An inhomogeneous optimal degeneration problem for Fano varieties

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Table of Contents

X : an n-dim. $\mathbb{Q}$-Fano variety, normal Fano variety with klt singularities.

Val_X : real valuations on $\mathbb{C}(X)$ (field of rational functions).

$X^{\mathrm{div}}_{\mathbb{Q}}$: set of divisorial valuations: $v = c \cdot \mathrm{ord}_E$, dense in Val_X.

$A_X(v)$: log discrepancy of valuations $v \in \mathrm{Val}_X$.

Assume $v = c \cdot \mathrm{ord}_E$ with $\mu : Y \to X$ and E is prime on Y.

$$\mathrm{vol}(\mathcal{F}_v^{(t)}) := \lim_{m \to +\infty} \frac{h^0(\mu^*(-mK_X) - tmcE)}{m^n/n!}.$$

Fact: $t \mapsto \mathrm{vol}(\mathcal{F}_v^{(t)})^{1/n}$: decreasing to 0, concave and differentiable (on $[0, \lambda_{\max}(v)]$ by Boucksom-Favre-Jonsson, Lazarsfeld-Mustaţă). Set

$$\begin{aligned} \tilde{\mathbf{S}}(v) &= -\log\left(\frac{1}{V}\int_0^{+\infty} e^{-t}(-d\mathrm{vol}(\mathcal{F}_v^{(t)}))\right) \\ &= -\log\left(1 - \frac{1}{V}\int_0^{+\infty} e^{-t}\mathrm{vol}(\mathcal{F}_v^{(t)})dt\right) \end{aligned}$$

$$\tilde{\beta}(v) = \begin{cases} A_X(v) - \tilde{\mathbf{S}}(v) & \text{if } A_X(v) < +\infty \\ +\infty & \text{if } A_X(v) = +\infty. \end{cases}$$

Theorem (Han-L.)

$\exists$ a quasi-monomial valuation v that achieves the minimum of $\tilde{\beta}(v)$.

Quasi-monomial $\iff$ $\mathrm{rank}_{\mathbb{Q}}(v) + \mathrm{trans.deg.}(v) = n$ (Ein-Lazarsfeld-Smith).

Theorem (Han-L.)

The minimizing valuation that induces a special $\mathbb{R}$-test configuration is unique. The central fibre (W, ξ) of this special $\mathbb{R}$-test configuration is K-semistable.

Conjecture

The minimizer is absolutely unique and induces a special $\mathbb{R}$-test configurations.

Main results II

Theorem (Han-L.)

If (W, ξ) is K-semistable, there exists a unique $\langle\xi\rangle$-equivariant special test configuration with K-polystable (Z, ξ).

The above results are proved using purely algebraic techniques.

Remark

Works of [Chen-Wang, Chen-Sun-Wang]+[Dervan-Székelyhidi] (which are based on analytic techniques) showed that, for smooth Fano manifolds, there exists a quasi-monomial valuation that achieves the minimum of $\tilde{\beta}(v)$ and induces a special $\mathbb{R}$-test configuration, while the uniqueness remained.

Combing above Theorems with [Chen-Sun-Wang, Dervan-Székelyhidi], we get:

Corollary (Chen-Sun-Wang's conjecture)

The Gromov-Hausdorff limit of normalized Kähler-Ricci flow on any Fano manifold is unique and does not depend on the choice of initial metrics.

Example: toric case

If P is the polytope of a toric Fano variety, P is reflexive. Set

$$\mathbb{T} \cong (\mathbb{C}^*)^n, \quad N_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}), \quad N_{\mathbb{R}} = N_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Any holomorphic vector field in $\xi \in N_{\mathbb{R}}$ corresponds to a toric valuation wt_ξ:

$$\mathrm{wt}_\xi \left(\sum_\alpha f_\alpha \right) = \min \left\{ \langle \alpha, \xi \rangle; f_\alpha \neq 0 \right\}.$$

Then

$$\tilde{\beta}(\mathrm{wt}_\xi) = C(n) \int_P e^{-\langle \xi, y \rangle} dy. \tag{1}$$

This function is strictly convex in ξ and there is a unique minimizer ξ_*.

Theorem (Wang-Zhu, Berman-Berndtsson)

There exists a Kähler-Ricci soliton, whose soliton vector field is ξ_*.

Indeed, (X, ξ_*) is ($\mathbb{T}$-uniformly) K-polystable (see Theorem 13 later).

Normalized volume: local analogue

$\mathrm{Val}_{X,x}$: space of real valuations centered at a klt singularity $x \in X$. Volume of valuations (Ein-Lazarsfeld-Smith): for any $v \in \mathrm{Val}_{X,x}$,

$$\mathrm{vol}(v) \quad = \quad \lim_{m \to +\infty} \frac{\dim_{\mathbb{C}} \mathcal{O}_{X,x}/\{f; v(f) \geq m\}}{m^n/n!}.$$

Normalized volume (**L.**'15): $\quad \widehat{\mathrm{vol}}(v) \quad = \quad A_X(v)^n \cdot \mathrm{vol}(v).$

Theorem (Blum, Xu)

There exists a minimizing valuation that is quasi-monomial.

Theorem (L.-Xu, L.-Wang-Xu)

The finitely generated minimizing valuation is unique, induces a degeneration to a K-semistable Fano cone. There is further a unique K-polystable degeneration.

Uniqueness is proved to be true in general (Xu-Zhuang).

Conjecture (L. '15)

The minimizing valuation is always finitely generated.

X : $\mathbb{Q}$-Fano variety. $v \in X_{\mathbb{Q}}^{\mathrm{div}}$.

$$\mathbf{S}(v) = \frac{1}{V}\int_0^{+\infty} t(-d\mathrm{vol}(\mathcal{F}_v^{(t)}))dt = \frac{1}{V}\int_0^{+\infty} \mathrm{vol}(\mathcal{F}_v^{(t)})dt.$$

$$\beta(v) = A_X(v) - \mathbf{S}(v), \qquad \delta(v) = \frac{A_X(v)}{\mathbf{S}(v)}.$$

Theorem (Fujita, L.)

