YTD conjecture for generalized Kähler solitons and applications

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1. Generalized Kähler solitons on Fano varieties

2. Application to Ricci-flat Kähler cone metric (revisit the work of Apostolov-Calderbank-Jubert-Lahdili)
X a projective manifold. $L \to X$ an ample holomorphic line bundle.

Kähler metric $\omega = \sqrt{-1} \sum_{i,j} \omega_{i\bar{j}} dz_i \wedge d\bar{z}_j > 0$ in $2\pi \cdot c_1(L)$.

$T \cong (S^1)^r$, with Lie algebra $N_\mathbb{R}$, acts holomorphically on $(X, L)$

Hamiltonian action: $\omega$ is $T$-invariant and there exists a moment map:

$$m : X \to N_\mathbb{R}^\vee \cong \mathbb{R}^r, \quad \iota_\xi \omega = d\langle m, \xi \rangle$$

for any $\xi \in N_\mathbb{R}$.

Atiyah-Guillemin-Sternberg: the image $m(X)$ is a convex polytope (which can be recovered by algebraic data using weight decompositions).

Duistermaat-Heckman measure: $DH_T := m_*(\omega^n)$ does not depend on the Kähler form in the same Kähler class. \Rightarrow For any smooth function $g$ on $\mathbb{R}^r$,

$$V_g := \int_X g(m) \omega^n = \int_P g(x) DH_T$$

is independent of $\omega \in [\omega]$. 

Generalized Kähler solitons on Fano varieties
Fano manifolds

- X Fano: \(-K_X := \wedge^n T_{\text{hol}} X\) is ample, \(c_1(X) = c_1(-K_X) > 0\).
- Local holomorphic frame: \(\partial_z = \partial_{z_1} \wedge \cdots \wedge \partial_{z_n}, dz = dz_1 \wedge \cdots \wedge dz_n\).
- Hermitian metric on \(-K_X\) \(\leftrightarrow\) volume form

\[
h_\varphi = h_0 e^{-\varphi} \quad \leftrightarrow \quad \Omega_\varphi = |\partial_z|_{h_\varphi}^2 (\sqrt{-1})^n dz \wedge d\bar{z}.
\]

Chern curvature:

\[
\text{Ric}(\Omega_\varphi) = \text{Ric}(\Omega_0) + \sqrt{-1} \partial \bar{\partial} \varphi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi = \omega_\varphi \in 2\pi c_1(X).
\]

Space of Kähler potentials:

\[
\mathcal{H}(\omega_0) = \left\{ \varphi \in C^\infty(X); \quad \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \right\}.
\]

- holomorphic vector field \(v\) \(\rightarrow\) a canonical Hamiltonian function:

\[
\theta_v(\varphi) = -\frac{\iota_v \Omega_\varphi}{\Omega_\varphi} \quad \Rightarrow \quad \iota_v \omega_\varphi = \sqrt{-1} \bar{\partial} \theta_v(\varphi).
\]

If \(v = \frac{1}{2}(-J\xi - \sqrt{-1}\xi)\), \(\theta_v(\varphi)\) is real \(\leftrightarrow\) \(\xi\) is Killing w.r.t. \(\omega_\varphi\).
**g-soliton equations**

- Moment map of $T = (S^1)^r$ action with respect to $\omega_\varphi$:

  $$m_\varphi : X \rightarrow N^\vee_R, \quad \langle m_\varphi, \xi \rangle = \theta_{v_\xi}(\varphi).$$

  $$v_\kappa = \frac{1}{2}(-J_\xi_\kappa - \sqrt{-1}\xi_\kappa): \kappa\text{-th generator of (C*)}^r$$

  $$m_\varphi = (\theta_{\kappa,0} + v_\kappa(\varphi))_{\kappa=1}^r : X \rightarrow \mathbb{R}^r.$$

- Let $g : P \rightarrow \mathbb{R} > 0$ be a smooth function. $g$-soliton equation:

  $$g(m_\varphi)(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-\varphi}\Omega_0$$

  $$\iff \text{Ric}(\omega_\varphi) - \omega_\varphi = \sqrt{-1}\partial\bar{\partial}\log g(m_\varphi).$$

