Non-Archimedean methods for canonical Kähler metrics

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Constant scalar curvature Kähler (cscK) metrics

- X a projective manifold. L: an ample line bundle

$\exists$ Hermitian metric $h = e^{-\psi}$ on L whose Chern curvature

$$\omega = \sqrt{-1}\partial\bar{\partial}\psi = \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

is a Kähler metric representing $c_1(L) \in H^{1,1}(X)$.

Ricci curvature $Ric(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}})$.

Scalar curvature: $S(\omega) = g^{i\bar{j}} R_{i\bar{j}}$.

Average scalar curvature: $\underline{S} = \frac{n c_1(X) \cdot c_1(L)^n}{c_1(L)^n}$.

- Yau-Tian-Donaldson Conjecture:

Existence of cscK Kähler metrics in $c_1(L) \iff (X, L)$ is K-stability

Mabuchi functionals

cscK metrics are exactly critical points of the Mabuchi functional:

$$\begin{aligned}\mathbf{M}(\varphi) &= -\int_0^1 dt \int_X \dot{\varphi}(S(\varphi_t) - \underline{S})\omega_{\varphi_t}^n \\ &= \mathbf{E}(\varphi) + \mathbf{E}^{\chi}(\varphi) + \mathbf{H}(\varphi).\end{aligned}$$

Berman-Berndtsson: cscK metrics are minimizers of $\mathbf{M}$.

Test configuration $(\mathcal{X}, \mathcal{L}) \longrightarrow$ geodesic ray $\Phi = \{\varphi(s)\}$.

$$\lim_{s \to +\infty} \frac{\mathbf{M}(\varphi_s)}{s} = \mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}).$$

Test configurations

• $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ a normal, flat family of projective varieties over $\mathbb{C}$. $\mathcal{L}$ a semi-ample $\mathbb{Q}$-line bundle.

An isomorphism $\tau : (\mathcal{X}^*, \mathcal{L}^*) \cong (X_{\mathbb{C}}, L_{\mathbb{C}}) = (X \times \mathbb{C}, p_1^* L)$.

The $\mathbb{C}^*$-action on $(\mathcal{X}^*, \mathcal{L}^*)$ extends to become a $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$.

• Example: product test configuration, projective transformation, flag ideals.

• Associated normal ample test configuration

Canonical compactification $(\mathcal{X}, \mathcal{L})/\mathbb{C} \to (\mathcal{X}, \mathcal{L})/\mathbb{P}^1$.

For any normal test configuration

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \frac{\underline{S}}{n+1}\mathcal{L}^{n+1} + \frac{1}{n}K^{\log}_{\mathcal{X}/\mathbb{P}^1}\cdot\mathcal{L}^n.$$

If $(\mathcal{X},\mathcal{L})$ is a product test configuration associated to the holomorphic vector field v, then

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) \;=\; \mathrm{Fut}(\mathrm{X},\mathrm{v}) = -\int_{\mathrm{X}}(\mathrm{S}(\omega)-\underline{\mathrm{S}})\theta_{\mathrm{v}}\omega^{\mathrm{n}}$$

where $\iota_v\omega = \sqrt{-1}\bar{\partial}\theta_v$.

Weight interpretation: Tian, Donaldson, Paul.

Toric case

X toric: there exists an effective $(\mathbb{C}^*)^n$-action with a dense open obit

Toric $X \longleftrightarrow$ Convex simplicial fans $\Sigma = \{\sigma_i\}$, $\bigcup_i \sigma_i = \mathbb{R}^n$.

$X_{\sigma_i} = \mathrm{Spec}\,(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n])$, $X = \bigcup_i X_{\sigma_i}$.

L a torus equivariant line bundle $\longleftrightarrow$ piecewise linear function Ψ on Σ: $\Psi|_{\sigma_i} = \langle u_i, \cdot \rangle$.

Suppose L is ample $\longleftrightarrow$ Ψ is strictly convex

$X \longleftrightarrow$ moment polytope $\Delta_L = \{u \in \mathbb{R}^n ; \langle u, v_k \rangle \geq \Psi(v_k)\}$.

Toric case: II

Test configuration $\leftrightarrow$ concave piecewise linear function θ over Δ_L.

Legendre transform: $\psi = (-\theta)^\vee$, $|\psi - \Psi|$ is bounded:

$$\begin{aligned} \psi(x) &= \sup\{\langle y, x\rangle + \theta(y), y \in \Delta_L\}, \\ \theta(y) &= \inf\{-\langle y, x\rangle + \psi(x), x \in \mathbb{R}^n\}. \end{aligned}$$

Donaldson's formula:

$$n! \cdot \mathbf{M}^{\mathrm{NA}} = \underline{S} \int_\Delta \theta dy - \int_{\partial\Delta} \theta d\sigma.$$

Toric case of YTD conjecture: Donaldson, Chen-Li-Sheng, ...

- Normal affine variety $X = \mathrm{Spec}(R)$, R a finitely generated $\mathbb{C}$-algebra.

Analytification with respect to trivial valuation on $\mathbb{C}$:

$$X^{\mathrm{NA}} = \{\text{multiplicative semi-norms on } R\} = \{\text{real valuations on subvarieties of X}\}.$$

- Projective variety $X \longrightarrow X^{\mathrm{NA}}$ by gluing affine pieces $X^{\mathrm{NA}} = \{\text{real valuations on subvarieties of X}\}$: Compact, Hausdorff, locally connected space.

Analytic functions on X^{NA} generated by $x \mapsto \log |f|_x^2$.

- Trivial valuation: $v_{\mathrm{triv}}(f) = 0$ for any $f \neq 0$.

Divisorial valuations: $q \cdot \mathrm{ord}_E$, $\mu : E \subset Y \to X$.

Quasi-monomial valuations: v_α.

Gauss extension

Gauss extension:

$$G(v)(\sum_k f_k t^k) = \min\{v(f_k) + k\}.$$

$\mathcal{X}$ a model of X with $\mathcal{X}_0 = \sum_i b_i E_i$. Define

$$v_i = r(b_i^{-1} \mathrm{ord}_{E_i}), \quad \mathrm{ord}_{E_i} = b_i \cdot G(v_i)$$

$(\mathcal{X}, \mathcal{X}_0)$ SNC $\to$ Dual complex $\Delta_{\mathcal{X}} \leftrightarrow$ Quasi-monomial valuations

$$X^{\mathrm{NA}} = \lim \Delta_{\mathcal{X}}.$$

Canonical contraction $p_{\mathcal{X}} : X^{\mathrm{NA}} \to \Delta_{\mathcal{X}}$.

Analytification of line bundles

- L^{NA}: Analytification of L.