$(X, -K_X)$ is K-semistable iff $\beta(v) \geq 0$ (i.e. $\delta(v) \geq 1$) for any $v \in X_{\mathbb{Q}}^{\mathrm{div}}$.

Theorem (Blum-Jonsson, Blum-Liu-Xu)

There exists a minimizing valuation of δ that is quasi-monomial.

Theorem (Blum-Liu-Zhou)

The minimizing valuations that induce special test configurations are in general not unique, but the central fibres have common special degenerations.

1. Concavity of log function implies $\tilde{\beta}(v) \geq \beta(v)$.
2. β is homogeneous: $\beta(av) = a\beta(v)$. Set $f(a) = \tilde{\beta}(av)$ on $[0, +\infty)$.

$$f(a) \text{ is strictly convex (for } v \neq v_{\mathrm{triv}}), \quad f'(0) = \beta(v).$$

Lemma (Properness)

For any $\epsilon > 0$, there exists $C = C(\epsilon)$ s.t. for any $a \in [0, +\infty)$,

$$f(a) \geq (A(v) - \epsilon)a) - \log a - C.$$

Corollary (Minimizing along a ray)

1. *$a \mapsto \tilde{\beta}(av)$ admits a unique minimum over $[0, +\infty)$.*
2. *X is K-semistable if and only if $\tilde{\beta}(v) \geq 0$.*

Define:

$$\tilde{\beta}_*(v) = \min_{a \in [0, +\infty)} \tilde{\beta}(av) = \tilde{\beta}(a_*(v)v).$$

Proof of properness

Recall that $f(x) := V^{-1/n}\mathrm{vol}(\mu^*(-K_X) - xE)^{1/n}$ is decreasing, concave on $[0, \lambda_{\max}(v))$, and differentiable. Fix $0 < \epsilon \ll 1$ s.t. $f(\epsilon) < f(0) = 1$. Set $C = -f'(\epsilon) > 0$, $T = \frac{1+C\epsilon}{C}$. Define a majorant:

$$\hat{f}(x) = \begin{cases} 1 & x \in [0, \epsilon] \\ 1 + C\epsilon - Cx & x \in (\epsilon, T] \\ 0 & x \in (T, +\infty). \end{cases}$$

Calculation shows that (with $v = \mathrm{ord}_E$):

$$\begin{aligned} e^{-\tilde{\mathbf{S}}(av)} &= 1 - \frac{1}{V}\int_0^{+\infty} \mathrm{vol}(-K_X - \frac{x}{a}E)e^{-x}dx = 1 - a\int_0^{+\infty} f^n(x)e^{-ax}dx \\ &\geq 1 - a\int_0^{\infty} \hat{f}^n(x)e^{-ax}dx = nCa^{-1}e^{-a\epsilon}(1 + O(a^{-1})). \end{aligned}$$

So $\tilde{\beta}(av) = A(av) + \log e^{-\tilde{\mathbf{S}}(av)} \geq (A(v) - \epsilon)a - \log a - O(1)$.

Remark (by the proof)

$v \leq C_1 v_0 \Longrightarrow \epsilon = \epsilon(C_1) \Longrightarrow a_*(v) \leq C(\tilde{\beta}(v), A(v), C_1)$.

$R_m = H^0(X, -mK_X), \quad R = \bigoplus_m R_m, \; N_m = \dim_{\mathbb{C}} R_m$.
Definition: A filtration $\mathcal{F} = \{\mathcal{F}^\lambda R_m\}_{\lambda\in\mathbb{R}, m\in\mathbb{N}}$ satisfies:

1. $\mathcal{F}^\lambda R_m \subseteq \mathcal{F}^{\lambda'} R_m$ for $\lambda' \leq \lambda$.
2. $\bigcap_{x<\lambda} \mathcal{F}^x R_m = \mathcal{F}^\lambda R_m$.
3. $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subseteq \mathcal{F}^{\lambda+\lambda'} R_{m+m'}$.
4. $\exists e \in \mathbb{R}$ s.t. $\mathcal{F}^\lambda R_m = R_m$ for $\lambda < -em$ and $\mathcal{F}^\lambda R_m = 0$ for $\lambda > em$.

Successive maxima: $\lambda_1^{(m)} \geq \cdots \geq \lambda_{N_m}^{(m)}$:

$$\lambda_j^{(m)} = \max\{\lambda; \dim_{\mathbb{C}} \mathcal{F}^\lambda R_m \geq j\}.$$

Volume of graded linear series: $\mathcal{F}^{(t)} = \{\mathcal{F}^{tm} R_m\}$:

$$\mathrm{vol}(\mathcal{F}^{(t)}) = \lim_{m\to+\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}^{tm} R_m}{m^n/n!}.$$

Convergence to the Duistermaat-Heckman measure (Boucksom-Chen):

$$\frac{n!}{m^n} \sum_j \delta_{\frac{\lambda_j^{(m)}}{m}} \xrightarrow{w} \mathrm{DH}(\mathcal{F}) = -d\mathrm{vol}(\mathcal{F}^{(t)}).$$

Any filtration can be approximated by a sequence of test configurations. Set

$$
\begin{aligned}
I^{\mathcal{F}}_{m,\lambda} &:= \mathrm{Image}\left(\mathcal{F}^{\lambda}R_m \otimes \mathcal{O}_X(-mL) \to \mathcal{O}_X\right) \text{ with } L = -K_X; \\
\tilde{\mathcal{I}}^{\mathcal{F}}_m &:= \sum_{\lambda} I^{\mathcal{F}}_{m,\lambda} t^{-\lambda} \quad \text{(a fractional ideal)} \\
\mathcal{X}^{\mathcal{F}}_m &:= (\mathrm{Bl}_{\tilde{\mathcal{I}}_m}(X \times \mathbb{C}))^{\nu}, \quad \mathcal{L}^{\mathcal{F}}_m = \pi^*(-K_X \times \mathbb{C}) - \frac{1}{m}E_m.
\end{aligned}
$$