  **Examples:**

  - $g = \exp(\langle x, \xi \rangle)$: Kähler-Ricci soliton;
  - $g = 1$: Kähler-Einstein.
  - $g(x) = 1 + \langle x - \bar{x}, \xi \rangle$: Mabuchi soliton
  - $g = (c + \langle x, \xi \rangle)^{-n-2}$: Ricci-flat Kähler cone metric (Apostolov et al.)
Futaki invariant and Matsushima type result

- Set $g_\varphi := g(m_\varphi)$ and $f_\varphi = \log g_\varphi$. For any holomorphic vector field $\nu$

  $$
  \text{Fut}_g(\nu) := \int_X \nu (-\log \frac{\omega_\varphi^n}{\Omega_\varphi} - f_\varphi) e^{g_\varphi} \omega_\varphi^n = -\int_X \theta_\nu(\varphi) g_\varphi \omega_\varphi^n.
  $$

  **Fact:** $\text{Fut}_g$ does not depend on the choice of $\omega \in 2\pi c_1(L)$.

  $\exists$ $g$-soliton in $\implies$ $\text{Fut}_g \equiv 0$.

- There is a generalized Matsushima reductivity result:

  **Theorem**

  If $(X, \mathbb{T})$ admits a $g$-soliton, then the following group is reductive:

  $$
  \text{Aut}(X, \mathbb{T}) = \{ \sigma \in \text{Aut}(X); t \circ \sigma = \sigma \circ t \ \forall t \in \mathbb{T} \}.
  $$

  (1)

  This follows from the identity:

  $$
  \text{aut}(X, \mathbb{T}) \cong \{ \theta \in C^\infty(X, \mathbb{C})^T; (\Delta + \nabla f)\theta = -\theta \}.
  $$

  (2)
Toric case

- $X$: toric manifold: $\mathbb{T} \cong (\mathbb{C}^*)^n$ has an open orbit $\iff$ moment polytope $P$
- $X$: toric Fano manifold $\iff$ reflexive polytope $P \ni 0$

\[ P : \ell_{n_i}(x) = \langle x, n_i \rangle \geq -1, \quad i = 1, \ldots, K. \]

- Holomorphic vector field $\iff \zeta \in \mathbb{R}^n$

\[ \text{Fut}_g(\zeta) = \int_P \langle x, \zeta \rangle gdx = 0 \iff g\text{-weighted barycenter} = 0. \]

If $g = f(\langle x, \xi \rangle)$ for a strictly convex $f$, this determines $\xi$ uniquely.

Examples: $f = \exp$ (KR soliton); $f = \frac{1}{(c+t)^d}$ for any $c > 0$ and $d > 0$.

Theorem (Han-L., a consequence of YTD theorem)

For toric manifolds, there exists a $g$-soliton iff the $g$-weighted barycenter$=0$.

- For Kähler-Ricci soliton this was proved by Wang-Zhu (2000).
Energy functionals and coercivity

- $g$-weighted functionals generalizing the unweighted case:

\[ E_g(\varphi) = \frac{1}{V_g}\int_0^1 dt \int_X \varphi \omega^n_t, \quad \Lambda_g(\varphi) = \frac{1}{V_g} \int_X \varphi g_{\varphi_0} \omega^n_0 \]

\[ I_g(\varphi) = \int_X \varphi (g_{\varphi_0} \omega^n_0 - g_{\varphi} \omega^n_0), \quad J_g(\varphi) = \Lambda_g(\varphi) - E_g(\varphi) \]

\[ L(\varphi) = -\log \left( \int_X e^{-\varphi} \Omega_0 \right), \quad D_g(\varphi) = -E_g(\varphi) + L(\varphi) \]

\[ M_g(\varphi) = \frac{1}{V_g} \int_X \log \frac{g_{\varphi} \omega^n_\varphi}{\Omega_0} g_{\varphi} \omega^n_\varphi - (I_g - J_g)(\varphi). \]

- Automorphisms: $\text{Aut}(X, \mathbb{T}) = \{ \sigma \in \text{Aut}(X); \sigma \cdot t = t \cdot \sigma, \forall t \in \mathbb{R} \}$

$\hat{T}$: maximal torus of $\text{Aut}(X, \mathbb{T})$; $T$: maximal compact torus of $\hat{T}$.