Local trivialization of L over $X \longrightarrow$ local trivialization of L^{NA} over X^{NA}.

Metrics on L^{NA}: $s_i = a_{ij} s_j \Rightarrow \log |s_i|^2 = \log |a_{ij}|^2 + \log |s_j|^2$.

- "Trivial" metrics on L^{NA}: $v \in X^{\mathrm{NA}}$, s a trivialization of L^{NA} near the center of v: $|s|_0^2 = 1$.
Relative potential $\log |s|^2 = \log |s|_0^2 + \phi$.

- A section $s \in H^0(X, L)$, $\log |s|^2(v) = -v(s)$.

Fubini-Study metrics $\phi = \frac{1}{m} \max_i \{\log |s_i|^2 + \lambda_i\}$.

- $\mathbb{R}^+$ action on metrics of $L^{\mathrm{NA}} \longleftrightarrow$ base change.

Metrics from test configurations: algebraic definition

- Test configuration $\Longrightarrow$ semi-positive Fubini-Study metric on L^{NA}.

$\mathcal{L} = p_1^* L_{\mathbb{C}} + D$ where $D = \sum_i c_i E_i$ is supported on $\mathcal{X}_0 = \sum_i b_i E_i$.

$$\phi_{\mathcal{L}} := \phi_{(\mathcal{X},\mathcal{L})}(v) := G(v)(D)$$

$v_i = r(b_i^{-1}\mathrm{ord}_{E_i})$:

$$\phi_{\mathcal{L}}(v_i) = G(v_i)(D) = b_i^{-1}\mathrm{ord}_{E_i}(\sum_i c_i E_i) = \frac{c_i}{b_i}.$$

- $\phi_{(\mathcal{X}_1,\mathcal{L}_1)} = \phi_{(\mathcal{X}_2,\mathcal{L}_2)} \Longleftrightarrow (\mathcal{X}_i, \mathcal{L}_i)$ are equivalent.
- Normalized blowup $\mathcal{X} = \overline{Bl_{\mathfrak{a}} X}$ for a flag ideal $\mathfrak{a}$, then

$$\phi_{\mathcal{X},\mathcal{L}}(v) = \phi_{\mathfrak{a}}(v) = -G(v)(\mathfrak{a}).$$

- $\phi_{\mathcal{L}}(v_{\mathrm{triv}}) = \sup \phi_{\mathcal{L}}$.

Metrics from test configurations: analytic definition

- $\mathcal{L} \to \mathcal{X}$ is semi-ample $\longrightarrow$ a smooth positively curved Hermitian metric $H_0 e^{-\tilde{\Phi}}$: $R(H_0) + \sqrt{-1}\partial\bar{\partial}\tilde{\Phi} \geq 0$.

Recall isomorphism $\tau : (\mathcal{X}^*, \mathcal{L}^*) \cong (X \times \mathbb{C}, p_1^* L)$.

A smooth sub-geodesic ray: $\tilde{\Phi} = (\tau^{-1})^* \tilde{\Phi} = \{\tilde{\varphi}(s)\}, s = -\log|t|^2$.

- Define: $\tilde{\Phi}^{\mathrm{NA}}(v) = -G(v)(\tilde{\Phi})$.

For a divisorial valuation $w = q \cdot \mathrm{ord}_E$ for $E \subset \mathcal{Y} \to \mathcal{X}$.

$$w(\tilde{\Phi}) = \sup\left\{c; \tilde{\Phi} \leq c \cdot q^{-1} \log|f|^2 + O(1)\right\}.$$

- As functions on X^{NA}, $\tilde{\Phi}^{\mathrm{NA}} = \phi_{\mathcal{L}}$.

- Fix a semipositive form $\Omega \in c_1(\mathcal{L})$. $\mathbb{D}_1 = \{|t| \leq 1\}$.

Solve complex Monge-Amère equation over $\pi^{-1}(\mathbb{D}_1)$:

$$(\Omega + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} = 0, \Phi|_{\partial\mathbb{D}_1} = 0.$$

$s = -\log|t|^2$, $\Phi = \{\varphi(s)\}$.

- It can also be characterized as an upper envelope:

$$\Phi = \sup{}^*\{\Psi : \Omega + \sqrt{-1}\partial\bar{\partial}\Psi \geq 0, \Psi|_{\partial\mathbb{D}_1} = 0\}.$$

Fact: $|\Phi - \tilde{\Phi}|$ is bounded $\Longrightarrow \Phi^{\mathrm{NA}} = \tilde{\Phi}^{\mathrm{NA}}$.

More regularity: Φ is $C^{1,1}$ away from $\mathcal{X}_0$: Phong-Sturm, Chu-Tosatti-Weinkove.

Monge-Ampère measure

Kähler potentials: $\mathcal{H}(L) = \{\varphi \in C^\infty(X); \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$.

For any $\varphi \in \mathcal{H}$,

$$\mathrm{MA}(\varphi) = \omega_\varphi^n = (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n$$

Bedford-Taylor: extension for bounded φ (integration by parts and Chern-Levine-Nirenberg estimates)

For general $\varphi \in \mathcal{PSH}$, one can define a non-pluripolar MA measure:

$$\mathrm{MA}(\varphi) = \lim_{k \to +\infty} \mathrm{MA}(\max(\varphi, -k)).$$

$\mathcal{PSH}_{\mathrm{full}} = \{\varphi \in \mathcal{PSH}; \int_X \mathrm{MA}(\varphi) = V\}$.

- Chambert-Loir introduced the non-Archimedean MA measure (in the theory of heights):

$\mathcal{X}_0 = \sum_i b_i E_i$, $v_i = b_i^{-1} r(\mathrm{ord}_{E_i}) \in Div(X) \subset X^{\mathrm{NA}}$.

$$\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{X},\mathcal{L}}) = \sum_i b_i(\mathcal{L}^n \cdot E_i)\delta_{v_i}.$$

There is a Hybrid space $\mathfrak{X}$ s.t. $\mathfrak{X}_1$ is X and $\mathfrak{X}_0$ is X^{NA}.
Convergence of measures $\mathrm{MA}(\varphi_t) \to \mathrm{MA}^{\mathrm{NA}}$.

Similarly mixed Monge-Ampere measure: $\phi_i = \phi_{\mathcal{L}_i}$

$$\mathrm{M}^{\mathrm{NA}}(\phi_1, \ldots, \phi_n) = \sum_i b_i(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot E_i)\delta_{v_i}$$

Toric case

Test configuration $\longleftrightarrow$ piecewise linear concave function θ on Δ_L
$\theta(y) = \min_j\{\langle y, v_j\rangle - t_j\}$.

Legendre transform $\psi = (-\theta)^\vee$ is piecewise linear function on $\mathbb{R}^n$ such that $|\psi - \Psi|$ is bounded.

