Moduli space of smoothable Kähler-Einstein Q-Fano varieties

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Uniformatization Theorem for Riemann Surfaces

Riemann surface: surface with a complex structure

Classification of closed Riemann surfaces :

Topology	Metric	Curvature
$\mathbb{S}^2 = \mathbb{CP}^1$	spherical	1
$\mathbb{T}^2 = \mathbb{C}/\Lambda$	flat	0
$\Sigma_g = \mathbb{B}^1/\pi_1(\Sigma_g)$	hyperbolic	-1

Notation: Σ_g closed oriented surface of genus $g \geq 2$.
$\mathbb{B}^1 = \{z \in \mathbb{C}; |z| < 1\}$.

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Isothermal Coordinate Theorem and Uniformization Theorem:

$$\mathcal{T}_g \cong \{\text{conformal or complex structures on } \Sigma_g\}.$$

$\mathcal{T}_g$ is a complex manifold of complex dimension $3g - 3$.
Tangent space: $H^1_J(\Sigma_g, T\Sigma_g) = H^0_J(\Sigma_g, K^2_{\Sigma_g})$.

Moduli space: $\mathcal{M}_g = \mathcal{T}_g/\mathrm{MCG}(\Sigma_g)$.
$\mathcal{M}_g$ is a complex orbifold.

Weil-Petersson metric: $g \geq 2$

Weil-Petersson metric ω_{WP}: For any hyperbolic metric h on Σ_g,

$$\omega_{\mathrm{WP}}(v) = \int_{\Sigma_g} |v|_h^2 \, d\mathrm{vol}_h, \quad v \in \mathcal{H}_J^1(\Sigma_g, T\Sigma_g; h).$$

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- ω_{WP} is a Kähler metric: $d\omega_{\mathrm{WP}} = 0$.
- The holomorphic sectional curvature of ω_{WP} is negative. $\mathcal{M}_g$ is Kobayashi hyperbolic.
- An hermitian line bundle $(L_{\mathrm{WP}}, h_{\mathrm{WP}})$ over $\mathcal{M}_g$ with Chern curvature $-\sqrt{-1}\partial\bar{\partial}\log h_{\mathrm{WP}} = \omega_{\mathrm{WP}}$.

Deligne-Mumford compactifications $\overline{\mathcal{M}}_g$, $g \geq 2$

- $\overline{\mathcal{M}}_g \backslash \mathcal{M}_g$ parametrizes stable curves.
  Constructed via relative MMP over a 1-dimensional base.
- Stable curves can be obtained as Gromov-Hausdorff limits
- ω_{WP} can be extended to $\overline{\mathcal{M}}_g$.
- The coarse moduli space $\overline{M}_g$ is a projective
  (Knudsen-Mumford)

Higher dimensional Kähler manifolds

X: complex manifold (transition functions are holomorphic);
J: $TX \to TX$ complex structure;
g: Riemannian metric s.t. $g(J\cdot, J\cdot) = g(\cdot, \cdot)$.

Kähler form: $\omega = g(\cdot, J\cdot)$. Using holomorphic coordinates $\{z^i\}$:

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad (g_{i\bar{j}}) > 0.$$

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Kähler condition: $d\omega = 0$. Consequences:

- ω determines the Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$.
- Locally, $\omega = \sqrt{-1}\partial\bar{\partial}\psi = \sqrt{-1}\sum_{i,j} \frac{\partial^2\psi}{\partial z^i \partial \bar{z}^j} dz^i \wedge d\bar{z}^j$.

Basic examples and curvature

Notation: $\mathbb{B}^n = \{z \in \mathbb{C}^n; |z| < 1\}$.

$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^* = \mathbb{C}^n \cup \mathbb{P}^{n-1}$.