- $F_g$ is reduced coercive if $\exists \gamma, C > 0$ s.t. $\forall \hat{T}$-invariant $\varphi \in \mathcal{H}(\omega_0)$

\[ F_g(\varphi) \geq \gamma \cdot \inf_{\sigma \in \hat{T}} J_g(\sigma^* \omega_\varphi) - C. \]
Analytic criterion

**Theorem (Han-L., generalize Tian, Tian-Zhu, Phong-Song-Sturm-Weinkove, Darvas-Rubinstein)**

The following are equivalent:

1. $(X, \mathbb{T})$ admits a $g$-soliton.
2. $D_g$ is reduced coercive.
3. $M_g$ is reduced coercive.

- $1 \rightarrow 2$: generalize the argument of Darvas-Rubinstein based on the convexity and uniqueness:
- $2 \rightarrow 3$: $D_g \leq M_g$.
- $3 \rightarrow 2$: A duality argument.
- $2 \rightarrow 1$: Variational argument

**Theorem (Berndtsson, BBEGZ)**

$L$ is convex along geodesic segment in $\mathcal{E}^1_g$. It is affine iff the geodesic is generated by a holomorphic vector field that commutes with $\mathbb{T}$.

2 $\rightarrow$ 3: $D_g \leq M_g$.
3 $\rightarrow$ 2: A duality argument.
2 $\rightarrow$ 1: Variational argument
\( \varphi \in L^1(\omega_0^n) \) is \( \omega_0 \)-psh if it is u.s.c. and \( \psi + \varphi \) is psh (\( \omega_0 = \sqrt{-1} \partial \bar{\partial} \psi \)).

\[
\text{PSH}(\omega_0) = \{ \varphi; \varphi \text{ is } \omega_0 \text{-psh} \}.
\]

Non-pluripolar product:

\[
(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = \lim_{j \to +\infty} (\omega_0 + \sqrt{-1} \partial \bar{\partial} \max\{\varphi, -j\})^n.
\]

\( g = \prod_{\kappa} (x_{\kappa} + c)^d_{\kappa} \) such that \( P + c(1, \ldots, 1) \in \mathbb{R}^r_{>0} \), then

\[
\int_X f g \varphi (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = \int_X (f^T)^{[d]}(\sum_{\kappa} \theta_{\kappa}(\varphi) \omega_{FS, \kappa} + \omega_{\varphi})^{n+d}.
\]

**General \( g \), find polynomials \( g_j \to g \) uniformly and set:**

\[
\int_X f g \varphi \omega_\varphi^n = \lim_{j \to +\infty} \int_X f(g_j) \varphi \omega_\varphi^n.
\]
Fibration construction

- $P \to B$: hol. line bundle with Hermitian metric $h_P$; $\tilde{P}$ circle bundle.
- $L \to X$: hol. line bundle with $\mathbb{C}^*$-action, $S^1$-invariant Hermitian metric $h_L$
- $(Y, F) = (P^* \times (X, L))/\mathbb{C}^* \to B$: associated holomorphic fibre bundle.
- Induced horizontal distribution and splitting $TY = \pi^* TB \bigoplus TX$.
- Induced Hermitian metric $h_F$ on $F$ whose Chern curvature at $\tilde{P}$:
  $$\sqrt{-1} \partial \bar{\partial} \log h_F = \pi^* (\theta_v \sqrt{-1} \partial \bar{\partial} \log h_P) + \sqrt{-1} \partial \bar{\partial} \log h_L.$$  

The right-hand-side is considered as an equivariantly closed form on $X$.

- Example:
  - $(P \to B) = (\mathbb{P}^{d+1} \setminus \mathbb{P}_\infty \to \mathbb{P}^k)$, $\tilde{P} \simeq S^{2d+1}$, $\mathbb{P}^* = \mathbb{C}^{d+1} \setminus \{0\}$.
  - Curvature: $\theta_v(\varphi) \omega_{FS} + \omega_\varphi$.

  - $S^{[\tilde{d}]} = S^{2d_1+1} \times \cdots \times S^{2d_r+1} \to \mathbb{P}^{[\tilde{d}]} = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_r}$.
    
    - $(X^{[\tilde{d}]}, L^{[\tilde{d}]}) = S^{[\tilde{d}]} \times (S^1)^r (X, L) \to \mathbb{P}^{[\tilde{d}]}$.
    
    - Curvature: $\sum_{\kappa=1}^r \theta_\kappa(\varphi) \omega_{FS, \mathbb{P}^{d_\kappa}} + \omega_\varphi$. 