Monge-Ampère measure for $\phi = \psi - \Psi$:

$$\mathrm{MA}^{\mathrm{NA}}(\phi) = \mathrm{MA}(\psi) = \sum_j m_j \delta_{v_j}.$$

$$m_j = \left|\{y \in \Delta; \theta(y) = \langle y, v_j\rangle - t_j\}\right|.$$

E-functional

For all $\varphi \in \mathcal{H}(L)$,

$$\mathbf{E}(\varphi) = \frac{1}{n+1} \sum_k \int_X \varphi \omega_\varphi^k \wedge \omega_0^{n-k}.$$

Primitive function for the Monge-Ampère operator

$$\frac{d}{dt} E(\varphi_t) = \int_X \dot{\varphi}_t \omega_{\varphi_t}^n.$$

Monotonicity:

$$\varphi_2 \geq \varphi_1 \Longrightarrow \mathbf{E}(\varphi_2) \geq \mathbf{E}(\varphi_1).$$

" $=$ " on the RH holds if and only if $\varphi_2 = \varphi_1$.

$\mathbf{E}^{\mathrm{NA}}$-functional

Non-Archimedean functional: $\phi(v_i) = b_i^{-1}\mathrm{ord}_{E_i}(\mathcal{L} - L_{\mathbb{P}^1})$.

$$\begin{aligned}
\mathbf{E}^{\mathrm{NA}}(\phi) &= \frac{1}{n+1}\sum_k \int_{X^{\mathrm{NA}}} \phi \cdot \mathrm{MA}(\phi^{[k]}, \phi_0^{[n-k]}) \\
&= \frac{1}{n+1}\sum_k \sum_i b_i^{-1}\mathrm{ord}_{E_i}(\mathcal{L} - L_{\mathbb{C}}) b_i \mathcal{L}^k \cdot L_{\mathbb{P}}^{n-k} \cdot E_i \\
&= \frac{1}{n+1}\sum_k (\mathcal{L} - L_{\mathbb{C}}) \cdot \mathcal{L}^k \cdot L_{\mathbb{C}}^{n-k} \\
&= \frac{1}{n+1}\mathcal{L}^{n+1}.
\end{aligned}$$

If $\mathcal{L}$ is ample over $\mathcal{X}$, then

$$\frac{\mathcal{L}^{n+1}}{n+1} = \lim_{m\to+\infty} \frac{h^0(\mathcal{X}, m\mathcal{L})}{m^{n+1}/n!} = \frac{\mathrm{vol}(\mathcal{L})}{n+1}.$$

Archimedean vs Non-Archemean

- Test configuration $(\mathcal{X}, \mathcal{L}) \longrightarrow \tilde{\Phi} = \{\tilde{\varphi}(s)\}$

$$\lim_{s\to+\infty} \frac{E(\tilde{\varphi}(s))}{s} = \mathbf{E}^{\mathrm{NA}}(\phi_{(\mathcal{X},\mathcal{L})}).$$

The left-hand side is: $\mathrm{LH} = \int_{\mathcal{X}_0} \theta_v \omega^n =: b_0$.

- Weight decomposition:

$H^0(\mathcal{X}, m\mathcal{L}) = \bigoplus_\lambda V_\lambda$, $H^0(\mathcal{X}_0, m\mathcal{L}_0) = \bigoplus_\lambda V'_\lambda$
$V_\lambda = t^{-\lambda}\bar{s}$ with $s \in H^0(X, mL)$. $\dim V'_\lambda \cong \dim V_\lambda - \dim V_{\lambda+1}$:

$$0 \to H^0(\mathcal{X}, m\mathcal{L} - \mathcal{X}_0) \to H^0(\mathcal{X}, m\mathcal{L}) \to H^0(\mathcal{X}_0, m\mathcal{L}_0) \to 0.$$

$$\begin{aligned} \dim H^0(\mathcal{X}, \mathcal{L}) &= \sum_\lambda \dim V_\lambda = \sum_\lambda \lambda(\dim V_\lambda - \dim V_{\lambda+1}) \\ &= b_0 \frac{m^{n+1}}{n!} + b_1 m^n + O(m^n) \end{aligned}$$

by equivariant Riemann-Roch.

J-functional

- For any $\varphi \in \mathcal{H}$,

$$\mathbf{\Lambda}(\varphi) = \int_X \varphi\omega_0^n, \quad \mathbf{J}(\varphi) = \int_X \varphi\omega_0^n - \mathbf{E}(\varphi) \geq 0.$$

$\mathbf{J}(\varphi) = 0 \Longleftrightarrow \varphi$ is constant.

- Non-Archimedean version:

$$\mathbf{\Lambda}^{\mathrm{NA}}(\phi_{\mathcal{L}}) \;=\; \mathcal{L} \cdot p_1^* L_{\mathbb{P}^1}^n = c_1(L)^n \cdot \phi_{\mathcal{L}}(v_{\mathrm{triv}}).$$

$$\mathbf{J}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \mathbf{\Lambda}^{\mathrm{NA}}(\phi_{\mathcal{L}}) - \mathbf{E}^{\mathrm{NA}}(\phi_{\mathcal{L}}).$$

$\mathbf{E}^\chi$-functional and $\mathbf{H}$-functional

$\chi = -Ric(\omega_0)$ a closed $(1,1)$-form that represent $c_1(K_X)$.
For any $\varphi \in \mathcal{H}(L)$

$$\mathbf{E}^\chi(\varphi) = \frac{d}{d\epsilon}\mathbf{E}_{\omega_0+\epsilon\chi}(\varphi) = n\int_X \varphi\chi \wedge \omega_\varphi^k \wedge \omega_0^{n-1-k}$$

Non-Archimedean version: $p_1 : X \times \mathbb{P} \to X$

$$(\mathbf{E}^\chi)^{\mathrm{NA}}(\phi) = \mathcal{L}^n \cdot p_1^* K_X.$$

Decomposition of Mabuchi functional: $\mathbf{M} = \mathbf{E} + \mathbf{E}^\chi + \mathbf{H}$

Entropy functional:

$$\mathbf{H}(\varphi) = \int_X \log \frac{\omega_\varphi^n}{\omega_0^n}\omega_\varphi^n.$$

log discrepancy of valuations

For divisorial valuation: $E \subset Y \to X$

$$A_X(\mathrm{ord}_E) = \mathrm{ord}_E(K_{Y/X}) + 1.$$

Quasi-monomial valuations

$$A_X(v_{\{E_i,\alpha_i\}}) = \sum_i \alpha_i A_X(E_i).$$

General $v \in X^{\mathrm{NA}}$, $A_X(v) = \sup_{\mathcal{X}} A_X(p_{\mathcal{X}}(v))$.