$\mathbb{B}^n$	$\omega_{\mathbb{B}^n} = -\sqrt{-1}\partial\bar{\partial}\log(1-\|z\|^2)$	$\mathbb{B}^n/\Gamma;\ \Gamma < \mathrm{PSU}(n,1)$
$\mathbb{C}^n$	$\omega_{\mathbb{C}^n} = \sqrt{-1}\partial\bar{\partial}\|z\|^2$	$\mathbb{C}^n/\Lambda;\ \Lambda \cong \mathbb{Z}^{2n}$
$\mathbb{P}^n$	$\omega_{\mathrm{FS}} = \sqrt{-1}\partial\bar{\partial}\log(1+\|z\|^2)$	$\mathbb{P}^n$

Kähler manifolds with constant holomorphic sectional curvatures:

$$R_{i\bar{j}k\bar{l}} = \mu(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}),\ \mu = -1, 0, 1.$$

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Curvature tensor: $R_{i\bar{j}k\bar{l}} = -\dfrac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{r\bar{q}}\dfrac{\partial g_{k\bar{q}}}{\partial z_i}\dfrac{\partial g_{r\bar{l}}}{\partial \bar{z}_j}$.

Ricci curvature

Ricci curvature: $R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{l}k\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}.$

Compact expression: $R_{i\bar{j}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{k\bar{l}}).$

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Ricci form is a (1,1)-form:

$$Ric(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{i\bar{j}} dz^i \wedge d\bar{z}^j =: -\sqrt{-1}\partial\bar{\partial} \log \omega^n$$

$Ric(\omega)$ represents the *first Chern class* of the complex manifold:

$$Ric(\omega) \in 2\pi c_1(X) \in H^{1,1}(X, \mathbb{Z}).$$

Kähler-Einstein metrics

Normalize the Einstein constant to $\mu = -1, 0$, or 1. **KE** equation:

$$Ric(\omega_\varphi) = \mu \cdot \omega_\varphi$$

$$\Updownarrow$$

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_\omega - \mu\varphi}\omega^n$$

$$\left(h_\omega \text{ satisfies: } Ric(\omega) - \mu\omega = \sqrt{-1}\partial\bar{\partial}h_\omega, \text{ and } \int_X e^{h_\omega}\omega^n = \int_X \omega^n. \right)$$

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$\mu = -1$	Solvable (Aubin, Yau)	$c_1(X) < 0$	Canonically polarized
$\mu = 0$	Solvable (Yau)	$c_1(X) = 0$	Calabi-Yau
$\mu = 1$	in general not solvable	$c_1(X) > 0$	Fano

Building blocks of projective manifolds

Minimal Model Program (some parts are still conjectural):

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow Y$$

1. $\kappa(X) = -\infty$. $X_k \to Y$ is a Mori fiber space with fiber being Fano variety of Picard number 1;
2. $0 \le \kappa(X) < n$. $X_k \to Y$ is a Calabi-Yau fiber space;
3. $\kappa(X) = n$. $Y = X^{\mathrm{can}}$ is a canonically polarized variety.

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3. $\dim_{\mathbb{C}} = 3$: 105 deformation families
4. Hypersurface in $\mathbb{P}^{n+1}$ of degree $\leq n+1$;

Obstructions to KE on Fano manifolds

First obstruction: KE $\Longrightarrow \mathrm{Aut}(X)$ is reductive (Matsushima).
Example: Rule out $\mathbb{P}^2$ blown-up one or two points:

- $\mathrm{Aut}(\mathbb{P}^2\sharp\overline{\mathbb{P}^2}) \cong \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{PGL}(3,\mathbb{C}) \right\}.$
- $\mathrm{Aut}(\mathbb{P}^2\sharp 2\overline{\mathbb{P}^2}) \cong \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{PGL}(3,\mathbb{C}) \right\}.$

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Conjecture (Yau-Tian-Donaldson)

Fano manifold X has KE $\Longleftrightarrow (X, -K_X)$ is K-polystable.

" $\Longrightarrow$ ": Proved by Tian and Berman;
" $\Longleftarrow$ ": Completed by Tian, Chen-Donaldson-Sun independently.

Futaki invariant

v: any holomorphic vector field. Recall: $Ric(\omega) - \omega = \sqrt{-1}\partial\bar{\partial} h_\omega$.

