Restricted volumes and K-stability

Chi Li

Department of Mathematics, Purdue University

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X : d-dimensional normal projective variety
L : a line bundle. Its volume is defined as:

$$\mathrm{vol}(L) = \limsup_{m \to +\infty} \frac{h^0(X, mL)}{m^d/d!}$$

Example: If L is ample (or just nef), then $\mathrm{vol}(L) = L^d$.

Theorem

The following conditions are equivalent:

1. $\mathrm{vol}(L) > 0$.
2. $\exists$ *a decomposition into* $\mathbb{Q}$*-divisors* $L = A + E$ *where* A *is ample and* E *is effective.*
3. $\kappa(X, L) = d$.

Definition: L is big if one of the above conditions are satisfied.

Properties of volume functional

• Volume only depends on the numerical class of line bundles/divisors:

$$L_1 \equiv L_2 \Longrightarrow \mathrm{vol}(L_1) = \mathrm{vol}(L_2).$$

$N^1(X) = \mathrm{Div}(X)/\mathrm{Num}(X)$: Néron-Severi group.

Theorem

The function $\xi \mapsto \mathrm{vol}(\xi)$ on $N^1(X)_{\mathbb{Q}}$ extends uniquely to a continuous function $\mathrm{vol} : N^1(X)_{\mathbb{R}} \to \mathbb{R}$.

• $\mathrm{vol}(\lambda\xi) = \lambda^d \mathrm{vol}(\xi)$ for any $\lambda > 0$.

• If $\mu : Y \to X$ is birational, then $\mathrm{vol}(\mu^* L) = \mathrm{vol}(L)$.

• Volume increases in effective directions: $\mathrm{vol}(\xi) \leq \mathrm{vol}(\xi + e)$ if $e \in N^1(X)_{\mathbb{R}}$ is effective.

Fujita's approximation theorem (from Lazarsfeld's book)

Theorem

Let L be a big line bundle on X. $\forall \epsilon > 0$, $\exists$ a projective birational morphism $\mu : Y \to X$ and a decomposition $\mu^* L = A + E$ in $N^1(Y)_{\mathbb{Q}}$ with A ample and E effective such that:

$$\mathrm{vol}_Y(A) > \mathrm{vol}_X(L) - \epsilon.$$

Corollary (limsup is lim)

$$\mathrm{vol}(L) = \lim_{m \to +\infty} \frac{h^0(X, mL)}{m^d/d!}.$$

Theorem (movable intersection number)

Let L be a big line on X. Let $\mathfrak{b}(|mL|)$ be the base ideal of $|mL|$. Let $\mu_m : X_m \to X$ be the (resolution of the) normalized blow-up of $\mathfrak{b}(|mL|)$ with exceptional divisor E_m. Then

$$\mathrm{vol}(L) = \lim_{m \to +\infty} \left(\mu_m^* L - \frac{1}{m} E_m \right)^d =: \langle L^d \rangle.$$

Restricted volume (Tsuji, Ein-Lazarsfeld-Mustață-Nakamaye-Popa)

Let $V \subset X$ be a subvariety of dimension r. Restricted volume:

$$\mathrm{vol}_{X|V}(L) = \limsup_{m \to +\infty} \frac{\dim_{\mathbb{C}} \mathrm{Im}(H^0(X, mL) \to H^0(V, mL))}{m^r / r!}.$$

If L is ample (or just nef), then $\mathrm{vol}_{X|V}(L) = L^r \cdot V$.

Base locus: $\mathbf{B}(L) = \bigcap_m \mathrm{Bs}(|mL|)_{\mathrm{red}}$ (not invariant on numerical class)

Augmented base locus: $\mathbf{B}_+(L) = \bigcap_{L = A+E} \mathrm{Supp}(E)$.

$\mathbf{B}_+(L)$ depends only on numerical class of L.