$\mathcal{X}_0 = \sum_i b_i E_i$, $v_i = b_i^{-1} r(\mathrm{ord}_{E_i}) \in X^{div} \subset X^{\mathrm{NA}}$

$$A_X(v_i) = b_i^{-1} A_X(r(\mathrm{ord}_{E_i})) = b_i^{-1}(A_{X_{\mathbb{C}}}(E_i) - 1)$$

Entropy and $\mathbf{H}^{\mathrm{NA}}$-functional

Decomposition formula:

$$\begin{aligned}\mathbf{M}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) &= \frac{\underline{S}}{n+1}\mathcal{L}^{n+1} + K_{\mathcal{X}/\mathbb{P}}\cdot\mathcal{L}^n \\ &= \underline{S}\frac{\mathcal{L}^{n+1}}{n+1} + K_{X_{\mathbb{P}}/\mathbb{P}}\cdot\mathcal{L}^n + K_{\mathcal{X}/X_{\mathbb{P}}}\cdot\mathcal{L}^n \\ &= \underline{S}\mathbf{E}^{\mathrm{NA}} + (\mathbf{E}^{\chi})^{\mathrm{NA}} + \mathbf{H}^{\mathrm{NA}}.\end{aligned}$$

$$\begin{aligned}\mathbf{H}^{\mathrm{NA}} &= K_{\mathcal{X}/X_{\mathbb{P}}}\cdot\mathcal{L}^n = \sum_i a_{X_{\mathbb{C}}}(E_i)E_i\cdot\mathcal{L}^n \\ &= \sum_i b_i(b_i^{-1}(A_{X_{\mathbb{C}}}(E_i)-1)(\mathcal{L}^n\cdot E_i) = \sum_i A_X(v_i)\mathrm{MA}^{\mathrm{NA}}(\mathcal{L})(v_i) \\ &= \int_{X^{\mathrm{NA}}} A_X(v)\mathrm{MA}^{\mathrm{NA}}(v).\end{aligned}$$

- Hironaka's resolution of singularities: there is a birational morphism $\pi : (\mathcal{Y}, \mathcal{Y}_0) \to (\mathcal{X}, \mathcal{X}_0)$ with $(\mathcal{Y}, \mathcal{Y}_0)$ SNC.

- Fix a volume form Ω on $\mathcal{Y}$. Decomposition:

$$\begin{aligned}\mathbf{M}(\varphi) &= \int_X \log \frac{\pi^*\omega_\varphi^n \wedge \sqrt{-1}dt \wedge d\bar{t}}{\Omega}\omega_\varphi^n + \int_X \frac{\Omega}{\omega_0 \wedge \sqrt{-1}dt \wedge d\bar{t}}\omega_0^n \\ &\quad + \int_X \varphi\sqrt{-1}\partial\bar{\partial}\log\Omega \wedge \sum_{k=0}^{n-1}\omega_\varphi^k \wedge \omega_0^{n-1-k} + \underline{S}\mathbf{E}(\varphi).\end{aligned}$$

Key estimate: First two term on RH $= O(\log s)$.

Geodesic rays in $\mathcal{E}^1$

- For any $\varphi \in \mathcal{PSH}(L)$, $\mathbf{E}(\varphi) = \inf\{\mathbf{E}(\tilde{\varphi}); \tilde{\varphi} \in \mathcal{H}(L), \tilde{\varphi} \geq \varphi\}$.

$\mathcal{E}^1(X, L) = \{\varphi \in \mathcal{PSH}(L); \mathbf{E}(\varphi) > -\infty\}$.
(Berman-Boucksom-Eyssidieux-Guedj-Zeriahi)

- Darvas: metric completion of $\mathcal{H}(L)$ with respect to the Finsler metric
$d_1(\varphi_0, \varphi_1) = \inf\left\{\int_X |\dot{\varphi}_t|\omega_{\varphi_t}^n; \{\varphi_t\}_{t\in[0,1]} \in C^\infty([0,1], \mathcal{H})\right\}$.

- $\mathbf{E}$ is affine along geodesic rays

Smooth geodesic ray $\Phi = \{\varphi(s)\} : (p_1^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} = 0$

$$\sqrt{-1}\partial_s\bar{\partial}_s\mathbf{E}(\varphi(s)) = \frac{1}{n+1}\int_{(X\times\mathbb{D})/\mathbb{D}} (\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} = 0.$$

Construction of geodesic rays

Ross-Witt-Nyström's construction of geodesic rays via test curves:
$\{[\psi_\lambda]\}$: a curve of singularity types of $PSH(L)$ concave in λ
$\varphi_{[\psi_\lambda]} = \sup{}^*\{u \in PSH(\omega_0); u \leq \varphi, u \leq \psi_\lambda + O(1)\}$

$$\Phi_{[\psi]} : \varphi(s) = \sup{}^*\{\phi_{[\psi_\lambda]} + \lambda s\} \quad \text{(partial Legendre transform)}$$

Example: for test configurations $(\mathcal{X}, \mathcal{L})$, $\psi_\lambda = \psi_{\{\frac{1}{m} V_{\lambda m}\}}$.

Example: $\psi \in \mathcal{PSH}_{\mathrm{full}}$, $[\psi_\lambda] = \max\{\psi, -\lambda\} \to \Phi_{[\psi]} := \Phi_{[\psi_\lambda]}$.

Singular psh metrics on $L^{\mathrm{NA}} \to X^{\mathrm{NA}}$:

$$\mathcal{PSH}^{\mathrm{NA}} = \{\text{decreasing limits of } \{\phi_m\} \subset \mathcal{H}^{\mathrm{NA}}\}.$$

Define

$$\mathbf{E}^{\mathrm{NA}}(\phi) = \inf\{\mathbf{E}^{\mathrm{NA}}(\tilde{\phi}); \tilde{\phi} \geq \phi, \tilde{\phi} \in \mathcal{H}^{\mathrm{NA}}\} = \lim_{m \to +\infty} \mathbf{E}^{\mathrm{NA}}(\phi_m).$$

Non-Archimedean version of $\mathcal{E}^1$:

$$(\mathcal{E}^1)^{\mathrm{NA}} = \{\phi \in \mathcal{PSH}^{\mathrm{NA}}; \mathbf{E}^{\mathrm{NA}}(\phi) > -\infty\}$$

Geodesic rays and $(\mathcal{E}^1)^{\mathrm{NA}}$

- $\Phi = \{\varphi(s)\} \subset \mathcal{E}^1$ a geodesic ray

$$\Phi^{\mathrm{NA}}(v) = -G(v)(\Phi).$$

- Two geodesic rays may define the same $\Phi^{\mathrm{NA}} \in (\mathcal{E}^1)^{\mathrm{NA}}$.