Generalized Kähler solitons on Fano varieties
Define:

\[ E^1 = \left\{ \varphi \in \text{PSH}(\omega_0) ; \int_X |\varphi| \omega^n_\varphi < +\infty \right\}, \quad \text{(Guedj-Zeriahi)} \]

\[ E^1_g = \left\{ \varphi \in \text{PSH}(\omega_0)^T ; \int_X |\varphi| g_\varphi \omega^n_\varphi < +\infty \right\} = (E^1)^T. \]

Strong topology: \( \varphi_j \to \varphi \) in strong topology iff it converges in \( L^1 \) and \( E_g(\varphi_j) \to E_g(\varphi) \) iff

\[ \varphi_j \to \varphi \text{ in } L^1, \quad \sup \varphi_j \to \sup \varphi, \quad I_g(\varphi, \varphi_j) \to 0. \]

**Theorem (generalized version of Berman-Boucksom-Eyssidieux-Guedj-Zeriahi)**

For any constant \( C > 0 \), the following subset is compact in strong topology:

\[ \left\{ \varphi \in E^1_g ; \int_X \log \frac{g_\varphi \omega^n_\varphi}{\Omega_0} g_\varphi \omega^n_\varphi < C, \sup \varphi = 0 \right\}. \]
Let $X \rightarrow \mathbb{P} = \mathbb{P}^{N_m-1}$ be the $\mathbb{T}$-equivariant Kodaira embedding via $|mL|$. 

$\sigma(s) = \exp(s\zeta), s \in [0, +\infty)$: one parameter subgroup of $GL(N_m, \mathbb{C})$. 

Limit scheme: $[\mathcal{X}_0] = \lim_{s \rightarrow +\infty} \sigma(s) \circ [X]$ and induced test configuration:

$$
\begin{align*}
\mathcal{X} &= \{(z, t) \in \mathbb{P}^{N_m-1} \times \mathbb{C}; z \in \sigma(-\log |t|^2) \circ X\} \\
\mathcal{L} &= (p_1^* \mathcal{O}_\mathbb{P}(1))^{1/m}.
\end{align*}
$$

Path in $\mathcal{H}(\omega_0)$: $\Phi = \{\varphi(s)\}$ with $\varphi(s) = \frac{1}{m} \log \frac{|\sigma(s) \cdot Z|^2}{|Z|^2}$.

For $\xi \in \mathbb{N}_\mathbb{R}$, twist $(\mathcal{X}, \mathcal{L}) \rightarrow (\mathcal{X}_\xi, \mathcal{L}_\xi) \Longleftrightarrow \sigma \cdot \exp(s\xi)$.

$\sigma$ commutes with $\mathbb{T} \rightarrow \mathbb{T} \times \mathbb{C}^*$ acts on $(\mathcal{X}_0, \mathcal{L}_0)$ and hence on $V_m = H^0(\mathcal{X}_0, mL_0)$ with weight decomposition:

$$
V_m = \bigoplus_{\alpha} V_{m,\alpha} = \bigoplus_{\alpha \in \mathbb{Z}^r} \bigoplus_{i \in \mathbb{Z}} V_{m,\alpha}(\lambda_i^{(m,\alpha)}).
$$
Non-Archimedean functionals

\[ E_{g}^{\text{NA}} = \frac{1}{V_{g}} \lim_{m \to +\infty} \frac{n!}{m^{n}} \sum_{\alpha, i} \lambda_{i}^{(m, \alpha)} \frac{g(\alpha)}{m} = \frac{1}{V_{g}} \int_{\mathcal{X}_{0}} \theta_{\zeta} g_{\varphi} \omega_{\varphi}^{n} \]

\[ \Lambda_{g}^{\text{NA}} = \lim_{m \to +\infty} \max_{\alpha, i} \frac{\lambda_{i}^{(m, \alpha)}}{m} = \sup_{\mathcal{X}_{0}}(\theta_{\zeta}), \quad \theta_{\zeta} = \frac{1}{m} \frac{\sum_{\alpha, i} \lambda_{i}^{(m, \alpha)} |s_{i}^{(m, \alpha)}|^{2}}{\sum_{\alpha, i} |s_{i}^{(m, \alpha)}|^{2}} \]

\[ J_{g}^{\text{NA}} = \Lambda_{g}^{\text{NA}} - E_{g}^{\text{NA}} \]

\[ L^{\text{NA}} = \text{lct}(\mathcal{X}, -(K_{\mathcal{X}} + \mathcal{L}); \mathcal{X}_{0}) - 1 \]

\[ = \sup \{ c; (\mathcal{X}, -K_{\mathcal{X}} - \mathcal{L} + (c + 1)\mathcal{X}_{0}) \text{ is sub-log-canonical} \} \]

\[ D_{g}^{\text{NA}} = -E_{g}^{\text{NA}} + L^{\text{NA}}. \]