Conversely, any test configuration is dominated by a blowup of flag ideals: $(\mathcal{X} = \mathrm{Bl}_{\mathcal{I}}(X \times \mathbb{C}), \mathcal{L} = \mu^* L_{\mathbb{C}} - E)$, and determines a finitely generated filtration:

$$
\mathcal{F}^{\lambda}_{(\mathcal{X},\mathcal{L})} R_m = \left\{ s \in R_m ; t^{-\lambda}\bar{s} \in H^0(\mathcal{X}, m\mathcal{L}) \right\}.
$$

Scaling of filtrations: $(a\mathcal{F})^{\lambda} R_m = \mathcal{F}^{\lambda/a} R_m$. For test configurations, if η is the holomorphic vector field generating the $\mathbb{C}^*$-action, then

$$
\mathcal{F}^{\lambda}_{(\mathcal{X},\mathcal{L},a\eta)} R_m = (a\mathcal{F}_{(\mathcal{X},\mathcal{L})})^{\lambda} R_m = \mathcal{F}^{\lambda/a}_{(\mathcal{X},\mathcal{L})} R_m.
$$

Base change and quotient correspond to scaling:

$$
\mathcal{F}_{(\mathcal{X}^{(d)},\mathcal{L}^{(d)},\eta^{(d)})} = \mathcal{F}_{(\mathcal{X},\mathcal{L},d\eta)}, \quad \mathcal{F}_{(\mathcal{X},\mathcal{L},\eta)/\mathbb{Z}_d} = \mathcal{F}_{(\mathcal{X},\mathcal{L},\eta/d)}.
$$

For test configurations (from blowing-up flag ideals):

$$\mathbf{E}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \frac{1}{V}\frac{\bar{\mathcal{L}}^{\cdot n+1}}{n+1},$$

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \mathrm{lct}(X\times\mathbb{C},\mathcal{I};(t))-1 = \inf_{v\in X^{\mathrm{div}}_{\mathbb{Q}}}(A_X(v)-G(v)(\mathcal{I})).$$

Generalized to filtrations:

$$\mathbf{E}^{\mathrm{NA}}(\mathcal{F}) = \frac{1}{V}\int_{\mathbb{R}}\lambda\cdot\mathrm{DH}(\mathcal{F}) = \lim_{m\to+\infty}\frac{n!}{m^n}\sum_j\frac{\lambda_j^{(m)}}{m}$$

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) = \lim_{m\to+\infty}\mathbf{L}^{\mathrm{NA}}(\mathcal{X}^{\mathcal{F}}_m,\mathcal{L}^{\mathcal{F}}_m)$$

$$\hat{\mathbf{L}}^{\mathrm{NA}}(\mathcal{F}) = \sup\{x;\mathrm{lct}(\mathcal{I}_\bullet^{\mathcal{F}^{(x)}})\geq 1\}\quad\text{(Xu-Zhuang)}.$$

Non-linear functional $\mathbf{H}^{\mathrm{NA}}(\mathcal{F}) = -\tilde{\mathbf{S}}^{\mathrm{NA}}(\mathcal{F})+\mathbf{L}^{\mathrm{NA}}(\mathcal{F})$, where

$$\tilde{\mathbf{S}}^{\mathrm{NA}}(\mathcal{F}) = -\log\left(\frac{1}{V}\int_{\mathbb{R}}e^{-\lambda}\mathrm{DH}(\mathcal{F})\right) = -\log\left(\lim_{m\to+\infty}\frac{n^n}{m^n}\sum_j e^{-\frac{\lambda_j^{(m)}}{m}}\right).$$

Scaling effects $\mathbf{E}^{\mathrm{NA}}(a\mathcal{F}) = a\cdot\mathbf{E}^{\mathrm{NA}}(\mathcal{F})$, $\mathbf{L}^{\mathrm{NA}}(a\mathcal{F}) = a\cdot\mathbf{L}^{\mathrm{NA}}(\mathcal{F})$ while:

$$\tilde{\mathbf{S}}^{\mathrm{NA}}(a\mathcal{F}) = -\log\left(\frac{1}{V}\int_{\mathbb{R}}e^{-a\lambda}\mathrm{DH}(\mathcal{F})\right).$$

Functionals for valuations and for special test configurations

For $v \in X_{\mathbb{Q}}^{\mathrm{div}}$, set $\mathcal{F}_v^{\lambda} R_m = \{s \in H^0(X, -mK_X); v(s) \geq \lambda\}$. Then

$$\mathbf{E}^{\mathrm{NA}}(\mathcal{F}_v) = S(v), \quad \tilde{\mathbf{S}}^{\mathrm{NA}}(\mathcal{F}_v) = \tilde{\mathbf{S}}(v); \quad \mathbf{L}^{\mathrm{NA}}(\mathcal{F}_v) \leq A_X(v);$$
$$\mathbf{D}^{\mathrm{NA}}(\mathcal{F}_v) \leq \beta(v), \quad \mathbf{H}^{\mathrm{NA}}(\mathcal{F}_v) \leq \tilde{\beta}(v).$$

$(\mathcal{X}^s, \mathcal{L}^s)$: special test configuration, i.e. $\mathcal{X}_0^s$ is $\mathbb{Q}$-Fano. $\mathbf{L}^{\mathrm{NA}}(\mathcal{X}^s, -K_{\mathcal{X}^s}) = 0$.