Given $\phi = \Phi^{\mathrm{NA}}$, there exists a unique maximal geodesic ray $\Phi(\phi)$ such that if a geodesic ray $\Phi'^{\mathrm{NA}} = \phi$, then $\Phi' \leq \Phi$.

- Characterization of maximal geodesic ray:

$$\mathbf{E}'^{\infty}(\Phi) = \mathbf{E}^{\mathrm{NA}}(\Phi^{\mathrm{NA}}) \Longleftrightarrow \Phi \text{ is maximal.}$$

Darvas, Berman-Boucksom-Jonsson: $\Phi_{[\psi]}$ is maximal if and only if $\psi \in \mathcal{E}^1$.

Varational criterion for the existence of cscK metric

Assume that $\mathrm{Aut}(X, L)$ is discrete.

• Existence cscK metric $\Longleftrightarrow$ **M** is coercive:
There exist $\delta, C > 0$ such that for any $\varphi \in \mathcal{H}(L)$

$$\mathbf{M}(\varphi) \geq \delta \mathbf{J}(\varphi) - C.$$

Chen-Cheng; Darvas-Rubinstein, Berman-Darvas-Lu

• Uniform K-stability: there exists $\delta > 0$ such that for any test configuration $(\mathcal{X}, \mathcal{L})$,

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \delta \mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}).$$

Construction of destabilizing geodesic rays

- If $\mathbf{M}$ is not δ-coercive, then there exists φ_j such that

$$\mathbf{H}(\varphi_j) - a\mathbf{J}(\varphi_j) - b \leq \mathbf{M}(\varphi_j) \leq \delta\mathbf{J}(\varphi_j) - j, \quad \sup\varphi_j = 0$$

$$\Longrightarrow \quad \mathbf{J}(\varphi_j) \geq -C - \mathbf{E}(\varphi_j) \geq \frac{b+j}{\delta+a} \to +\infty.$$

- Connect 0 and φ_j by a geodesic segment $\Phi_j = \{\varphi_j(s)\}_{s\in[0,S_j]}$ of distance equal to $S_j = -\mathbf{E}(\varphi_j) \to +\infty$.

- Convexity of $\mathbf{M}$ along $\Phi_j \Rightarrow \mathbf{H}(\varphi_j(s)) \leq Cs$ (uniform entropy bound).

- As $j \to +\infty$, Φ_j converges to a geodesic ray $\Phi = \{\varphi(s)\} \subset \mathcal{E}^1$ satisfying:

$$\mathbf{M}'^{\infty}(\Phi) \leq 0, \quad \mathbf{J}'^{\infty}(\Phi) = 1, \quad \sup\Phi = 0.$$

Multiplier ideal approximation

- Multiplier ideal sheaf of $m\Phi$:

$\mathcal{J}_{X_{\mathbb{C}}}(m\Phi) = \{f \in \mathcal{O}_{X\times\mathbb{C}}; |f|^2 e^{-m\Phi}\text{is locally integrable}\}$.

- Nadel: $\mathcal{J}(m\Phi)$ is a coherent ideal sheaf.

- Relative Nadel vanishing: for $m > 0$ and A very ample

$R^p\pi_*\mathcal{O}(K_X + (n+1)A + \otimes\mathcal{O}(mL) \otimes \mathcal{J}(m\Phi)) = 0$.

- Global generation: $\exists m_0 > 0$ such that $\mathcal{O}((m+m_0)L) \otimes \mathcal{J}(m\Phi)$ is relatively globally generated when $m \geq m_0$.

Normalized blowup of $\mathcal{J}(m\Phi)$: $\pi_m : \mathcal{X}_m \to X_{\mathbb{C}}$ with exceptional divisor E_m. $\mathcal{L}_m = \pi_m^* L_{\mathbb{C}} - \frac{1}{m+m_0} E_m$ is semi-ample over $\mathcal{X}_m$.

- Φ is maximal $\Leftrightarrow \lim_{m\to+\infty} \Phi_{\mathcal{L}_m} = \Phi$

$$\Leftrightarrow \mathbf{E}'^{\infty}(\Phi) = \lim_{m\to+\infty} \mathbf{E}^{\mathrm{NA}}(\phi_m) = \mathbf{E}^{\mathrm{NA}}(\Phi^{\mathrm{NA}})$$

$$\Leftrightarrow \mathbf{J}'^{\infty}(\phi) = \lim_{m\to+\infty} \mathbf{J}^{\mathrm{NA}}(\phi_m).$$

Destabilizing geodesic rays are maximal

- $\Phi \to \Phi^{\mathrm{NA}} \to$ maximal $\hat{\Phi}$ with $\hat{\Phi} \geq \Phi$ and $\hat{\Phi}^{\mathrm{NA}} = \Phi^{\mathrm{NA}}$.

$$\text{for any } \alpha > 0, \quad +\infty > \int_{X\times\mathbb{D}} e^{\alpha(\hat{\Phi}-\Phi)} \omega_0^n dt \wedge d\bar{t} = \int_0^{+\infty} e^{-s} ds \int_X e^{\alpha(\hat{\varphi}(s)-\varphi(s))} \omega_0^n.$$

$$\log \int_X e^{\alpha(\hat{\varphi}(s)-\varphi(s))} \frac{\omega_0^n}{\omega_\varphi^n} \omega_\varphi^n \geq \int_X \alpha(\hat{\varphi}(s)-\varphi(s))\omega_\varphi^n - \mathbf{H}(\varphi) \geq C\alpha(\mathbf{E}(\hat{\varphi}(s)) - \mathbf{E}(\varphi(s))) - \mathbf{H}(\varphi).$$

$$\mathbf{E}'^{\infty}(\hat{\Phi}) - \mathbf{E}'^{\infty}(\Phi) \leq \frac{1}{C\alpha}(\mathbf{H}'^{\infty}(\Phi) + C) \stackrel{\alpha\to+\infty}{\longrightarrow} 0.$$

$\Longrightarrow \mathbf{E}(\hat{\varphi}(s)) \equiv \mathbf{E}(\varphi(s)) \Longrightarrow \Phi = \hat{\Phi}$ is maximal.

D-functional

- X: Fano manifold, $L = -K_X$ is ample.

$$\mathbf{D}(\varphi) = -\mathbf{E}(\varphi) + \mathbf{L}(\varphi), \quad \mathbf{L}(\varphi) = -\log\left(\int_X e^{-\varphi}\Omega_0\right)$$

Euler-Lagrange equation for **D**:

$$\omega_\varphi^n = \frac{e^{-\varphi}\Omega_0}{\int_X e^{-\varphi}\Omega_0} \Longleftrightarrow Ric(\omega_\varphi) = \omega_\varphi$$

$\exists$ a Kähler-Einstein metric $\Leftrightarrow$ **D** is coercive $\Leftrightarrow$ **M** is coercive.