Futaki invariant: $\mathrm{Fut}_X(v) = \int_X v(h_\omega)\omega^n$. $(\mathrm{Fut}_X : \mathfrak{h} \to \mathbb{C})$

Theorem (Futaki)

- $\mathrm{Fut}_X(v)$ *is independent of* $\omega \in 2\pi c_1(X)$.
- X *Kähler-Einstein* $\Longrightarrow \mathrm{Fut}_X \equiv 0$.

Interpretation: $(\bar{\partial} - \sqrt{-1} i_v)(\omega + \mathrm{div}_\Omega(v)) = 0$ $\quad (\Omega = e^{h_\omega}\omega^n)$

$$\mathrm{Fut}_X(v) = \frac{1}{n+1}\int_X (\omega + \mathrm{div}_\Omega v)^{n+1}.$$

Equivariant cohomology (⇒ localization formula)

Special degeneration

Definition (Special Degeneration)

A $\mathbb{C}^*$-equivariant degeneration of Fano manifolds over $\mathbb{C}$:

$$\begin{tikzcd}
(X \times \mathbb{C}^*, -K_X) \arrow[r, hook] \arrow[d] & (\mathcal{X}, -K_{\mathcal{X}/\mathbb{C}}) \arrow[d] & (\mathcal{X}_0, -K_{\mathcal{X}_0}) \arrow[l, hook'] \arrow[d] \\
\mathbb{C}^* \arrow[r] & \mathbb{C} & \{0\} \arrow[l]
\end{tikzcd}$$

such that $\mathcal{X}_0$ is an irreducible normal Fano variety with Kawamata-log-terminal (klt) singularities.

$\mathbb{C}^*$-action $\rightsquigarrow$ holomorphic vector field v on $\mathcal{X}_0$. For special degenerations, define:

$$\mathrm{Fut}(\mathcal{X}, -K_{\mathcal{X}/\mathbb{C}}) = -\mathrm{Fut}_{\mathcal{X}_0}(v)$$

K-stability

Definition (K-polystability, Tian '97)

$\mathrm{Fut}(\mathcal{X}, K_{\mathcal{X}}^{-1}) \geq 0$ *for any specicial degeneration* $\mathcal{X}$ *of* X, *with equality holds iff* $\mathcal{X} \cong X \times \mathbb{C}$.

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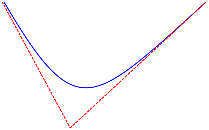
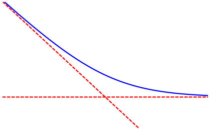
Imitating Hilbert-Mumford numerical criterion in GIT:

$$\text{Slope at infinity} \longleftrightarrow \mathrm{Fut}(\mathcal{X}, \mathcal{L}).$$

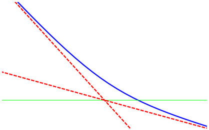
Stability from functionals (variational point of view):



(d) Stable



(e) Semistable



(f) Unstable

Generalization by Donaldson using general test configuration.

Canonically polarized case: KSBA compactification

Generalization of Deligne-Mumford compactification developped by Kollár-Shepherd-Barron-Alexeev.
Four aspects of the construction:

1. Properness: stable varieties (semi-log-canonical singularities)
2. Boundedness: Hacon-McKernan-Xu
3. Separatedness: relatively easy
4. Local openness: Kollár

Extra properties:

- Projectivity: Kollár, Fujino.
- By Berman-Huenancia and Odaka: Canonically polarized case: Kähler-Einstein = Stable varieties = K-stable
- Expect: all stable varieties can be obtained from GH limits.

Fano case: Properness

$(\mathcal{X}, \mathcal{L}) \to (C, 0)$: flat family of polarized projective varieties.
$(\mathcal{X}^\circ, \mathcal{L}^\circ) \cong (\mathcal{X}^\circ, K^{-1}_{\mathcal{X}^\circ/C^\circ}) \to C^\circ$: family of smooth Fano manifolds.
The special fiber $\mathcal{X}_0$ can be very bad.