Lemma (L.'15, using Boucksom-Hisamoto-Jonsson)

For any special test configuration $(\mathcal{X}^s, -K_{\mathcal{X}^s})$, $v_{\mathcal{X}_0^s} := \mathrm{ord}_{\mathcal{X}_0}|_{\mathbb{C}(X)}$ *satisfies* $\mathcal{F}_{(\mathcal{X}^s, -K_{\mathcal{X}^s})} = \mathcal{F}_v(-A_X(v))$. *As a consequence,* $\mathbf{D}^{\mathrm{NA}}(\mathcal{X}^s, -K_{\mathcal{X}^s}) = \beta(v_{\mathcal{X}_0^s})$.

Shift of filtrations: $\mathcal{F}(\sigma)^{\lambda} R_m = \mathcal{F}^{\lambda - \sigma m} R_m$.
Similarly:

$$\begin{aligned}
\mathbf{H}^{\mathrm{NA}}(\mathcal{X}^s, -K_{\mathcal{X}^s}) &= \hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{X}^s, -K_{\mathcal{X}^s}) = -\tilde{\mathbf{S}}^{\mathrm{NA}}(\mathcal{F}_{(\mathcal{X}^s, -K_{\mathcal{X}^s})}) \\
&= \log\left(\frac{1}{V}\int_{\mathbb{R}} e^{-\lambda}(-d\mathrm{vol}(\mathcal{F}_v(-A_X(v))^{(\lambda)}))\right) \\
&= A_X(v) + \log\left(\frac{1}{V}\int_{\mathbb{R}} e^{-\lambda}(-d\mathrm{vol}(\mathcal{F}_v^{(\lambda)}))\right) = \tilde{\beta}(v_{\mathcal{X}_0^s}).
\end{aligned}$$

Theorem (Han-L.)

$\forall$ test configuration $(\mathcal{X}, \mathcal{L})$, $\exists$ a special test configuration $(\mathcal{X}^s, \mathcal{L}^s)$ s.t. $\mathbf{H}^{\mathrm{NA}}(\mathcal{X}^s, \mathcal{L}^s) \leq \mathbf{H}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$. The equality holds iff $(\mathcal{X}, \mathcal{L})$ is already special.

• Use scaling to take care of the base change:

$$\mathbf{F}^{\mathrm{NA}}(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}, \eta^{(d)}/d) = \mathbf{F}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, \eta).$$

• Use derivative formula to derive monotonicity formula for $\mathbf{H}^{\mathrm{NA}}$ along the MMP devised in [**L.**-Xu, '14].

Example: $(\mathcal{X}, \mathcal{X}_0)$ is log canonical and run $K_{\mathcal{X}/\mathbb{C}}$-MMP with rescaling w.r.t. $\mathcal{L}$. Assume $K_{\mathcal{X}} + \mathcal{L} = \sum_i e_i E_i$ with $e_1 \leq \cdots \leq e_k$.

$$\mathcal{L}_\lambda := \frac{K_{\mathcal{X}/\mathbb{C}} + \lambda \mathcal{L}}{\lambda - 1}, \quad \frac{d}{d\lambda}\mathcal{L}_\lambda = -\frac{1}{(\lambda-1)^2}(K_{\mathcal{X}/\mathbb{C}} + \mathcal{L}).$$

Then $\mathbf{L}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{\lambda}{\lambda - 1} e_1$ and with $\tilde{\mathbf{S}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = -\log \mathbf{Q}$,

$$\begin{aligned}\frac{d}{d\lambda}\mathbf{H}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= -\frac{1}{(\lambda-1)^2} e_1 + \frac{1}{(\lambda-1)^2}\frac{\sum_i e_i \mathbf{Q}_i}{\mathbf{Q}} \\ &= \frac{\sum_i (e_i - e_1)\mathbf{Q}_i}{(\lambda-1)^2 \mathbf{Q}} \geq 0.\end{aligned}$$

Derivative formula for $\tilde{\mathbf{S}}$

$$\begin{aligned}\mathbf{Q} &:= e^{-\mathbf{S}} = \frac{1}{V}\int_{\mathbb{R}} e^{-\lambda}\mathrm{DH}(\mathcal{F}) \\ &= \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\frac{1}{V}\int_{\mathbb{R}}\lambda^k\mathrm{DH}(\mathcal{F}) =: \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\mathbf{E}_k^{\mathrm{NA}}.\end{aligned}$$

Proposition (Intersection and Derivative formula)

- *Intersection formula (generalizing Mumford's formula $k=1$):*

$$\mathbf{E}_k^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \frac{1}{V}\frac{k!n!}{(n+1)!}\left(\bar{\mathcal{L}}^{[k-1]}\right)^{\cdot n+k}.$$

- *If* $\frac{d}{dt}\mathcal{L}(t) = \sum_i e_i E_i$, *then*

$$\frac{d}{dt}\tilde{\mathbf{S}}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \frac{\sum_i e_i \mathbf{Q}_i}{\mathbf{Q}},$$

where $\mathbf{Q}_i(\mathcal{X},\mathcal{L}) = \frac{1}{V}\int_{E_i} e^{-\theta}\omega_{\mathrm{FS}}^n > 0.$

Existence of minimizers: based on Blum-Liu-Xu's techniques

Corollary (Together with (2))

$$\inf_{v\in \mathrm{Val}_X} \tilde{\beta}(v) = \inf_{\mathcal{F}} \mathbf{H}^{\mathrm{NA}}(\mathcal{F}) = \inf_{(\mathcal{X}^s,\mathcal{L}^s)} \mathbf{H}^{\mathrm{NA}}(\mathcal{X}^s,\mathcal{L}^s) = \inf_{\mathcal{F}} \hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{F}).$$

Theorem (Blum-Liu-Xu, using Birkar's work)

There exists $N = N(n)$ such that for any special test configuration $(\mathcal{X}^s,\mathcal{L}^s)$ of $\mathbb{Q}$-Fano variety X, $v_{\mathcal{X}^s_0}$ is a log canonical place of an N-complement.