Theorem (ELMNP)

1. *If $V \not\subseteq \mathbf{B}_+(L)$, then $\mathrm{vol}_{X|V}(L) > 0$.*
2. *$\mathrm{vol}_{X|V}(L)$ depends only on the numerical equivalence class of L.*
3. *$\mathbf{B}_+(L)$ is the union of all positive dimensional subvarieties V such that $\mathrm{vol}_{X|V}(L) = 0$.*

Assume that $V \not\subseteq \mathbf{B}(L)$. Let $\mu_m : X_m \to X$ be a resolution of the base ideal $\mathfrak{b}_m = \mathbf{B}(|mL|)$. We have decomposition:

$$\mu_m^*|mL| = |M_m| + E_m.$$

Let $\tilde{V}_m$ be the strict transform of V. The asymptotic intersection number of L and V:

$$\|L^r \cdot V\| := \limsup_{m \to +\infty} \frac{M_m^r \cdot \tilde{V}_m}{m^r}.$$

Theorem (Generalized Fujita approximation theorem, ELMNP)

If $V \not\subseteq \mathbf{B}_+(L)$, then

$$\mathrm{vol}_{X|V}(L) = \|L^r \cdot V\|.$$

Theorem (Positive intersection product, Boucksom-Favre-Jonsson)

$\forall$ big class ξ, $\exists\ \langle \xi^{d-1}\rangle \in N^1(X)^*$ satisfying: for any $\gamma \in N^1(X)$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(\xi + t\gamma) = d\langle \xi^{d-1}\rangle \cdot \gamma$$

Theorem (BFJ)

If V is a prime divisor, then $\mathrm{vol}_{X|V}(L) = \langle L^{d-1}\rangle \cdot V = \| L^{r-1} \cdot V \|$.

$\forall$ big $\xi \in N^1(X)$ and $1 \le p \le d$, BFJ defined a positive intersection product (where $\mathfrak{X} = \varprojlim_{\pi} X_\pi$ (ranges over blowups $\pi : X_\pi \to X$) is the Riemann-Zariski space):

$$\langle \xi^p \rangle = \sup\{\beta^p ; \beta \in CN^1(\mathfrak{X}) \text{ nef and } \beta \le \xi\} \in N^p(\mathfrak{X}).$$

Example: $\langle \xi^d \rangle \in N^d(\mathfrak{X}) \cong \mathbb{R}$ is the volume;
$\langle \xi \rangle \in N^1(\mathfrak{X})$ is the collection of positive part of the divisorial Zariski decompositions of $\pi^*\xi$ (Nakayama, Boucksom) $\{P(\pi^*\xi)\}_\pi$

Example: Let V be a prime divisor. If $\mu_m : X_m \to X$ is the (resolution of the) normalized blowup of $\mathfrak{b}(|mL|)$, then

$$\langle L^{d-1}\rangle \cdot V = \lim_{m\to+\infty} (L_m^{d-1}) \cdot \mu_m^* V = \sup_{\mu^* L = A+E} A^d \cdot \mu^* V.$$

- If L is ample, then by Hirzebruch-Riemann-Roch formula:

$$h^0(mL) = L^d \frac{m^d}{d!} + \frac{1}{2} K_X \cdot L^{d-1} \frac{m^{d-1}}{(d-1)!} + O(m^{d-2}).$$

This is still true if L is big and nef by Fujita's vanishing theorem.

- In general if L is big, we only have $h^0(mL) = \mathrm{vol}(L) \frac{m^d}{d!} + O(m^{d-1})$. Cutkosky-Srinivas constructed a 3-fold example such that $h^0(mL)$ is a polynomial of degree 3 in $[m(2 - \frac{\sqrt{3}}{3})]$.

- If L admits a rational Zariski decomposition $L = P + N$, then $h^0(mL) = h^0(\lfloor mP \rfloor)$ is polynomial with 1st coefficient given by $K_X \cdot P^d$.

Definition

For any big line bundle L, define $\mathfrak{r}_1(X, L) := \mathfrak{r}_1(X, L) = \langle \mathcal{L}^{d-1} \rangle \cdot K_X$.

Lemma

If $\mu : Y \to X$ is any birational morphism, we have $\mathfrak{r}_1(X, L) = \mathfrak{r}_1(Y, \mu^* L)$.

I would like to propose the following conjecture

Conjecture

For any big line bundle L over a (smooth) projective variety, there exists a sequence of birational morphism $\mu_m : X_m \to X$ and a decomposition $\mu^* L = L_m + E_m$ with L_m ample and E_m effective such that

$$\lim_{m \to +\infty} L_m^d = \mathrm{vol}(L), \quad \text{and} \quad \lim_{m \to +\infty} \mathfrak{r}_1(L_m) = \mathfrak{r}_1(L).$$

Lemma

If L admits a birational Zariski decomposition, i.e. if there exists a birational morphism $\mu : Y \to X$ such that $\mu^* L$ admits a Zariski decomposition, then the above conjecture is true. In particular, if X is toric (or more generally spherical variety), then the conjecture is true.