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Theorem (L.-Xu, '12)

There exists a $\mathbb{Q}$-Fano filling after base change:

$$\begin{tikzcd}
\mathcal{X}^s \arrow[r, dashed] \arrow[d] & \mathcal{X} \times_C C' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}^\circ \arrow[l, hook'] \arrow[d] \\
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Moreover, $\mathrm{CM}(\mathcal{X}^s/C', -K_{\mathcal{X}^s}) \leq \deg(\phi) \cdot \mathrm{CM}(\mathcal{X}/C, \mathcal{L})$.

- Use Minimal Model Program to simplify the family
- Keep track of the CM-degree (which generalizes Futaki invariant)

Fano case: Separatedness

Compare 2 flat families of $\mathbb{Q}$-Fano with isomorphic generic fibres:

$$\begin{tikzcd}
\mathcal{X} \arrow[d] \arrow[rr, dashed, bend left] & \mathcal{X}^\circ = \mathcal{X}'^\circ \arrow[l, hook'] \arrow[r, hook] \arrow[d] & \mathcal{X}' \arrow[d] \\
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Question of separatednes: $\mathcal{X} \cong \mathcal{X}'$? Answer: In general fails:

- Smooth $\dim_{\mathbb{C}} = 3$: Mukai-Umemura's example.
- Singular $\dim_{\mathbb{C}} = 2$: infinitely many singular del-Pezzo degenerations of $\mathbb{P}^2$ (Hacking-Prokhorov).

Issue in the construction of compact moduli in the Fano case

1. Boundedness fails without restrictions on the singularities
2. Separatedness fails
3. Continuous automorphism group
4. Tian's conjecture: the moduli space of Kähler-Einstein manifolds is quasi-projective

Surprisingly, by adding "Kähler-Einstein" condition, these issues can be solved simultaneously.

Algebraic structure on Gromov-Hausdorff limit

$\{(X_i, \omega_i)\}$: Fano Kähler manifolds. $Ric(\omega_i) = \omega_i$. Then:

- $\mathrm{Diam}(X_i, \omega_i) \leq D(n) = \sqrt{2n-1} \cdot \pi$ (Myers Theorem)
- $\mathrm{Vol}(B_r(x))/\mathrm{Vol}(B_r(\underline{0})) \searrow$ as $r \nearrow$ (Bishop-Gromov)

Gromov compactness $\Longrightarrow (X_i, \omega_i) \xrightarrow{GH} (X_\infty, \omega_\infty)$.

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Proposition (Tian, also L. '12)

Partial C^0-estimate $\Longrightarrow X_\infty$ has an algebraic structure.

Proof: Skoda-Siu's theorem on finite generation.

Conjecture (Tian's partial C^0-estimate Conjecture)

There exists $m = m(n)$ and $\delta = \delta(n) > 0$ such that $\rho_m \geq \delta$.

$\{s_i\}_{i=1}^{N_m}$: O.N. basis of $H^0(X, K_X^{-m})$ under the L^2-inner product.

Bergman kernel: $$\rho_m(z) = \sum_{i=1}^{N_m} |s_i|^2_{h^{\otimes m}}(z)$$

Properness of "KE Fano" under Gromov-Hausdorff limits

Theorem (Donaldson-Sun, Tian)

Tian's partial C^0-estimate conjecture holds. As a consequence, X_∞ is a normal Fano variety.

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$$\begin{array}{cc} \omega_t^n = e^{-u_i}\Omega_i, & \Omega_i = \left(\sum_{j=1}^{N_m} |s_j^{(i)}|^2\right)^{-1/m} \\ \Downarrow & \Downarrow \\ \omega_\infty^n = e^{-u_\infty}\Omega_\infty, & \Omega_\infty = \left(\sum_{j=1}^{N_m} |s_j^{(\infty)}|^2\right)^{-1/m} \end{array}$$

$$e^{-u_i} = V\frac{(\rho_m^{(i)})^{1/m}}{\int_{X_i}(\rho_m^{(i)})^{1/m}\Omega_i} \Longrightarrow e^{-u_\infty} = V\frac{(\rho_m^{(\infty)})^{1/m}}{\int_{X_\infty}(\rho_m^{(\infty)})^{1/m}\Omega_\infty}.$$

$$\int_{X_\infty}\Omega_\infty = \int_{X_\infty} e^{u_\infty}\omega_\infty^n < +\infty \Longrightarrow X_\infty \text{ has Klt singularities.}$$