- (Uniform) K-stability $\Longleftrightarrow$ (uniform) Ding-stability

Valuative criterion for integrability

- Let $\mathfrak{a}$ be an ideal sheaf on X, let $\mu : Y \to X$ be a resolution of $\mathfrak{a}$: $\mu^*\mathfrak{a} = \mathcal{O}_Y(-\sum_i b_i E_i)$ with $\sum_i E_i$ a SNC divisor.

For $c > 0$, $f \in \mathcal{J}(\mathfrak{a}^c) := \mu_*(\mathcal{O}(K_{Y/X} - [c\sum_i b_i E_i]))$
$\Leftrightarrow \exists \epsilon > 0$ such that $A(E_i) + \mathrm{ord}_{E_i}(f) - (1+\epsilon)cb_i > 0, \forall i$
$\Leftrightarrow A(v) + v(f) - (1+\epsilon)cv(\mathfrak{a}) > 0$ for any $v \in X^{div}$.

- $f \in \mathcal{J}(m\Phi) \Longleftrightarrow$ there exists $\epsilon > 0$ such that $A_{X_\mathbb{C}}(w) + w(f) - (1+\epsilon)w(\Phi) > 0$ for any $w \in X_\mathbb{C}^{div}$.

- Φ_m geodesic ray associated to $(\mathcal{X}_m, \mathcal{L}_m)$. Comparison:

$$G(v)(\mathcal{J}(m\Phi)) \leq G(v)(m\Phi) \leq G(v)(\mathcal{J}(m\Phi)) + A(v).$$

$$\Longrightarrow \qquad \phi_m(v) \geq \phi(v) \geq \phi_m(v) - \frac{A(v)}{m}.$$

Slope of $\mathbf{D}$ and $\mathbf{D}^{\mathrm{NA}}$ functional

- Berndtsson: $\mathbf{L}$ is convex in s along the geodesic ray $\Phi = \{\varphi(s)\}$.

$$\mathbf{L}'^{\infty}(\Phi) = \sup\left\{c : e^{-(c+1)\log|t|^2}\mathbf{L}(\varphi(s))dt \wedge d\bar{t} \text{ is locally integrable}\right\}.$$

$$\text{Fubini: } \int_{\mathbb{D}} e^{-(c+1)\log|t|^2}\mathbf{L}(\varphi(s)) = \int_{X\times\mathbb{D}} e^{-\Phi-(c+1)\log|t|^2} dt \wedge d\bar{t}.$$

- Valuative characterization of $\mathcal{J}$:

$$\begin{aligned} & 1 \in \mathcal{J}(\Phi + (c+1)\log|t|^2) \\ \Longleftrightarrow\ & A_{X_{\mathbb{C}}}(G(v)) - (1+\epsilon)(G(v)(\Phi) + c + 1) > 0 \ \ \forall v \in X^{div} \\ \Longleftrightarrow\ & c < A_{X_{\mathbb{C}}}(G(v)) - 1 - G(v)(\Phi) = A_X(v) - G(v)(\Phi). \end{aligned}$$

So $\mathbf{L}'^{\infty}(\Phi) = \mathbf{L}^{\mathrm{NA}}(\Phi^{\mathrm{NA}}) = \inf_v(A_X(v) + \Phi^{\mathrm{NA}}(v))$.

Approximation of $\mathbf{D}^{\mathrm{NA}}$

$\mathbf{D}'^{\infty}(\Phi) = -\mathbf{E}'^{\infty}(\Phi) + \mathbf{L}'^{\infty}(\Phi) \leq 0$ is equivalent to

$$\mathbf{L}'^{\infty}(\Phi) = \inf_{v}(A_X(v) - G(v)(\Phi)) \leq \mathbf{E}'^{\infty}(\Phi) = -1$$

- There exists v_1 s.t. $A_X(v_1) - G(v_1)(\Phi) < -1 + \epsilon$.

For $m \gg 1$, $A_X(v_1) - G(v_1)(\Phi_m) < -1 + \epsilon$

$$\Longrightarrow \mathbf{L}^{\mathrm{NA}}(\phi) < -1 + \epsilon.$$

- Recall that maximality of Φ implies

$$\mathbf{J}^{\mathrm{NA}}(\mathcal{L}_m) = -\mathbf{E}^{\mathrm{NA}}(\mathcal{L}_m) \longrightarrow \mathbf{J}'^{\infty}(\Phi) = -\mathbf{E}'^{\infty}(\Phi) = 1.$$

$\Longrightarrow$ when $m \gg 1$, $\mathbf{D}'^{\infty}(\Phi_m) \leq \epsilon$ and $\mathbf{J}'^{\infty}(\Phi_m) > 1 - \epsilon$.

$\Longrightarrow$ contradicts with uniform stability with slope when $0 < \epsilon \ll \delta$.

Continuous automorphism groups

- $\mathbb{T} \cong (\mathbb{C}^*)^r$: a maximal torus in $\mathrm{Aut}(X, L)^\circ$. Reduced coercive:

$$\mathbf{M}(\varphi) \geq \delta \mathbf{J}_{\mathbb{T}}(\varphi) - C.$$

$$\mathbf{J}_{\mathbb{T}}(\varphi) = \inf_{\sigma \in \mathbb{T}} \mathbf{J}(\sigma^* \omega_\varphi), \quad \mathbf{J}(\varphi) = \mathbf{J}(\omega_\varphi) = \int_X \varphi \omega_\varphi^n - \mathbf{E}(\varphi).$$

- Non-Archimedean twisting: $N_{\mathbb{Q}} = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

$\xi \in N_{\mathbb{Q}} \to$ product test configuration $(\mathcal{X}_\xi, \mathcal{L}_\xi) \to \theta_\xi = \phi_{\mathcal{L}_\xi}$.

$$\phi_\xi(v) = \phi(v_\xi) + \theta_\xi(v), \quad \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\phi) = \inf_{\xi \in N_{\mathbb{Q}}} \mathbf{J}^{\mathrm{NA}}(\phi_\xi).$$

$$\mathbf{M}^{\mathrm{NA}}(\phi_\xi) = \mathbf{M}^{\mathrm{NA}}(\phi) + \mathrm{Fut}(\xi), \quad \mathbf{D}^{\mathrm{NA}}(\phi_\xi) = \mathbf{D}^{\mathrm{NA}}(\phi) + \mathrm{Fut}(\xi).$$