• Nakayama: there are examples of big line bundles which do not have birational Zariski decomposition.

Definition

A test configuration of (X, L) is the following data:

- *A flat family of projective varieties $\pi : \mathcal{X} \to \mathbb{C}$ with a $\mathbb{C}^*$-action such that π is $\mathbb{C}^*$-equivariant.*
- *A semiample $\mathbb{Q}$-line bundle $\mathcal{L}$ such that $\mathbb{C}^*$ lifts to act on $\mathcal{L}$.*
- *There is a $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \times_{\mathbb{C}} \mathbb{C}^* \cong X \times \mathbb{C}^*$.*

Equivalent data:

- $m \in \mathbb{N}$ and an embedding $X \to \mathbb{P}(H^0(X, mL)^*) \cong \mathbb{P}(\mathbb{C}^{N_m - 1})$.
- A one parameter $\mathbb{C}^*$-subgroup $\exp(s\eta) \in PGL(N_m, \mathbb{C})$
- Limit in the Hilbert scheme: $[\mathcal{X}_0] = \lim_{s \to +\infty}[X]$.
- $(\mathcal{X}, \mathcal{L}) = (X \times \mathbb{C}^* \cup \mathcal{X}_0, m^{-1}H_{\mathbb{P}^{N_m - 1}}) \subset \mathbb{P}^{N_m - 1} \times \mathbb{C}$

Set $(\bar{\mathcal{X}}, \bar{\mathcal{L}}) = (\mathcal{X}, \mathcal{L}) \cup (X, L) \times \{\infty\}$: the natural compactification.
• $\mathcal{X}$ is dominating if there is a $\mathbb{C}^*$-equivariant birational morphism $\rho : \mathcal{X} \to X \times \mathbb{C}$. By resolution of singularities, we can always achieve this.

Definition: For any test configuration $(\mathcal{X}, \mathcal{L})$, define:

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^{\cdot n} + \frac{\underline{S}}{n+1} \bar{\mathcal{L}}^{\cdot n+1}$$

where $K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} = K_{\mathcal{X}} + (\mathcal{X}_0)_{\mathrm{red}} - \pi^*(K_{\mathbb{P}^1} + \{0\})$, $\underline{S} = \frac{n c_1(X) \cdot L^{\cdot n-1}}{L^{\cdot n}}$.

• Base change $(\mathcal{X}', \mathcal{L}') := (\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}, t \mapsto t^b} \mathbb{C}$ with reduced central fibre s.t.

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{1}{b} \mathbf{M}^{\mathrm{NA}}(\mathcal{X}', \mathcal{L}') = \frac{1}{b} \left(K_{\bar{\mathcal{X}'}/\mathbb{P}^1} \cdot \bar{\mathcal{L}'}^{\cdot n} + \frac{\underline{S}}{n+1} \bar{\mathcal{L}'}^{\cdot n+1} \right).$$

• Translational invariance:

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0).$$

• If $\mathcal{X}_0$ is smooth and $\mathcal{L}$ is relative ample, $\mathbf{M}^{\mathrm{NA}}$ reduces to the Futaki invariant $\mathrm{Fut}(\eta)$ of central fibre (an obstruction to the existence of constant scalar curvature Kähler (cscK) metrics).

Non-Archimedean **J** functional of a test configuration:

$$\mathbf{J}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \bar{\mathcal{L}} \cdot L_{\mathbb{P}^1}^{n+1} - \frac{1}{n+1}\bar{\mathcal{L}}^{\cdot n+1} > 0 \quad \text{if } (\mathcal{X},\mathcal{L}) \text{ non-trivial}$$

Definition

(X, L) is uniformly K-stable if $\exists \gamma > 0$ s.t. for any test configuration $(\mathcal{X},\mathcal{L})$,

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}).$$

Conjecture (Uniform version of Yau-Tian-Donaldson (YTD) conjecture)

Assume $\mathrm{Aut}(X, L)_0$ is discrete. There exists a constant scalar curvature Kähler (cscK) metrics in $c_1(L)$ if and only if (X, L) is uniformly K-stable.

There is a version when $\mathrm{Aut}(X, L)_0$ is not discrete.

• The direction: cscK $\Longrightarrow$ uniform K-stability is known.

• YTD conjecture is known to hold when X is Fano and $L = -K_X$, even for all singular Fano varieties (Tian, Berman, Chen-Donaldson-Sun, Berman-Boucksom-Jonsson, **L.**-Tian-Wang, Hisamoto, **L.**, ...).