Gromov-Hausdorff limits of pairs

Conical Kähler-Einstein metric:

$$Ric(\omega(\beta)) = \beta\omega + 2\pi(1-\beta)\{D\}.$$

Theorem (Tian, Chen-Donaldson-Sun)

X_i: n-dim'l Fano manifold; $D_i \in \frac{1}{m}|-mK_{X_i}|$ smooth divisors; $\omega_i(\beta_i)$: conical KE on $(X_i, (1-\beta_i)D_i)$. If $\beta_i \to \beta_\infty \in (0,1)$, then, by passing to a subsequence,

1. $(X_i, (1-\beta_i)D_i; \omega_i(\beta_i)) \xrightarrow{GH} (Y, (1-\beta_\infty)E; \omega(\beta_\infty))$
2. *There exist embeddings $T_i : X_i \to \mathbb{P}^N$ and $T_\infty : Y \to \mathbb{P}^N$, such that $(T_i(X_i), T_i(D_i)) \to (T_\infty(Y), T_\infty(E))$ as projective varieties.*

Separatedness and local openness

Theorem (L.-Wang-Xu, '14)

$\mathcal{X} \to (C, 0)$ a flat family over a smooth pointed curve, satisfying

1. *$-K_{\mathcal{X}/C}$ is $\mathbb{Q}$-Cartier and relatively ample;*
2. *for any $t \in C^\circ := C \backslash \{0\}$, $\mathcal{X}_t$ is smooth and $\mathcal{X}_0$ is klt;*
3. *$\mathcal{X}_0$ is K-polystable.*

Then

- (i) *$\exists$ a Zariski open neighborhood U of $0 \in C$, s.t. $\mathcal{X}_t$ is K-semistable (resp. K-stable if $\mathrm{Aut}(\mathcal{X}_0)$ is discrete) for all $t \in U$.*
- (ii) *For any flat $\mathcal{X}' \to C'$ satisfying (1)-(3) as above, and $\mathcal{X}' \times_C C^\circ \cong \mathcal{X} \times_C C^\circ$, we have $\mathcal{X}'_0 \cong \mathcal{X}_0$;*
- (iii) *$\mathcal{X}_0$ admits a weak Kähler-Einstein metric. If $\mathcal{X}_t$ is K-polystable, then $\mathcal{X}_0$ is the Gromov-Hausdorff limit $\mathcal{X}_t$ endowed with the Kähler-Einstein metric for any $t \to 0$.*

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Related works by Spotti-Sun-Yao.

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Theorem (L.-Wang-Xu, '14)

$\exists$ *proper algebraic moduli space* $\overline{\mathcal{M}}$ *of K-polystable, smoothable, Fano varieties.*

- X (weak) KE $\Rightarrow \mathrm{Aut}(X)$ is reductive. (CDS, BBEGZ)
- Locally K-polystable slice = GIT moduli
- Glue: $\overline{\mathcal{M}} = \bigcup_{i=1}^{I} (\mathcal{U}_{z_i} /\!\!/ G_{z_i})$ using languages of algebraic stacks

Related work by Odaka

On projectivity of moduli spaces

$\mathcal{M}^-$: moduli space of canonically polarized manifolds
$\overline{\mathcal{M}^-}$: Kollár-Shepherd-Barron-Alexeev compactification

- Viehweg: $\mathcal{M}^-$ is quasi-projective (nef K_X is enough)
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Constraint: Use of canonical metrics $\longleftrightarrow$ Weil-Petersson geometry

Theorem (L.-Wang-Xu, '15)

The moduli space $\mathcal{M}$ parametrizing smooth Kähler-Einstein Fano manifolds is quasi-projective.