- Set: $W = \mathbb{P}(H^0(X, \mathcal{O}_X(-NK_X)^*)$, H the universal divisor on $X \times W$, $D = \frac{1}{N}H$, $Z = \{w \in W; \mathrm{lct}(X_w, D_w) = 1\}$ locally closed in W. Fix $z \in Z$ and $g : Y_z \to X$ a log resolution. $K_Y + D_{Y_z} = g^*(K_X + D_z)$.
  $\mathcal{S} := \mathrm{QM}(Y_z, D_{Y_z}) \bigcap \{v \in \mathrm{Val}_X; A_X(v) = 1\}$. By Corollary 2, $\forall v \in \mathcal{S}$, $\exists a_*(v) \geq 0$ s.t. $\tilde{\beta}(a_*(v)v) = \inf_{a>0} \tilde{\beta}(av) = \tilde{\beta}_*(v)$.
- Izumi's estimate: $\mathcal{S} \ni v \leq C_1 \cdot \mathrm{ord}_F$ with $F = \cap_i D_{Y_z,i}$.
  Remark 2 $\Rightarrow \{a_*(v); v \in \mathcal{S}\}$ uniformly bounded.
  $v \mapsto \tilde{\beta}(v)$ is continuous (Blum-Jonsson, Blum-Liu-Xu).
  $\Rightarrow \exists v_z^* \in \mathcal{S}$ s.t. $b_z := \tilde{\beta}(v_z^*) = \min_{v\in\mathcal{S}} \tilde{\beta}_*(v) = \min\{\tilde{\beta}(v); v \in \mathrm{QM}(Y_z, D_z)\}$.
- Decompose $Z = \cup_i Z_i$ s.t. $Z_i' \to Z_i$ étale s.t. $(X_{Z_i'}, D_{Z_i'})$ admits fiberwise log resolutions. Use Hacon-McKernan-Xu's invariance of log plurigenera to show that b_z is independent of $z \in Z_i$. $\min_i b_{z_i}$ is thus achieved.

Define a semi-valuation $\bar{v}_{\mathcal{F}} : \bigoplus_m R_m \to \mathbb{C}$ by

$$\bar{v}_{\mathcal{F}}(\sum_m s_m) = \min_m \left\{ \max\{\lambda; s_m \in \mathcal{F}^\lambda R_m\}, s_m \neq 0 \right\}.$$

Let $\Gamma(\mathcal{F}) \subset \mathbb{R}$ be the group generated by $\{\lambda_i^{(m)} - \lambda_{N_m}^{(m)}; m \in \mathbb{N}\}$.
The extended Rees algebra and associated graded ring of $\mathcal{F}$:

$$\mathcal{R}(\mathcal{F}) := \bigoplus_{m\geq 0} \bigoplus_{\lambda\in\Gamma(\mathcal{F})} t^{-\lambda}\mathcal{F}^\lambda R_m, \quad \mathrm{Gr}(\mathcal{F}) := \bigoplus_{m\geq 0} \bigoplus_{\lambda\in\Gamma(\mathcal{F})} \mathcal{F}^\lambda R_m / \mathcal{F}^{>\lambda} R_m.$$

Definition:
• An $\mathbb{R}$-test configuration ($\mathbb{R}$-TC) is a finitely generated filtration and $X_0 := \mathrm{Proj}(\mathrm{Gr}(\mathcal{F}))$ has dimension n. $\mathcal{F}$ is special if X_0 is a $\mathbb{Q}$-Fano variety.
• We call $\mathrm{rank}(\Gamma(\mathcal{F})) =: \mathrm{rank}(\mathcal{F})$ the rank of $\mathcal{F}$. If $\mathrm{rank}(\mathcal{F}) = 1$, then we get the usual test configuration.

Lemma

If $\mathrm{Gr}(\mathcal{F})$ is integral, then $\mathcal{F} = \mathcal{F}_v(-\sigma)$ for some $v = v_{\mathcal{F}} \in \mathrm{Val}_X$ and $\sigma \in \mathbb{R}$.

Fact: Any $\mathbb{R}$-TC is induced by a one parameter $\mathbb{R}$-subgroup of $PGL(N_\ell)$ with a generating holomorphic vector field ξ, for some embedding $X \hookrightarrow \mathbb{P}^{N_\ell - 1}$, s.t. $\lim_{s\to+\infty} \exp(s\xi)\cdot[X] = [X_0]$.

Assume that $\mathcal{F}$ is generated $\mathcal{F}R_\ell$. Let $\{w_1, \dots, w_k\}$ be distinct values of (normalized) successive maxima of $\mathcal{F}^\lambda R_\ell$. $\{\zeta_1, \dots, \zeta_r\}$ the subset of maximal $\mathbb{Q}$-linearly independent subset. Then

$$w_j = \sum_{p=1}^{r} r_{jp}\zeta_p = \langle \alpha_j, \xi\rangle \quad \text{with } \xi = \zeta/D, \quad \alpha_j = D\cdot \vec{r}_j \in \mathbb{Z}^r.$$

We get identity:

$$\mathrm{Gr}_\ell(\mathcal{F}) = \bigoplus_{m\geq 0}\bigoplus_{\lambda} \mathcal{F}^\lambda R_{m\ell}/\mathcal{F}^{>\lambda}R_{m\ell} = \bigoplus_{m\geq 0}\bigoplus_{\alpha\in M_{\mathbb{Z}}} R'_{m,\alpha}$$

Central fibre: $X_0 = \mathrm{Proj}(\mathrm{Gr}_\ell(\mathcal{F}))$ admits a holomorphic vector field ξ generating a torus $\mathbb{T} \cong (\mathbb{C}^*)^r$-action.