• K-stability for Fano varieties is an active subject in algebraic geometry...

A model of (X, L) is the same data as a test configuration except that we don't require $\mathcal{L}$ to be semiample.
For any model $(\mathcal{X}, \mathcal{L})$, choose $c \gg 1$ such that $\bar{\mathcal{L}}_c = \bar{\mathcal{L}} + cX_0$ is a big line bundle over $\bar{\mathcal{X}}$. Define non-Archimedean functionals (independent of $c \gg 1$):

$$\begin{aligned} \mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \langle \bar{\mathcal{L}}_c^n \rangle \cdot K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} + \frac{S}{n+1} \langle \bar{\mathcal{L}}_c^{n+1} \rangle \\ \mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \langle \bar{\mathcal{L}}_c \rangle \cdot L^n_{\mathbb{P}^1} - \frac{1}{n+1} \langle \bar{\mathcal{L}}_c^{n+1} \rangle. \end{aligned}$$

Theorem (L. '20, based on many people's works)

If (X, L) is uniformly K-stable for all models, i.e. $\exists \gamma > 0$ such that $\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ for all models, then (X, L) admits a cscK metric.

If Conjecture 1 is true, then we can remove the words "for all models" and get YTD conjecture 2, because we can then find a sequence of test configurations $(\mathcal{X}_m, \mathcal{L}_m)$ such that (up to base change):

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{m \to +\infty} \mathbf{M}^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m), \quad \mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{m \to +\infty} \mathbf{J}^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m).$$

- $(X^{\mathrm{NA}}, L^{\mathrm{NA}})$: (Berkovich) analytification w.r.t. to the trivially valued field $\mathbb{C}$.
- Points of X^{NA}: real valuations on functional fields of subvarieties.
- $X^{\mathrm{div}}_{\mathbb{Q}}$: set of divisorial valuations on $\mathbb{C}(X)$, dense in X^{NA}.
- smooth PSH metrics on $L^{\mathrm{NA}} \longleftrightarrow$ (equiv. classes of) test configurations

They are represented by relative (non-Archimedean) potential ($G(v)$ is the Gauss extension $X^{\mathrm{div}}_{\mathbb{Q}} \to (X \times \mathbb{C})^{\mathrm{div}}_{\mathbb{Q}}$)

$$(\phi_{\mathcal{L}} - \phi_{\mathrm{triv}})(v) = G(v)(\mathcal{L} - \rho^* L_{\mathbb{C}}).$$

- non-Archimedean Monge-Ampere measure (by Chambert-Loir): assume $\mathcal{X}_0 = \sum_i b_i E_i$, $x_i = b_i^{-1} r(\mathrm{ord}_{E_i})$ where r is the restriction of valuations via embedding $\mathbb{C}(X) \to \mathbb{C}(X \times \mathbb{C})$.

$$\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \sum_i b_i(\mathcal{L}^n \cdot E_i)\delta_{x_i}.$$

- Mixed Monge-Ampere measure for several line bundles defined similarly.

- $\mathbf{M}^{\mathrm{NA}}$ has a decomposition: for any test configuration $(\mathcal{X}, \mathcal{L})$

$$\mathbf{M}^{\mathrm{NA}}(\phi_{\mathcal{L}}) \quad = \quad K_{\bar{\mathcal{X}}/X_{\mathbb{P}^1}} \cdot \bar{\mathcal{L}}^n + \rho^* K_X \cdot \bar{\mathcal{L}}^n + \frac{\underline{S}}{n+1} \bar{\mathcal{L}}^{n+1}.$$

- non-Archimedean entropy

$$\begin{aligned} K^{\log}_{\bar{\mathcal{X}}/X_{\mathbb{P}^1}} \cdot \bar{\mathcal{L}}^n &= \sum_i (A_{X_{\mathbb{C}}}(E_i) - 1) E_i \cdot \bar{\mathcal{L}}^n = \sum_i A_X(b_i^{-1} r(E_i)) b_i (\mathcal{L}^n \cdot E_i) \\ &= \int_{X^{\mathrm{NA}}} A_X(x) \mathrm{MA}^{\mathrm{NA}}(\phi) \end{aligned}$$

$A_{X \times \mathbb{C}}(E_i) = \mathrm{ord}_{E_i}(K_{\mathcal{X}/X_{\mathbb{C}}}) + 1$ is the log discrepancy of E_i, similar for $A_X(\cdot)$.