Proof: Continuity method

Continuity method: use the log version.
Define the set of parameters:

$$\begin{aligned}\mathbf{B}(\mathcal{X},\mathcal{D}) \;&=\; \{\beta\in(0,\mathfrak{B}]\,|(\mathcal{X}_t,(1-\beta)\mathcal{D}_t) \text{ has conical KE } \omega_t(\beta)\\ &\qquad \text{and } (\mathcal{X}_t,\mathcal{D}_t;\omega_t(\beta)) \xrightarrow{\text{GH}} (\mathcal{X}_0,\mathcal{D}_0;\omega_0(\beta))\Big\}.\end{aligned}$$

Need to prove $\mathbf{B}(\mathcal{X},\mathcal{D})$ satisfies
1: Non-empty; 2: Open; 3: Closed.

Proof: Non-emptiness of $\mathbf{B}(\mathcal{X}, \mathcal{D})$

Lemma

There exists $\epsilon = \epsilon(n) > 0$ *such that if*

1. $(\mathcal{X}, (1-\epsilon)\mathcal{D})$ *and* $(\mathcal{X}', (1-\epsilon)\mathcal{D}')$ *are two families of klt log-Fano varieties with* $\mathcal{D}^{(')} \in \frac{1}{m}\left|-mK_{\mathcal{X}^{(')}/C^{(')}}\right|$ *irreducible;*
2. $(\mathcal{X}, \mathcal{D}) \times_C C^\circ \cong (\mathcal{X}', \mathcal{D}') \times_C C^\circ$,

then $(\mathcal{X}', \mathcal{D}') \cong (\mathcal{X}, \mathcal{D})$.

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Corollary

Any klt log-Fano pair $(X, (1-\epsilon)D)$ *with* $D \in \frac{1}{m}|-mK_X|$ *irreducible is K-stable, and hence has conical KE.*

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Corollary

$\mathbf{B}(\mathcal{X}, \mathcal{D})$ *is non-empty.*

Proof: key algebraic lemma on action of reductive group

G: a reductive group acting on $\mathbb{P}^N$;
$z : C \to \mathbb{P}^N$ be an arc with $z(0) = z_0$;
$\overline{O} = \lim_{t \to 0} \overline{O_{z(t)}}$ with $O_{z(t)} = G \cdot z(t)$.

Lemma (Key Algebraic Lemma)

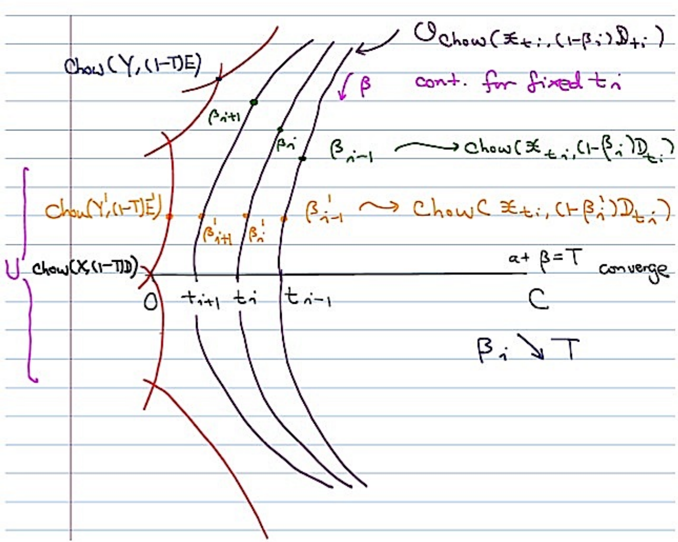
Suppose the stabilizer of $z_0 \in \mathbb{P}^N$ *is reductive. Then there is a* G*-invariant Zariski open neighborhood of* $z_0 \in U \subset \mathbb{P}^N$ *satisfying:*

$$\overline{O} \cap U = \bigcup_{\substack{O_p \subset \overline{O} \\ \overline{O}_p \cap O_{z_0} \neq \emptyset}} O_p \cap U,$$

i.e. the closure of the G-orbit of any point in $\overline{O}$ *near* z_0 *contains* $g \cdot z_0$ *for some* $g \in G$.