Theorem (Generalized slope formula, proof uses fibration construction)

For each \( F \in \{ E_{g}, J_{g}, D_{g} \} \), we have the identity:

\[ F^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{s \to +\infty} \frac{F(\varphi(s))}{s}. \]
(\mathcal{X}, \mathbb{T}) is reduced uniformly $g$-Ding stable if there exists $\gamma > 0$ such that for any $\tilde{\mathbb{T}}$-equivariant test configuration $(\mathcal{X}, \mathcal{L})$:

$$D^\text{NA}_g(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot J^\text{NA}_{g, \tilde{\mathbb{T}}}(\mathcal{X}, \mathcal{L})$$

where

$$J_{g, \tilde{\mathbb{T}}}(\mathcal{X}, \mathcal{L}) = \inf_{\xi \in \tilde{\mathbb{N}}_\mathbb{R}} J^\text{NA}_g(\mathcal{X}_\xi, \mathcal{L}_\xi).$$

It is $g$-(Ding)-polystable if $D^\text{NA}(\mathcal{X}, \mathcal{L}) \geq 0$ and equality holds iff it is induced by a holomorphic vector field.

Theorem (Liu-Xu-Zhuang, Blum-Liu-Xu-Zhuang)

$(\mathcal{X}, \mathbb{T})$ is reduced uniformly $g$-Ding-stable if and only if it is polystable with respect to $\tilde{\mathbb{T}}$. 
Theorem (Generalizing K. Fujita and L., based on L. -Xu)

\((X, \mathbb{T})\) is reduced uniformly g-Ding stable if and only if it is so for special test configurations (the central fibre is a normal Fano variety).

The proof uses Minimal Model Program and the fibration construction.

- \(X^\text{div}_Q\): set of divisorial valuations. \(v = \text{ord}_E \in X^\text{div}_Q\).
- \(A_X(v)\): log discrepancy of valuation; \(S_g(v) = \frac{1}{V_g} \int_0^{+\infty} \text{vol}_g(F_v^t) dt\).

Theorem (Han-L., generalizing Fujita and L.)

\((X, \mathbb{T})\) is reduced uniformly stable if and only if there exists \(\delta > 1\) such that for any \(\mathbb{T}\)-invariant valuation \(v \in X^\text{div}_Q\), there exists a \(\xi \in \tilde{N}_\mathbb{R}\) such that

\[A_X(v_\xi) - \delta \cdot S_g(v_\xi) \geq 0.\]

Proved by studying filtration associated to the valuation \(v = r(\text{ord}_{X_0})\).
Yau-Tian-Donaldson conjecture

Theorem (Han-L. , generalization of Berman-Boucksom-Jonsson, L. )

\((X, \mathbb{T})\) admits a g-soliton metric if and only if \((X, \mathbb{T})\) is reduced uniformly g-Ding stable over special test configurations.

\[\implies\] Analytic criterion + slope formula

\[\iff\] Proof by contradiction. Suppose that \(M_g\) is not reduced coercive.

- Take a destabilizing sequence and construct a destabilizing geodesic ray, based on convexity and compactness
- Blow up multiplier ideal sheaf to construct test configurations
- Approximate slopes of energy functional by non-Archimedean functionals of test configurations and get a contradiction.

Problem: Prove partial \(C^0\)-estimates for the Bakry-Emery Ricci curvature tensor \(Ric(\omega_{\varphi}) + \sqrt{-1}\partial\bar{\partial}f_{\varphi}\) under appropriate assumptions.
**log Fano case**

- $X$: projective normal variety; $D$: an effective Weil divisor satisfying: $-(K_X + D)$ is $\mathbb{Q}$-Cartier and ample.

$(X, D)$ has klt singularities if for any $v = \text{ord}_E \in X^{\text{div}}_\mathbb{Q}$, $A(X, D)(v) = \text{ord}_D(K_{X'} / X) + 1$ where $E$ is an ordinary divisor on $X' \to X$.