Y : a $\mathbb{Q}$-Fano variety Y that admits an effective $\mathbb{T} \cong (\mathbb{C}^*)^r$ action.
$N_\mathbb{Z} = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T})$, $N_\mathbb{R} = N_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$, $M_\mathbb{R} = N_\mathbb{R}^\vee$. $P \subset M_\mathbb{R}$ moment polytope.
$g : P \to \mathbb{R}_{>0}$ a smooth function, $V_g := n! \int_P g(y)dy$. Ex.: $g_\xi(y) = e^{-\langle \xi, y \rangle}$.

Definition

$(Y, \mathbb{T})$ is g-K-semistable if $\forall\mathbb{T}$-equivariant (weakly) special TC:

$$\mathrm{Fut_g}(\mathcal{Y}, -\mathrm{K}_\mathcal{Y}) := -\mathbf{E}_\mathrm{g}^\mathrm{NA}(\mathcal{Y}_0, \eta) \geq 0, \quad \textit{where}$$

$$\mathbf{E}_g^{\mathrm{NA}}(\mathcal{Y}_0,\eta) = \frac{1}{V_g} \lim_{m\to+\infty} \frac{n!}{m^n} \sum_{i,\alpha} \frac{\mu_{\alpha,i}^{(m)}}{m} g\left(\frac{\alpha}{m}\right) = \frac{1}{V_g} \int_{\mathcal{Y}_0} \theta_\eta(\varphi_{\mathrm{FS}}) g(\mathbf{m}_{\varphi_{\mathrm{FS}}}) \omega_{\mathrm{FS}}^n.$$

$(Y, \mathbb{T})$ is moreover g-K-polystable if $= 0$ only if $(\mathcal{Y}, \mathcal{L})$ is a product TC.

Theorem (Valuative criterion)

$(Y, \mathbb{T})$ is g-K-semistable if and only if $\beta_g = A_X(v) - S_g(v) \geq 0$.

Theorem (Han-**L.**, Generalized YTD)

$(Y, \mathbb{T})$ is $\mathrm{Aut}(Y, \mathbb{T})$-uniformly g-K-stable if and only if there is a solution to the g-soliton equation: $g(\mathbf{m}_\varphi)(\mathrm{dd^c}\varphi)^n = e^{-\varphi}$.

For $\xi \in N_{\mathbb{R}}$, set $g_\xi(y) = e^{-\langle \xi, y\rangle}$. We say that (Y, ξ) is K-semi(-poly)stable if $(Y, \mathbb{T})$ is g_ξ-K-semi(-poly)stable.

Theorem

Let $\mathcal{F}$ be a special $\mathbb{R}$-test configuration. Then $v_{\mathcal{F}}$ is a minimizer of $\tilde{\beta}$ if and only if the central fibre (X_0, ξ) is K-semistable.

Idea of proof: If $(\mathcal{Y}, \mathcal{L})$ is a $\mathbb{T}$-equivariant special test configuration of $(Y, \xi) := (\mathcal{X}_{\mathcal{F}}, \xi_{\mathcal{F}})$, then there exists a family of $\mathbb{R}$-special test configurations $\mathcal{F}_s$ (generated by a family of holomorphic vector field η_s) with corresponding valuations $v_s \in \mathrm{Val}_X$ such that:

$$\frac{d}{ds}\bigg|_{s=0} \tilde{\beta}(v_s) = \mathrm{Fut}_{g_\xi}(\mathcal{Y}_0, \eta'(0)) = \mathrm{Fut}_{g_\xi}(\mathcal{Y}, -K_{\mathcal{Y}}).$$

Assume that there are two minimizing special $\mathbb{R}$-test configurations $\mathcal{F}_i, i = 0, 1$ with central fibre $W^{(i)}$.

Step 1: Consider the initial term degeneration $\mathcal{F}'_1$ of $\mathcal{F}_1$ to $W^{(0)}$.

Step 2: Show that $\hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{F}_1) \geq \hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{F}'_1)$ and $\hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{F}_0) = \hat{\mathbf{H}}^{\mathrm{NA}}(\mathrm{wt}_{\xi_0})$.

Step 3: Consider the rescaling of twist $\mathcal{F}'_s = s\mathcal{F}'_{\frac{1-s}{s}\xi_0}$, which interpolates between $\mathcal{F}'_1$ and $\mathcal{F}'_0 := \mathcal{F}_{\mathrm{wt}_{\xi_0}}$. Prove that $\hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{F}'_s)$ is strictly convex in $s \in [0, 1]$ unless $\mathcal{F}'_1$ is equivalent to $\mathcal{F}'_0$.

Step 4: We know that $(W^{(0)}, \xi_0)$ is K-semistable. So $\mathcal{F}'_0 = \mathcal{F}_{\mathrm{wt}_{\xi_0}}$ obtains the minimum of $\hat{\mathbf{H}}^{\mathrm{NA}}_{W^{(0)}}$. Step 2 implies that $\mathcal{F}'_1$ also obtains the minimum. Step 3 implies $\mathcal{F}'$ is equivalent to $\mathcal{F}'_0$.

Step 5: Note that $d_2(\mathcal{F}_0, \mathcal{F}_1) = d_2(\mathcal{F}'_0, \mathcal{F}'_1) = 0$, which by Boucksom-Jonsson's characterization of equivalent filtrations implies $\phi_{\mathcal{F}_1} = \phi_{\mathcal{F}_0}$.

Step 6: We know that $\mathcal{F}_i = \mathcal{F}_{v_i}(-\sigma_i)$. Prove that $\phi_{\mathcal{F}_{v_1}} = \phi_{\mathcal{F}_{v_2}} + c$ implies $v_1 = v_2$.

