R}}^+$. $\langle v_\xi \rangle \cong (\mathbb{C}^*)^d$ for $d > 1$ (main interest).

Ricci-flat Kähler cone (Martelli-Sparks-Yau)

- s: no-where vanishing section of $|mK_Y|$, $\mathbb{T}$-equivariant: $t \circ s = t^{\alpha} s$.

  Canonical volume form: $dV_Y = \left(\sqrt{-1}^{m(n+1)^2} s \wedge \bar{s}\right)^{1/m}$.

- Ricci-flat Kähler cone equation:

  $$(\sqrt{-1}\partial\bar{\partial} r^2)^{n+1} = dV_Y \quad \Longleftrightarrow \quad Ric(\hat{\omega}) = 0.$$

  Normalization of Reeb vector fields:

  $$\mathfrak{L}_{r\partial_r} dV_Y = 2(n+1) dV_Y \quad \Longleftrightarrow \quad \mathfrak{L}_{v_{\hat{\xi}}} s = (n+1)s$$

- Regular example:
  X a Fano manifold. h_{KE}: a KE Hermitian metric on $-K_X$.
  Then $r = h_{\mathrm{KE}}^{\frac{1}{n+1}}$ is a radius function of Ricci-flat Kähler cone on $Y = \mathrm{Spec}\left(\bigoplus_{m\in\mathbb{N}} H^0(X, mL)\right)$ for any $L = \gamma^{-1}(-K_X)$ with $\gamma > 0$.
  For example, $X = \mathbb{P}^n$, $L = \frac{1}{n+1}(-K_X)$, $Y = \mathbb{C}^{n+1}$.

- Sasaki manifold $S = \{r = 1\}$. CR structure $\mathfrak{D} = JT_{\mathbb{R}}S \cap T_{\mathbb{R}}S$

  Contact form: $\eta = -Jd\log r$ determined by:

$$\eta(\hat{\xi}) = 1, \quad \eta|_{\mathfrak{D}} = 0.$$

  Kähler form: $\hat{\omega} = \sqrt{-1}\partial\bar{\partial}r^2 = d(r^2\eta) = r^2 d\eta + 2rdr \wedge \eta$.

  $\hat{\omega}^{n+1} = 2(n+1)r^{2n+1}dr \wedge (d\eta)^n \wedge \eta$.

  $dV_S = dV_S^{\hat{\xi}} = \iota_{\partial_r} dV_Y \longrightarrow dV_Y = r^{2n+1}dr \wedge dV_S$.

- Rewrite the Ricci-flat equation:

$$(\sqrt{-1}\partial\bar{\partial}r^2)^{n+1} = dV_Y \quad \Longleftrightarrow \quad (d\eta)^n \wedge \eta = dV_S.$$

Deformation of Reeb vector fields

- For $\hat{\xi} \in \hat{N}_{\mathbb{R}}^{+}$, set $\mathcal{R}^{\hat{\xi}} = \left\{ r = r_0 e^{\varphi/2} \text{ is a radius function w.r.t. } \hat{\xi} \right\}$.
- Fix a reference $\hat{\chi} \in \hat{N}_{\mathbb{R}}^{+}$ and a radius function r_0 with respect to $\hat{\chi}$.

$$\hat{\xi} \in \hat{N}_{\mathbb{R}}^{+} \longrightarrow r := r_0^{\hat{\xi}} \in \mathcal{R}^{\hat{\xi}}$$

$$J(r\partial_r) = \hat{\xi}, \quad S = \{r = 1\} = \{r_0 = 1\}.$$

Transformation of Reeb vector fields and contact forms:

$$\hat{\xi} = \eta(\hat{\xi})\hat{\chi} + \xi^h \quad \Longrightarrow \quad \begin{cases} \eta = \eta_{r_0^{\hat{\xi}}} = \eta_0(\hat{\xi})^{-1}\eta_0 \\ \\ r\partial_r = \eta(\hat{\xi}) r_0 \partial_{r_0} + J(\xi^h). \end{cases}$$

- Ricci-flat equation to g-soliton (Apostolov et al. via Tanaka-Webster):

$$\begin{aligned} (d\eta)^n \wedge \eta = dV_S \quad &\Leftrightarrow \quad \eta_0(\hat{\xi})^{-n-1}(d\eta_0)^n \wedge \eta_0 = \eta_0(\hat{\xi}) dV_S^{\hat{\chi}} \\ &\Leftrightarrow \quad \eta_0(\hat{\xi})^{-n-2}(d\eta_0)^n \wedge \eta_0 = dV_S^{\hat{\chi}}. \end{aligned}$$

Reduce to g-soliton equation on Fano orbifolds

- $(X, D) = Y/\langle v_{\hat{\chi}} \rangle$: Fano orbifold and an orbifold l.b. $L \to X$ satisfying:

$$-(K_X + D) = L.$$

The no-vanishing section of $|K_Y|$ and dV_Y:

$$s = dz \wedge dw, \quad dV_Y = (\sqrt{-1})^{n^2+1} dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}.$$