- **Example 1**: Orbifold $(X, D)$, for any $p \in X$, $\exists$ a neighborhood $U$ s.t.

$$ U \cong D / \Gamma, \quad D = \sum_i (1 - d_i^{-1}) D_i $$

where $\Gamma$ is a finite group that acts linearly on $U$ and $D_i$ is the set of points with non-trivial stabilizers isomorphic to $\mathbb{Z}_{d_i}$.

- **Example 2**: $(S, \Delta)$ is log Fano variety, $-(K_S + \Delta) = \gamma L$ with $\gamma > 0$, then the cone singularity $(X = \text{Spec}(\bigoplus_m H^0(S, mL)), D = \text{Spec}(\bigoplus_m H^0(\Delta, mL)))$ is klt.

**Theorem (Han-L.)**

$(X, D, \mathbb{T})$ admits a $g$-soliton iff it is reduced uniformly $g$-Ding stable.
Choose resolution of singularities \( \mu : Y \to X \) with SNC exceptional divisors \( \{E_i\} \). There exist \( \beta_i = A(X,D)(E_i) > 0 \) and a positive perturbation \( P = \mu^*L - \sum \theta_iE_i \):

\[
-K_{X'} - \sum \left(1 - \beta_i + \frac{\epsilon}{1 + \epsilon} \theta_i\right)E_i = \mu^*(-K_X - D) - \frac{\epsilon}{1 + \epsilon} \sum \theta_iE_i
\]

\[
-(K_{X'} + B_\epsilon) = L_\epsilon = \frac{1}{1 + \epsilon} (\mu^*L + \epsilon P) > 0.
\]

Idea to overcome the difficulty caused by singularities: carry out the construction on \( (X', B_\epsilon) \) and let \( \epsilon \to 0 \).

- Construct a destabilizing geodesic ray.
- Use multiplier ideals of perturbed sub-geodesic rays to construct destabilizing test configurations of \( (X', B_\epsilon) \).
- Use valuative criterion to prove the uniform stability of \( (X', B_\epsilon) \).
- Prove uniform convergence estimates as \( \epsilon \to 0 \) to get contradiction.
Kähler cone metric

- $Y = \text{Spec}(R)$: affine variety isolated singularity $o \in Y$.

- $\hat{T} \cong (\mathbb{C}^*)^{r+1}$-action $\Rightarrow$ Weight decomposition: $R = \sum_{\hat{\alpha} \in \mathbb{Z}^{r+1}} R_{\hat{\alpha}}$.

- $\hat{T}$-invariant Radius function: $r : Y \to \mathbb{R}_{\geq 0}$, $S = \{r = 1\}$.

  Kähler form: $\hat{\omega} = \sqrt{-1} \partial \bar{\partial} r^2$, $\hat{\omega}(\cdot, J\cdot) = dr^2 + r^2 g_S$.

- Reeb vector field: $J(r \partial_r) \in \hat{N}_{\mathbb{R}}^+ = \left\{ \xi \in \hat{N}_{\mathbb{R}}; \langle \hat{\alpha}, \hat{\xi} \rangle > 0 \right\}$ (Reeb cone).

  - Quasi-regular: $\hat{\xi} \in \mathcal{N}_{\mathbb{Q}}^+$. $\langle v_{\hat{\xi}} \cong \mathbb{C}^* \text{ and } Y/\langle v_{\hat{\xi}} \rangle = (X, D) \text{ is an orbifold}.$

  - Regular: if $\langle v_{\xi} \rangle$ acts freely on $Y^*$. $Y/\langle v_{\xi} \rangle$ is a projective manifold.

  - Irregular: $\hat{\xi} \not\in \mathcal{N}_{\mathbb{R}}^+$. $\langle v_{\hat{\xi}} \rangle \cong (\mathbb{C}^*)^d$ for $d > 1$ (main interest).

Application to Ricci-flat Kähler cone metric, (revisit the work of Apostolov-Calderbank-Jubert-Lahdili)
Ricci-flat Kähler cone (Martelli-Sparks-Yau)

- $s$: no-where vanishing section of $|mK_Y|$, $\mathbb{T}$-equivariant: $t \circ s = t^\alpha s$.

  Canonical volume form: $dV_Y = \left(\sqrt{-1}^{m(n+1)^2} s \wedge \bar{s}\right)^{1/m}$.