$\mathcal{F}_0$: a special $\mathbb{R}$-TC with central fibre (W, ξ_0). $R' := R(W, -K_W)$.
$\mathcal{F}_1$: another filtration. For $f \in R_m$ with $\langle \alpha, \xi_0 \rangle = v_{\mathcal{F}_0}(f)$, set:

$$\mathbf{in}_{\mathcal{F}_0}(f) = t^{-\langle \alpha, \xi \rangle} \bar{f}(0) := f' \in \mathcal{F}^{\langle \alpha, \xi_x \rangle} R_m / \mathcal{F}_0^{> \langle \alpha, \xi_0 \rangle} R_m.$$

$\forall \lambda \in \mathbb{R}$, take the Gröbner base type degeneration:

$$\mathcal{F}_1'^{\lambda} R'_m := \mathrm{Span}_{\mathbb{C}} \left\{ \mathbf{in}_{\mathcal{F}_0}(f); f \in \mathcal{F}_1^{\lambda} R_m \right\} \subset R'_m.$$

Note that $\mathcal{F}'_0 = \mathcal{F}_{\mathrm{wt}_{\xi_0}} R'$. Other key facts/properties:

1. Preservation of (relative) successive maxima implies:

$$\tilde{\mathbf{S}}^{\mathrm{NA}}(\mathcal{F}_i) = \tilde{\mathbf{S}}^{\mathrm{NA}}(\mathcal{F}'_i), i = 0, 1, \quad d_p^X(\mathcal{F}_0, \mathcal{F}_1) = d_p^W(\mathcal{F}'_0, \mathcal{F}'_1).$$

2. $\hat{\mathbf{H}}_X^{\mathrm{NA}}(\mathcal{F}_0) = \hat{\mathbf{H}}_W^{\mathrm{NA}}(\mathcal{F}_{\mathrm{wt}_{\xi_0}}) = \tilde{\beta}(v_{\mathcal{F}_0})$.
3. Lower semicontinuity of log canonical threshold in families implies:

$$\hat{\mathbf{L}}^{\mathrm{NA}}(\mathcal{F}_1) \geq \hat{\mathbf{L}}^{\mathrm{NA}}(\mathcal{F}'_1), \quad \hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{F}_1) \geq \hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{F}'_1).$$

Newton-Okounkov bodies, concave transform

Fix a faithful $\mathbb{Z}^n$-valuation $\mathfrak{v}$. Define the Newton-Okounkov body of the graded linear series $\mathcal{F}^{(t)} = \{\mathcal{F}^{tm} R_m\}$:

$$\Delta(\mathcal{F}^{(t)}) = \overline{\bigcup_{m=1}^{+\infty} \mathfrak{v}(\mathcal{F}^{(tm)} R_m)}.$$

For any filtration $\mathcal{F}$, define the concave transform on $\Delta = \Delta(-K_X)$:

$$G_{\mathcal{F}}(y) = \sup\{t \in \mathbb{R}; y \in \Delta(\mathcal{F}^{(t)})\}.$$

Theorem (Boucksom-Chen)

$\mathrm{vol}(\mathcal{F}^{(t)}) = n! \cdot \Delta(\mathcal{F}^{(t)})$; $-d\mathrm{vol}(\mathcal{F}^{(t)}) = (G_{\mathcal{F}})_* dy$.

If X admits an effective $\mathbb{T} \cong (\mathbb{C}^*)^r$-action, then we can choose the valuation $\mathfrak{v}$ that is adapted to the $\mathbb{T}$-action: for any $f \in \mathbb{C}(X)_\alpha$,

$$\mathfrak{v}(f) = (\alpha, \mathfrak{v}^{r+1}(f), \ldots, \mathfrak{v}^n(f)).$$

Let $\mathcal{F}$ be a $\mathbb{T}$-equivariant filtration. Set (**L.**)

$$\mathcal{F}_\xi^\lambda R_{m,\alpha} = \mathcal{F}^{\lambda - m\langle\alpha,\xi\rangle} R_m.$$

Lemma (Yao, Han-L.)

If $\mathfrak{v}$ is a $\mathbb{Z}^n$-valuation adapted to $\mathbb{T}$-action, then we have:

$$G_{\mathcal{F}_\xi}(y) = G_{\mathcal{F}}(y) + \langle y, \xi\rangle.$$

Set $\mathcal{F}_s = s\mathcal{F}_{\frac{1-s}{s}\xi}$. Then

$$G_{\mathcal{F}_s}(y) = (1-s)\langle y, \xi\rangle + sG_{\mathcal{F}}(y).$$

Rescaling and twist formula $\Rightarrow$ (compare (1))

$$\hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{F}_s) = s\hat{\mathbf{L}}^{\mathrm{NA}}(\mathcal{F}) + \log\left(\frac{1}{V}\int_\Delta e^{-(1-s)\langle y,\xi\rangle - sG_{\mathcal{F}}(y)} dy\right)$$

is convex with respect to $s \in [0,1]$.

Derivative formula: $\left.\frac{d}{ds}\right|_{s=0} \hat{\mathbf{H}}^{\mathrm{NA}}(\mathcal{F}_s) = \tilde{\beta}(\mathcal{F}).$

Non-Archimedean metric associated to filtrations:

$$(\phi_{\mathcal{F}} - \phi_{\mathrm{triv}})(v) = \lim_{m\to+\infty} -\frac{1}{m} G(v)(\tilde{\mathcal{I}}_m^{\mathcal{F}}).$$

d_p-distance of two filtrations $\mathcal{F}_0$ and $\mathcal{F}_1$: $\exists\{s_1^{(m)}, \dots, s_{N_m}^{(m)}\}$ compatible with both $\mathcal{F}_i R_m$, $i = 0, 1$. Assume $s_j^{(m)} \in \mathcal{F}_i^{\mu_{j,i}^{(m)}} \setminus \mathcal{F}_i^{>\mu_{j,i}^{(m)}}$.
The following limit exists (Chen-McLean, Boucksom-Jonsson)

$$d_p(\mathcal{F}_0, \mathcal{F}_1) := \lim_{m\to+\infty} \left(\frac{n!}{m^n} \sum_{j=1}^{N_m} |\mu_{j,1}^{(m)} - \mu_{j,0}^{(m)}|^p \right)^{1/p}.$$

Theorem (Boucksom-Jonsson)

For any $p \in [1, +\infty)$, $d_p(\mathcal{F}_0, \mathcal{F}_1) = 0$ if and only if $\phi_{\mathcal{F}_0} = \phi_{\mathcal{F}_1}$.