Reduced uniform K-stability:

$$\mathbf{M}^{\mathrm{NA}}(\phi) \geq \delta \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\phi) \qquad (\Longrightarrow \mathrm{Fut} \equiv 0).$$

YTD with automorphism

- Not reduced coercive $\Leftrightarrow$ maximal geodesic ray Φ s.t.

$$\mathbf{D}^{\mathrm{NA}}(\phi) \leq 0, \quad \mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\phi) = 1, \quad \phi(v_{\mathrm{triv}}) = 0.$$

Then

$$\mathbf{L}'^{\infty}(\Phi) = \inf_{v}(A_X(v) + \phi(v)) \leq \mathbf{E}'^{\infty}(\Phi) = -1.$$

$\Rightarrow \mathbf{L}^{\mathrm{NA}}(\phi_m) < -1 + \epsilon.$

- $\mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\phi_m) \to \mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\phi) = 1.$

Key point: there exists $C > 0$ s.t.
$\mathbf{J}_{\mathbb{T}}(\phi_m) = \inf_{|\xi| \leq C} \mathbf{J}^{\mathrm{NA}}(\phi_{m,\xi}), \quad \mathbf{J}_{\mathbb{T}}(\phi) = \inf_{|\xi| \leq C} \mathbf{J}^{\mathrm{NA}}(\phi_{\xi})$

- Destabilizing geodesic ray $\Phi \longrightarrow$ Multiplier ideal approximation $\phi_m = \phi_{(\mathcal{X}_m, \mathcal{L}_m)}$.

Maximality of Φ: $\mathbf{J}^{\mathrm{NA}}(\phi_m) \to \mathbf{J}^{\mathrm{NA}}(\Phi^{\mathrm{NA}}) = \mathbf{J}'^{\infty}(\Phi)$.

- Problem: we do NOT know:

$$\mathbf{M}^{\mathrm{NA}}(\phi_m) \to \mathbf{M}^{\mathrm{NA}}(\Phi^{\mathrm{NA}}) = \mathbf{M}'^{\infty}(\Phi).$$

- Partial solutions:

(i) $\mathbf{M}'^{\infty}(\Phi) \geq \mathbf{M}^{\mathrm{NA}}(\Phi^{\mathrm{NA}})$.

(ii) There exists models $(\mathcal{X}_m, \mathcal{L}_m)$ such that

$$\mathbf{M}^{\mathrm{NA}}(\phi_m) \to \mathbf{M}^{\mathrm{NA}}(\Phi^{\mathrm{NA}}), \quad \mathbf{J}^{\mathrm{NA}}(\phi_m) \to \mathbf{J}^{\mathrm{NA}}(\Phi^{\mathrm{NA}}).$$

Models and envelopes

- Test configurations weaken to models: $\mathcal{L} \to \mathcal{X}$ not assumed semi-ample. Then $\mathcal{L} \to \mathcal{X}$ is (relatively) big.
- $\mu_m : \mathcal{X}_m \to \mathcal{X}$ blow of $\mathfrak{b}(m\mathcal{L})$ with exceptional divisor E_m,
- Set $\mathcal{L}_m = \mu_m^* L_{\mathbb{C}} - \frac{1}{m} E_m$

$$\phi_{\mathcal{L}} = \lim_{m \to +\infty} \phi_{\mathcal{L}_{m!}} \in (\mathcal{E}^1)^{\mathrm{NA}} \subset \mathcal{PSH}^{\mathrm{NA}}.$$

- $\mathcal{L} = p_1^* L + D$ defines $\to$ FS metric: $f_{\mathcal{L}}(v) = G(v)(D)$.

$D = \sum_i c_i E_i$, $\mathcal{X}_0 = \sum_i b_i E_i$, $v_i = b_i^{-1} r(\mathrm{ord}_{E_i})$.

$$\begin{aligned} \phi_{\mathcal{L}} &= P(f_{\mathcal{L}}) = \sup{}^* \{\tilde{\phi} \in \mathcal{PSH}^{\mathrm{NA}}, \tilde{\phi} \le f_{\mathcal{L}}\} \\ &= \sup{}^* \left\{ \tilde{\phi} \in \mathcal{PSH}^{\mathrm{NA}}, \tilde{\phi}(v_i) \le \frac{c_i}{b_i}, \forall i \right\}. \end{aligned}$$

- Toric case: convex envelop of continuous functions

Volumes of big line bundles

- Volume of big line bundles:

$$\mathrm{vol}(\mathcal{L}) = \lim_{m\to+\infty} \frac{h^0(m\mathcal{L})}{m^{n+1}/(n+1)!} = \lim_{m\to+\infty} \mathcal{L}_m^{n+1}.$$

- vol functional is differentiable in the cone of big classes.

$$\langle \mathcal{L}^n \rangle \cdot D = \frac{1}{n+1} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathrm{vol}(\mathcal{L}+\epsilon D).$$

- $\langle \mathcal{L}^n \rangle \cdot D = \lim_{m\to+\infty} \mathcal{L}_m^n \cdot \tilde{D}$ where $\tilde{D}$ is the strict transform under μ_m.

- $\mathrm{vol}(\mathcal{L}) = \langle \mathcal{L}^n \rangle \cdot \mathcal{L}$.

Monge-Ampère measures for models

- For any model $(\mathcal{X}, \mathcal{L})$ with $\mathcal{X}_0 = \sum_i b_i E_i$,

$$\mathrm{MA}(\phi_{\mathcal{L}}) = \sum_i b_i \langle \mathcal{L}^n \rangle \cdot E_i \; \delta_{v_i}.$$

- Boucksom-Favre-Jonsson: conversely for $\sigma = \sum_i m_i \delta_{v_i}$ with $\sum_i m_i = c_1(L)^n$, then there exists a model $(\mathcal{X}, \mathcal{L})$ such that

$$\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \sigma.$$

which satisfies

$$\phi = \sup{}^* \left\{ \tilde{\phi} \in \mathcal{PSH}^{\mathrm{NA}}; \tilde{\phi}(v_i) \leq \phi(v_i) \right\}.$$

Functionals for models

The functionals for test configurations generalized for models:

$$\mathbf{E}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \frac{1}{n+1}\langle \mathcal{L}^{n+1}\rangle = \frac{1}{n+1}\mathrm{vol}(\mathcal{L}).$$

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{\underline{S}}{n+1}\langle \mathcal{L}^{n+1}\rangle + \langle \mathcal{L}^{n}\rangle \cdot K_{\mathcal{X}/\mathbb{P}^1}.$$

$$\mathbf{J}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \phi_{\mathcal{L}}(v_{\mathrm{triv}}) - \mathbf{E}^{\mathrm{NA}}(\phi_{\mathcal{L}}).$$

$$\mathbf{H}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \int_{X^{\mathrm{NA}}} A_X(v)\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}}).$$