- Ricci-flat Kähler cone equation:
  \[
  (\sqrt{-1} \partial \bar{\partial} r^2)^{n+1} = dV_Y \iff \text{Ric}(\hat{\omega}) = 0.
  \]

Normalization of Reeb vector fields:
\[
\mathcal{L}_{r \partial_r} dV_Y = 2(n+1)dV_Y \iff \mathcal{L}_{v_{\xi}} s = (n+1)s
\]

- Regular example:
  $X$ a Fano manifold. $h_{KE}$: a KE Hermitian metric on $-K_X$.

  Then $r = h_{KE}^{\frac{1}{n+1}}$ is a radius function of Ricci-flat Kähler cone on
  $Y = \text{Spec} \left( \bigoplus_{m \in \mathbb{N}} H^0(X, mL) \right)$ for any $L = \gamma^{-1}(-K_X)$ with $\gamma > 0$.

  For example, $X = \mathbb{P}^n$, $L = \frac{1}{n+1}(-K_X)$, $Y = \mathbb{C}^{n+1}$.

Application to Ricci-flat Kähler cone metric, (revisit the work of Apostolov-Calderbank-Jubert-Lahdili)
• Sasaki manifold $S = \{ r = 1 \}$. CR structure $\mathcal{D} = JT_R S \cap T_R S$

Contact form: $\eta = -Jd \log r$ determined by:

$$\eta(\hat{\xi}) = 1, \quad \eta|_{\mathcal{D}} = 0.$$  

Kähler form: $\hat{\omega} = \sqrt{-1} \partial \bar{\partial} r^2 = d(r^2 \eta) = r^2 d\eta + 2r dr \wedge \eta$.

$\hat{\omega}^{n+1} = 2(n + 1)r^{2n+1} dr \wedge (d\eta)^n \wedge \eta$.

$dV_S = dV_{\hat{\xi}} = \iota_{\partial_r} dV_Y \longrightarrow dV_Y = r^{2n+1} dr \wedge dV_S$.

• Rewrite the Ricci-flat equation:

$$(\sqrt{-1} \partial \bar{\partial} r^2)^{n+1} = dV_Y \iff (d\eta)^n \wedge \eta = dV_S.$$
Deformation of Reeb vector fields

- For \( \hat{\xi} \in \hat{N}_R^+ \), set \( \mathcal{R}\hat{\xi} = \left\{ r = r_0 e^{\phi/2} \right\} \).

- Fix a reference \( \hat{\chi} \in \hat{N}_R^+ \) and a radius function \( r_0 \) with respect to \( \hat{\chi} \).

\[
\hat{\xi} \in \hat{N}_R^+ \rightarrow r := r_0 \hat{\xi} \in \mathcal{R}\hat{\xi}
\]

\[
J(r \partial_r) = \hat{\xi}, \quad S = \{ r = 1 \} = \{ r_0 = 1 \}.
\]

Transformation of Reeb vector fields and contact forms:

\[
\hat{\xi} = \eta(\hat{\xi}) \hat{\chi} + \xi^h \quad \Rightarrow \quad \begin{cases} 
\eta = \eta_{r_0} = \eta_0(\hat{\xi})^{-1} \eta_0 \\
r \partial_r = \eta(\hat{\xi}) r_0 \partial_{r_0} + J(\xi^h).
\end{cases}
\]

- Ricci-flat equation to \( g \)-soliton (Apostolov et al. via Tanaka-Webster):

\[
(d\eta)^n \wedge \eta = dV_S \quad \Leftrightarrow \quad \eta_0(\hat{\xi})^{-n-1}(d\eta_0)^n \wedge \eta_0 = \eta_0(\hat{\xi})dV_{\hat{\chi}} \\
\Leftrightarrow \quad \eta_0(\hat{\xi})^{-n-2}(d\eta_0)^n \wedge \eta_0 = dV_{\hat{\chi}}.
\]

Application to Ricci-flat Kähler cone metric, (revisit the work of Apostolov-Calderbank-Jubert-Lahdili)
Reduce to g-soliton equation on Fano orbifolds

- \((X, D) = Y/\langle \nu_\hat{\chi} \rangle\): Fano orbifold and an orbifold l.b. \(L \to X\) satisfying:

\[-(K_X + D) = L.\]

The no-vanishing section of \(|K_Y|\) and \(dV_Y\):

\[s = dz \wedge dw, \quad dV_Y = (\sqrt{-1})^{n^2+1} dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}.\]

- \(r = r_0 e^{\varphi/2}\): radius function w.r.t. \(\hat{\chi}\) ↔ orbifold metric \(h = h_0 e^{-\varphi}\) on \(-K_X\).