Lemma (Han-L.)

For $v_0, v_1 \in \mathrm{Val}_X$, $\phi_{\mathcal{F}_{v_0}} = \phi_{\mathcal{F}_{v_1}} + c$ iff $v_0 = v_1$.

Cone: $C := C(X, -K_X) = \mathrm{Spec}_{\mathbb{C}}(R)$ with $R = \bigoplus_m H^0(X, -mK_X) =: \bigoplus_m R_m$.

$\mathfrak{a} = \bigoplus_m \bigoplus_{\alpha \in M_{\mathbb{Z}}} \mathfrak{a}_\alpha$: $\mathbb{T}$-equivariant homogeneous primary ideal.

$$\mathrm{colen}_g(\mathfrak{a}) := \sum_{m \geq 0} \sum_{\alpha} g\left(\frac{\alpha}{m}\right) \dim_{\mathbb{C}} R_{m,\alpha}/\mathfrak{a}_{m,\alpha},$$

$$\mathrm{mult}_g(\mathfrak{a}) := \lim_{k \to +\infty} \frac{\mathrm{colen}_g(\mathfrak{a}^k)}{k^{n+1}/(n+1)!}.$$

$\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}$: graded sequence of $\mathbb{C}^* \times \mathbb{T}$-invariant primary ideals.

$$\mathrm{mult}_g(\mathfrak{a}_\bullet) := \lim_{k \to +\infty} \frac{\mathrm{colen}_g(\mathfrak{a}_k)}{k^{n+1}/(n+1)!} = (n+1)! \int_{\bar{P}^c} g(y) dy.$$

Equivariant g-volume: for any $v \in \mathrm{Val}_{C,o}^{\mathbb{C}^* \times \mathbb{T}}$, set:

$$\mathrm{vol}_g(v) := \mathrm{mult}_g(\mathfrak{a}_\bullet(v)).$$

g-normalized volume:

$$\widehat{\mathrm{vol}}_g(v) = \begin{cases} A_X(v)^n \cdot \mathrm{vol}_g(v), & A_X(v) < +\infty \\ +\infty & A_X(v) = +\infty. \end{cases}$$

Properties of g-normalized volume

Similar properties as normalized volumes: e.g. $\widehat{\mathrm{vol}}_g(\lambda v) = \widehat{\mathrm{vol}}_g(v)$ and g-version of Liu's identities relating to g-version of de-Fernex-Ein-Mustaţă:

$$\inf_{\bar{v}} \widehat{\mathrm{vol}}_g(\bar{v}) = \inf_{\alpha} \mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}_g(\mathfrak{a}) = \inf_{\mathfrak{a}_\bullet} \mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{mult}_g(\mathfrak{a}_\bullet).$$

For any $v \in X_{\mathbb{Q}}^{\mathrm{div}}$ and $\tau > 0$, set $\bar{v}_\tau$:

$$\bar{v}_\tau\Big(\sum_m s_m t^m\Big) = \min_m \left(v(s_m) + \tau m\right).$$

Formula for g-volume:

$$\mathrm{vol}_g(\bar{v}_\tau) = \frac{1}{\tau^{n+1}} V_g - (n+1)\int_0^{+\infty} \mathrm{vol}_g(\mathcal{F}_v R^{(x)}) \frac{dx}{(x+\tau)^{n+2}}.$$

Theorem (Han-L., modeled on L., L.-Liu-Xu)

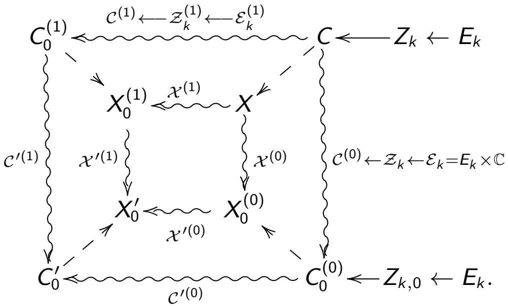
(X, ξ) is g-K-semistable if and only if ord_X minimizes $\widehat{\mathrm{vol}}_g$ over $\mathrm{Val}_{C,o}^{\mathbb{C}^*\times\mathbb{T}}$.

Proof.

Consider $w_s := \overline{(sv)}_{(1-s)A_X(v)}$ and $f(s) = \widehat{\mathrm{vol}}(w_s)$. Then $f(s)$ is convex in $s \in [0,1]$ and $f'(0) = C \cdot \beta_g(v)$. Then apply Theorem 12. □

Uniqueness of K-polystable degenerations, modeled on **L.**-Wang-Xu

Assume that (X, ξ) is K-semistable and admits two polystable degeneration via two $\mathbb{T} = \langle \xi \rangle$-equivariant special test configuration $(\mathcal{X}^{(i)}, -K_{\mathcal{X}^{(s)}})$. Then as in [**L.**-Wang-Xu], using the help of $\widehat{\mathrm{vol}}_g$ and [BCHM], we have:



Using K-polystability of $(X_0^{(i)}, \xi), i = 0, 1$,

$$\mathrm{Fut}_\xi(\mathcal{X}'^{(1)}) = 0 = \mathrm{Fut}_\xi(\mathcal{X}'^{(0)}) = 0 \implies \mathrm{X}_0^{(1)} = \mathrm{X}_0' = \mathrm{X}_0^{(0)}.$$

Thanks for your attention!