An existence result

- Existence of cscK if (X, L) uniformly K-stable for models:

$$\mathbf{M}^{\mathrm{NA}}(\phi_{\mathcal{L}}) \geq \delta \mathbf{J}^{\mathrm{NA}}(\phi_{\mathcal{L}}).$$

- $\mathbf{M}'^{\infty}(\Phi) \geq \mathbf{M}^{\mathrm{NA}}(\phi) = \mathbf{H}^{\mathrm{NA}} + (\mathbf{E}^{\chi})^{\mathrm{NA}} + \underline{S}\mathbf{E}^{\mathrm{NA}}.$
- there exists measures $\sigma_k = \sum_i m_{k,i}\delta_{v_i}$ such that

$$\sigma_k \to \mathrm{MA}(\phi), \quad \int_{X^{\mathrm{NA}}} A_X(v)\sigma_k \to \int_{X^{\mathrm{NA}}} A_X(v)\mathrm{MA}(\phi).$$

Set $\phi_k = \mathrm{MA}^{-1}(\mathfrak{m}_k)$. Then $\phi_k = \phi_{\mathcal{L}_k}$ for model $(\mathcal{X}_k, \mathcal{L}_k)$ and

$$\mathbf{J}^{\mathrm{NA}}(\phi_k) \to \mathbf{J}^{\mathrm{NA}}(\phi), \quad \mathbf{M}^{\mathrm{NA}}(\phi_k) \to \mathbf{M}^{\mathrm{NA}}(\phi).$$

Regularization conjecture

- Conjecture: there exist test configurations $(\mathcal{X}_m, \mathcal{L}_m)$ s.t.

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{L}_m) \to \mathbf{M}^{\mathrm{NA}}(\phi_{\mathcal{L}}), \quad \mathbf{J}^{\mathrm{NA}}(\mathcal{L}_m) \to \mathbf{J}^{\mathrm{NA}}(\phi_{\mathcal{L}}).$$

- For natural candidates such as $\mathcal{L}_m = \pi_m^* - \frac{1}{m} E_m$, we know $\mathbf{F}^{\mathrm{NA}}(\mathcal{L}_m) \to \mathbf{F}^{\mathrm{NA}}(\phi_{\mathcal{L}})$ for $\mathbf{F} \in \{\mathbf{E}, \mathbf{E}^{\chi}, \mathbf{J}\}$ and

$$\liminf_{m \to +\infty} \mathbf{H}^{\mathrm{NA}}(\mathcal{L}_m) \geq \mathbf{H}^{\mathrm{NA}}(\phi_{\mathcal{L}}).$$

Divisorial stability

- Weighted divisorial data $\mathfrak{v} = \{(v_i, t_i \leq 0)\}_i$.

The envelope: $\phi_{\mathfrak{v}} = \sup{}^*\{\phi; \phi(v_i) \leq t_i \ \forall i\}$ satisfies

$$\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathfrak{v}}) = \sum_i m_i \delta_{v_i}, \quad \mathbf{H}^{\mathrm{NA}}(\phi_{\mathfrak{v}}) = \sum_i m_i A(v_i).$$

- Choose a birational morphism $\mu : Y \to X$ s.t. $v_i = q_i \cdot \mathrm{ord}_{F_i}$ for a prime divisor $F_i \subset Y$.

$$\mathbf{E}_L^{\mathrm{NA}}(\phi_{\mathfrak{v}}) = \int_0^{+\infty} -s \cdot d\mathrm{vol}_X(\mu^* L - \sum_i \max\{(t_i + s), 0\} q_i^{-1} E_i).$$

$$(\mathbf{E}^{\chi})^{\mathrm{NA}}(\phi_{\mathfrak{v}}) = \frac{d}{d\epsilon} \mathbf{E}^{\mathrm{NA}}_{L+\epsilon[\chi]}(\phi_{\mathfrak{v}}).$$

Dirac measure

$\mathfrak{v} = \{\delta_v\}$, $v = \mathrm{ord}_E$. Green function:

$$\phi_v = \sup{}^*\{\phi; \phi(v) \leq 0\}.$$

$$\begin{aligned}
\mathbf{H}^{\mathrm{NA}}(\phi_v) &= A_X(v) \\
\mathbf{E}_L^{\mathrm{NA}}(\phi_v) &= \int_0^{+\infty} -s \cdot d\mathrm{vol}_X(\mu^* L - sE) \\
&= \int_0^{+\infty} \mathrm{vol}_X(\mu^* L - sE) ds \\
(\mathbf{E}^{\chi})^{\mathrm{NA}}(\phi_v) &= n \int_0^{+\infty} \langle (L - sE)^{n-1} \rangle \cdot K_X ds.
\end{aligned}$$

Fano case: $L = -K_X$

(Uniform) K-stability $\Longleftrightarrow$ (Uniform K-stability for special test configurations (Li-Xu, Fujita)

Special test configuration $\longrightarrow \mathcal{X}_0 = E$ is an irreducible Fano variety $\longrightarrow v = r(\mathrm{ord}_E) = q \cdot \mathrm{ord}_F$

$$
\begin{aligned}
& (\mathbf{E}^{\mathcal{X}})^{\mathrm{NA}}(\phi) + \underline{S} \cdot \mathbf{E}^{\mathrm{NA}}(\phi) \\
= \;& n \int_0^{+\infty} \langle (L - sF)^{n-1} \rangle \cdot K_X ds + n \int_0^{+\infty} \langle (L - sF)^n \rangle ds \\
= \;& n \int_0^{+\infty} \langle (L - sF)^{n-1} \rangle \cdot (-sF) ds = \int_0^{+\infty} s \frac{d}{ds} \mathrm{vol}(L - sF) ds \\
= \;& -\int_0^{+\infty} \mathrm{vol}(L - sF) ds.
\end{aligned}
$$

Valuative criterion

- For any $v = \mathrm{ord}_F \in X^{div}$, set

$$\beta(v) := \mathbf{M}^{\mathrm{NA}}(\phi_v) = A_X(F) - S_{-K_X}(F).$$

$$S_{-K_X}(v) = \frac{1}{\mathbf{V}} \int_0^{+\infty} \mathrm{vol}_X(L - sF) ds.$$

K-stability $\Leftrightarrow \beta(v) > 0$ for any $v \in X^{div}$.
Uniform-K-stability $\Longleftrightarrow \delta(X) := \inf_{v \in X^{div}} \frac{A_X(v)}{S_{-K_X}(v)} > 1$.

- Uniform K-stability $\Longleftrightarrow$ K-stability (Liu-Xu-Zhuang)

Thanks for your attention!