- Non-Archimedean Monge-Ampère energy:

$$\begin{aligned} \bar{\mathcal{L}}^{n+1} &= \sum_{i=0}^{n+1} (\bar{\mathcal{L}}^{k+1} \cdot L_{\mathbb{P}^1}^{n-k} - \bar{\mathcal{L}}^k \cdot L_{\mathbb{P}^1}^{n-k+1}) = \sum_0^{n+1} (\bar{\mathcal{L}} - L_{\mathbb{P}^1}) \cdot \bar{\mathcal{L}}^k \cdot L_{\mathbb{P}^1}^{n-k} \\ &= \sum_{k=0}^{n+1} \int_{X^{\mathrm{NA}}} (\phi_{\mathcal{L}} - \phi_{\mathrm{triv}}) \mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}}^{[k]}, \phi_{\mathrm{triv}}^{[n-k]}) = (n+1) \mathbf{E}^{\mathrm{NA}}(\mathcal{L}). \end{aligned}$$

Extension to models

Boucksom-Favre-Jonsson defined Monge-Ampère measure for any continuous PSH metric, which is a non-Archimedean analogue of Bedford-Taylor's theory in classical pluripotential theory. They also solved non-Archimedean Monge-Ampère equations, as a non-Archimedean analogue of Calabi-Yau's theorem.

- models $\Longrightarrow$ continuous PSH metrics on L^{NA}: $\mu : \mathcal{X}_m \to \mathcal{X}$ blowup of $\mathfrak{b}(|m\mathcal{L}|)$ with exceptional divisor E_m, $\mathcal{L}_m = \mu_m^*\mathcal{L} - \frac{1}{m}E_m$

$$\phi_{\mathcal{L}} = \lim_{m\to+\infty} \phi_{\mathcal{L}_m}.$$

- Monge-Ampère measure for big models (**L.** '20):

$$\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}_c}) = \sum_i b_i(\langle \mathcal{L}_c^n\rangle \cdot E_i)\delta_{x_i}.$$

- Non-Archimedean Mabuchi functional of models:

$$\begin{aligned}
\mathbf{M}^{\mathrm{NA}}(\phi_{\mathcal{L}}) &= \int_{X^{\mathrm{NA}}} A_X(x)\mathrm{MA}^{\mathrm{NA}}(\phi) + (\mathbf{E}^{K_X})^{\mathrm{NA}}(\phi) + \underline{S}\mathbf{E}^{\mathrm{NA}}(\phi).\\
&= \langle \bar{\mathcal{L}}_c^n\rangle \cdot K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} + \frac{\underline{S}}{n+1}\langle \bar{\mathcal{L}}_c^{n+1}\rangle\\
&= \mathfrak{r}_1(\bar{\mathcal{L}}_c) + \frac{\underline{S}}{n+1}\mathrm{vol}(\bar{\mathcal{L}}_c) + 2L^n.
\end{aligned}$$

Archimedean Mabuchi functional: $e^{-\varphi}$ a PSH metric on $L \to X$

$$\mathbf{M}(\varphi) = \int_X \log \frac{(\mathrm{dd^c}\varphi)^n}{\omega_0^n} \omega_0^n + \mathbf{E}^{Ric(\omega_0)}(\varphi) + \underline{S}\mathbf{E}(\varphi).$$

• For any $\phi = \phi_{\mathcal{L}}$ there exists a unique geodesic rays $\Phi = \{\varphi(s)\}$ in the space of positively curved Hermitian metrics such that $\Phi_{\mathrm{NA}} = \phi$ where $(\Phi_{\mathrm{NA}} - \phi_{\mathrm{triv}})(v) = G(v)(\Phi)$ is the Lelong number with respect to any divisorial valuation v.

Conjecture

For any (continuous) metric $\phi = \phi_{\mathcal{L}}$ associated to models, we have:

$$\lim_{s \to +\infty} \frac{\mathbf{M}(\varphi(s))}{s} = \mathbf{M}^{\mathrm{NA}}(\phi).$$

This is true for $(\mathbf{E}^{K_X})^{\mathrm{NA}}$ and $\mathbf{E}^{\mathrm{NA}}$. It's known that LHS$\geq$RHS (**L.**). The conjecture for the entropy part is known for test configurations. The conjecture is again implied by Conjecture 1!

Thanks for your attention!