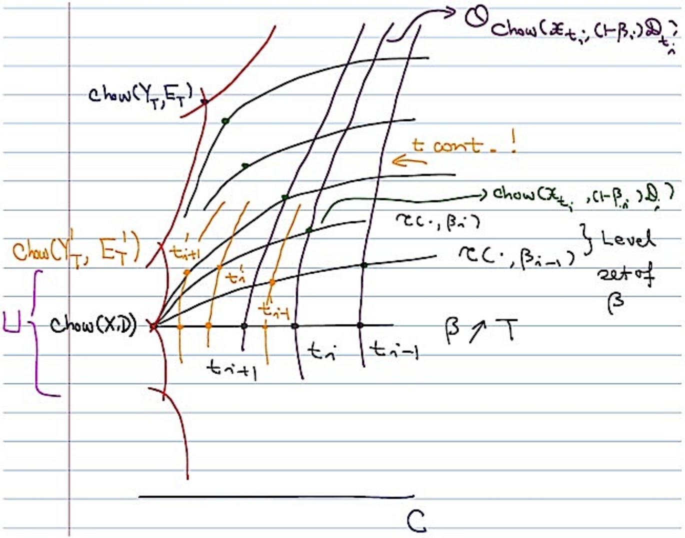
Proved by Luna Slice Theorem. The condition of reductivity is needed.

Proof: Openness of $\mathbf{B}(\mathcal{X}, \mathcal{D})$



Picture by X. Wang

Proof: Closedness of $\mathbf{B}(\mathcal{X}, \mathcal{D})$



Picture by X. Wang

Zariski openness of K-semistable points

This essentially follows from the two properties of the following invariant:

$$\mathrm{kst}(\mathcal{X}_t, \mathcal{D}_t) = \sup\{\beta \in [0, \mathfrak{B}]\,|(\mathcal{X}_t, (1-\beta)\mathcal{D}_t) \text{ is K-semistable}\}$$

As a function of $t \in C$, we have

1. kst is a constructible function. This follows GIT argument because K-stability can also be formulated as CM-stability.
2. kst is lower semi-continuous. This can be proved using similar arguments as above (also proved by Spotti-Sun-Yao).

Structure near K-polystable orbit

1. Smoothable K-semistable $\mathbb{Q}$-Fano varieties are bounded;
2. There exists a neighborhood U^{ks} s.t. any K-semistable $z \in U^{\mathrm{ks}}$ degenerates to $\hat{z} \in U^{\mathrm{kps}}$

Theorem

$\exists$ *a Zariski open neighborhood* U *of* $Chow(X)$, *such that* $Chow(Y) \in U$ *is GIT-polystable with respect to* $\mathrm{Aut}(X)$ *if and only if* Y *is K-polystable.*

Remark: case of smooth X is due to Donaldson, Brönnle and Székelyhidi.

- Glue local GIT moduli to get global moduli by the work of Alper and others.

CM line bundle

Let $(\mathcal{X}, -K_{\mathcal{X}/C}) \to C$ be flat family of polarized $\mathbb{Q}$-Fano varieties. CM line bundle over the base C (Tian, Fujiki-Schumacher):

$$L_{\mathrm{CM}} = \frac{1}{2^{n+1}} \det \left(\pi_! [-(K_{\mathcal{X}/C}^{-1} - K_{\mathcal{X}/C})^{n+1}] \right), .$$

Knudsen-Mumford expansion:

$$\det \left(\pi_* (K_{\mathcal{X}/C}^{-k}) \right) = -L_{\mathrm{CM}} \frac{k^{n+1}}{(n+1)!} + O(k^n).$$

Futaki invariant as CM weight:

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = -\frac{1}{n+1} (K_{\bar{\mathcal{X}}/\mathbb{P}^1}^{-1})^{n+1}$$