- $r = r_0 e^{\varphi/2}$: radius function w.r.t. $\hat{\chi} \leftrightarrow$ orbifold metric $h = h_0 e^{-\varphi}$ on $-K_X$. $\eta = (\partial - \bar{\partial}) \log h$: connection form, $Ker(\eta)$ is the horizontal distribution.

$$(d\eta)^n \wedge \eta = \omega^n \wedge d\psi, \quad dV_S^{\hat{\chi}} = \iota_{\partial_r} dV_Y|_S = 2d\psi \wedge \Omega_\varphi.$$

$$v_{\hat{\xi}} = (n+1)w\partial_w + v_{\tilde{\xi}} = (n+1)w\partial_w + \theta_{v_\xi} w\partial_w + v_\xi^h \quad \Longrightarrow \quad \eta(\hat{\xi}) = 1 + \frac{\theta_{v_\xi}}{n+1}.$$

$$\eta(\hat{\xi})^{-n-2}(d\eta)^n \wedge \eta = dV_S^{\hat{\chi}} \quad \Longleftrightarrow \quad (n+1+\theta_{v_\xi})^{-n-2}\omega_\varphi^n = e^{-\varphi}\Omega_0.$$

Stability of affine cones vs. weighted stability

- $\mathcal{Y}$: a special test configuration of Y.
  $(\mathcal{X}, \mathcal{D}) := \mathcal{Y}_0/\langle v_{\hat{\chi}}\rangle$ a special TC of (X, D)
  Volume of $\hat{\xi} = \frac{1}{n+1}\hat{\chi} + \xi \in \hat{N}^+_{\mathbb{R}}(\mathcal{Y}_0)$:

$$\begin{aligned}\mathrm{vol}(\hat{\xi}) &= \lim_{p\to+\infty} \frac{\dim_{\mathbb{C}} R/\{\mathrm{wt}_{\hat{\xi}} \geq p\}}{p^{n+1}/(n+1)!} = \int_{\mathbb{R}} \frac{-d\mathrm{vol}(\mathcal{F}^{(t)}_{\xi})}{(1+t)^{n+1}} \\ &= \int_{\mathcal{X}_0} \frac{\omega^n}{(1+\theta_{v_\xi})^{n+1}} = \int_{P_0} \frac{\mathrm{DH}_{\hat{T}}(\mathcal{X}_0)}{(1+\langle x, \xi\rangle)^{n+1}}.\end{aligned}$$

- As a consequence, with $g = (1+\langle x, \xi\rangle)^{-n-2}$ and $\mathrm{Fut}_\xi \equiv 0$, we have:

$$\begin{aligned}\mathbf{D}^{\mathrm{NA}}(\mathcal{Y}) &= \frac{1}{\mathrm{V}_g} D_\zeta \mathrm{vol}(\hat{\xi}) = \frac{1}{\mathrm{V}_g}\int_{\mathcal{X}_0} \frac{-\theta_\zeta \omega^n}{(1+\theta_{v_\xi})^{n+2}} \\ &= \mathbf{D}^{\mathrm{NA}}_g(\mathcal{X}, \mathcal{D}, \mathcal{L})\end{aligned}$$

Stability of Y (Collins-Székelyhidi) $\longleftrightarrow$ g-Ding stability of (X, D)

Toric case

- The YTD theorem for toric g-solitons (Theorem 2) implies

Theorem (Futaki-Ono-Wang)

A toric affine variety with isolated singularity admits a Ricci-flat Kähler cone.

- An irregular example: $X = \mathrm{Bl}_p\mathbb{P}^2$. Y: cone $C(X, -K_X)$.

  P: the trapezoid; moment cone of Y is the standard cone over P.

  Reeb cone: spanned by $\langle -1,-1,1\rangle, \langle 1,1,1\rangle, \langle 1,0,1\rangle, \langle 0,1,1\rangle$.

  (normalized) Reeb vector field $\hat{\xi} = (a, b, 1)$.

$$\mathrm{vol}(\hat{\xi}) = \mathrm{vol}\left(\mathcal{C}^* \cap \{\langle x, \hat{\xi}\rangle \leq 1\}\right) = \int_P \frac{dx}{(ax_1 + bx_2 + 1)^3}$$

  obtains the minimum at $a = b = \frac{4-\sqrt{13}}{3} = a_*$

  $\Rightarrow \exists$ g-soliton with $g = (a_*(x_1 + x_2) + 1)^{-3}$

  $\Rightarrow \exists$ Ricci-flat Kähler cone metric with Reeb vector field given by $3 \cdot (a_*, a_*, 1)$.

Thanks for your attention!