\(\eta = (\bar{\partial} - \partial) \log h\): connection form, \(\text{Ker}(\eta)\) is the horizontal distribution.

\[(d\eta)^n \wedge \eta = \omega^n \wedge d\psi, \quad dV_\hat{\chi}^S = \nu_{\partial_r} dV_Y|_S = 2d\psi \wedge \Omega_\varphi.\]

\(v_\hat{\xi} = (n+1)w_\partial w + v_\hat{\xi} = (n+1)w_\partial w + \theta v_\xi w_\partial w + v_\xi^h \quad \Rightarrow \quad \eta(\hat{\xi}) = 1 + \frac{\theta v_\xi}{n+1}.\]

\[\eta(\hat{\xi})^{-n-2}(d\eta)^n \wedge \eta = dV_\hat{\chi}^S \quad \iff \quad (n + 1 + \theta v_\xi)^{-n-2} \omega^n_\varphi = e^{-\varphi} \Omega_0.\]
Stability of affine cones vs. weighted stability

- $\mathcal{Y}$: a special test configuration of $Y$.
  \((\mathcal{X}, \mathcal{D}) := \mathcal{Y}_0/\langle v_\hat{\chi} \rangle\) a special TC of \((X, D)\)

Volume of $\hat{\xi} = \frac{1}{n+1} \hat{\chi} + \xi \in \hat{N}_\mathbb{R}^+(\mathcal{Y}_0)$:

$$\text{vol}(\hat{\xi}) = \lim_{p \to +\infty} \frac{\dim \mathbb{C} R/\{\text{wt} \hat{\xi} \geq p\}}{p^{n+1}/(n+1)!} = \int_{\mathbb{R}} \frac{-d\text{vol}(\mathcal{F}^{(t)}_{\hat{\xi}})}{(1 + t)^{n+1}}$$

$$= \int_{\mathcal{X}_0} \frac{\omega^n}{(1 + \theta v_\xi)^{n+1}} = \int_{P_0} \frac{\text{DH}_\mathcal{F}(\mathcal{X}_0)}{(1 + \langle x, \xi \rangle)^{n+1}}.$$

- As a consequence, with $g = (1 + \langle x, \xi \rangle)^{-n-2}$ and $\text{Fut}_\xi \equiv 0$, we have:

$$D^\text{NA}(\mathcal{Y}) = \frac{1}{V_g} D_\zeta \text{vol}(\hat{\xi}) = \frac{1}{V_g} \int_{\mathcal{X}_0} \frac{-\theta_\zeta \omega^n}{(1 + \theta v_\xi)^{n+2}}$$

$$= D^\text{NA}_g(\mathcal{X}, \mathcal{D}, \mathcal{L}).$$

Stability of $Y$ (Collins-Székelyhidi) $\iff$ $g$-Ding stability of $(X, D)$
The YTD theorem for toric $g$-solitons (Theorem 2) implies

**Theorem (Futaki-Ono-Wang)**

A toric affine variety with isolated singularity admits a Ricci-flat Kähler cone.

- An irregular example: $X = \text{Bl}_p \mathbb{P}^2$. $Y$: cone $C(X, -K_X)$.
  - $P$: the trapezoid; moment cone of $Y$ is the standard cone over $P$.
  - Reeb cone: spanned by $\langle -1, -1, 1 \rangle, \langle 1, 1, 1 \rangle, \langle 1, 0, 1 \rangle, \langle 0, 1, 1 \rangle$.
  - (normalized) Reeb vector field $\hat{\xi} = (a, b, 1)$.

\[
\text{vol}(\hat{\xi}) = \text{vol} \left(C^* \cap \{ \langle x, \hat{\xi} \rangle \leq 1 \} \right) = \int_P \frac{dx}{(ax_1 + bx_2 + 1)^3}
\]

obtains the minimum at $a = b = \frac{4 - \sqrt{13}}{3} = a^*$

$\Rightarrow \exists g$-soliton with $g = (a^*(x_1 + x_2) + 1)^{-3}

\Rightarrow \exists$ Ricci-flat Kähler cone metric with Reeb vector field given by $3 \cdot (a^*, a^*, 1)$. 

Application to Ricci-flat Kähler cone metric, (revisit the work of Apostolov-Calderbank-Jubert-Lahdili)
Thanks for your attention!