Weil-Petersson metric

$\mathcal{X} \to C$ a flat family of Kähler-Einstein manifolds

$$\omega_{\mathrm{WP}}(v) = \int_X |v|^2_\omega \omega^n, v \in \mathcal{H}^1(X, TX; \omega).$$

1. $$\omega_{\mathrm{WP}} = -\int_{\mathcal{X}/C} \omega^{n+1}.$$

   Here $\omega = -\sqrt{-1}\partial\bar{\partial}\log\{\omega_t^n\}$ is a (1,1)-form on the total space $\mathcal{X}$ with $\omega|_{\mathcal{X}_t} = \omega_t$.

2. $\omega_{\mathrm{WP}} = -\sqrt{-1}\partial\bar{\partial}\log h_{\mathrm{WP}}$. Here h_{WP} is a Quillen-type metric on the determinant line bundle L_{CM}.

Hermtian metric on L_{CM} via Deligne's pairing

- CM line bundle via Deligne pairings

$$L_{\mathrm{CM}} \cong -\langle -K_{\mathcal{X}/C}, \cdots, -K_{\mathcal{X}/C} \rangle_{\mathrm{DP}} \to C.$$

- Monge-Ampère equations: $\omega_t^n = e^{-u_t}\Omega_t$, where

$$\Omega_t = \left(\sum_{i=1}^{N_m} |s_i|^2 \right)^{-1/m}, \quad e^{-u_t} = \frac{\rho_m^{1/m}}{\int_{\mathcal{X}_t} \rho_m^{1/m} \Omega_t}.$$

- Change of metric formula Define: $h_{\mathrm{DP}} := h_{\Omega_t} e^{-\mathfrak{U}_t}$:

$$\mathfrak{U}_t = -\sum_{k=0}^{n} \int_{\mathcal{X}_t} u_t \omega_t^k \wedge \check{\omega}_t^{n-k}, \quad \check{\omega}_t = -\sqrt{-1}\partial\bar{\partial} \log \Omega_t$$

h_{DP} is defined even for singular $\mathcal{X}_t$!

Continuation of construction

- Continuous extension of Hermtian metric h_{DP} to locus of singular fibers, depending on:

Lemma (Continuity Lemma)

$$\lim_{t \to 0} \int_{\mathcal{X}_t} \Omega_t = \int_{\mathcal{X}_0} \Omega_0.$$

Proof: Calculation by lifting to resolution (weak semistable reduction) + Inversion of adjunction.

- Descent of metric to the line bundle $\overline{\mathcal{M}}$ by the functoriality of Deligne's pairing

  Futaki invariant=0 $\Longrightarrow \mathrm{Aut}(\mathcal{X}_t)_0$ acts trivially on $L_{\mathrm{CM}}|_{\{t\}}$

  $\Longrightarrow h_{\mathrm{DP}}$ is invariant up to the action of $\mathrm{Aut}(\mathcal{X}_t)_0$

Quasi-Projectivity criterion

Using h_{DP}, it's easy to check that the assumption of the following criterion is satisfied.

Theorem (L.-Wang-Xu, '15)

M: a normal proper algebraic space of finite type over $\mathbb{C}$;
L: a line bundle on M; $M^\circ \subset M$ an open subspace.
Assume: $L^m \cdot Z \geq 0$ for any m-dimensional irreducible subspace and the strict inequality holds for any Z meeting M°.
Then for sufficiently large power k, $|L^k|$ induces a rational map which is an embedding when restricted on M°.

Proof: 1. Reduce to the case when M is projective;
2. Results of Nakamaye and Birkar:

Null locus = augmented stable base locus.

Some open questions

- Is $\overline{\mathcal{M}}$ projective? Is the line bundle L_{CM} ample?
- What are some properties of $\overline{\mathcal{M}}$? Is $\mathcal{M}$ rationally connected?
- What are the behaviors of the limiting metric structure (X_∞, d_∞) near X_∞^{sing}?

Thanks for your attention!