Kähler-Einstein metrics and K-stability

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Riemann surface: surface with a complex structure

Classification of closed Riemann surfaces :

Topology	Metric	Curvature
$\mathbb{S}^2 = \mathbb{CP}^1$	spherical	1
$\mathbb{T}^2 = \mathbb{C}/\Lambda$	flat	0
$\Sigma_g = \mathbb{B}^1/\pi_1(\Sigma_g)$	hyperbolic	-1

Notation: Σ_g closed oriented surface of genus $g \geq 2$.
$\mathbb{B}^1 = \{z \in \mathbb{C}; |z| < 1\}$.

Generalization for higher dimensional complex manifolds?

We will restrict to the class of Kähler manifolds, in particular projective manifolds.

Kähler manifolds and Kähler metrics

X: complex manifold (transition functions are holomorphic);
J: $TX \to TX$ complex structure;
g: Riemannian metric s.t. $g(J\cdot, J\cdot) = g(\cdot,\cdot)$.

Kähler form: $\omega = g(\cdot, J\cdot)$. Using holomorphic coordinates $\{z^i\}$:

$$\omega = \sqrt{-1}\sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad (g_{i\bar{j}}) > 0.$$

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Kähler condition: $d\omega = 0$. Consequences:

- ω determines the Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$.
- Locally, $\omega = \sqrt{-1}\partial\bar{\partial}\psi = \sqrt{-1}\sum_{i,j} \frac{\partial^2\psi}{\partial z^i \partial \bar{z}^j} dz^i \wedge d\bar{z}^j$.

Basic examples and curvature

Notation: $\mathbb{B}^n = \{z \in \mathbb{C}^n; |z| < 1\}$.

$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^* = \mathbb{C}^n \cup \mathbb{P}^{n-1}$.

$\mathbb{B}^n$	$\omega_{\mathbb{B}^n} = -\sqrt{-1}\partial\bar{\partial}\log(1-\lvert z\rvert^2)$	$\mathbb{B}^n/\Gamma;\ \Gamma < \mathrm{PSU}(n,1)$
$\mathbb{C}^n$	$\omega_{\mathbb{C}^n} = \sqrt{-1}\partial\bar{\partial}\lvert z\rvert^2$	$\mathbb{C}^n/\Lambda;\ \Lambda \cong \mathbb{Z}^{2n}$
$\mathbb{P}^n$	$\omega_{\mathrm{FS}} = \sqrt{-1}\partial\bar{\partial}\log(1+\lvert z\rvert^2)$	$\mathbb{P}^n$

Kähler manifolds with constant holomorphic sectional curvatures:

$$R_{i\bar{j}k\bar{l}} = \mu(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}),\ \mu = -1, 0, 1.$$

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Curvature tensor: $R_{i\bar{j}k\bar{l}} = -\dfrac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{r\bar{q}}\dfrac{\partial g_{k\bar{q}}}{\partial z_i}\dfrac{\partial g_{r\bar{l}}}{\partial \bar{z}_j}.$

Ricci curvature

Ricci curvature: $$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{l}k\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}.$$

Compact expression: $$R_{i\bar{j}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log\det(g_{k\bar{l}}).$$

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Ricci form is a (1,1)-form:

$$Ric(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{i\bar{j}} dz^i \wedge d\bar{z}^j =: -\sqrt{-1}\partial\bar{\partial} \log \omega^n$$

$Ric(\omega)$ represents the *first Chern class* of the complex manifold:

$$Ric(\omega) \in 2\pi c_1(X) \in H^{1,1}(X, \mathbb{Z}).$$

Kähler potentials

Lemma ($\partial\bar{\partial}$-Lemma)

Smooth $\omega_i \in [\omega], i = 1,2 \Rightarrow \omega_2 = \omega_1 + \sqrt{-1}\partial\bar{\partial}\varphi$ with $\varphi \in C^\infty(X)$.

$$\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi = \sqrt{-1}\sum_{i,j}\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) dz^i \wedge d\bar{z}^j .$$

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$$\begin{aligned} \omega_\varphi \text{ positive definite} \quad &\longleftrightarrow \quad \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \\ &\longleftrightarrow \quad (g_{i\bar{j}} + \varphi_{i\bar{j}}) > 0. \end{aligned}$$

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Compare to conformal case: $g_2 = e^f g_1$ for $f \in C^\infty(X)$.

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$$\iff \det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) = e^{h_\omega - \mu\varphi}\det(g_{i\bar{j}}).$$

$$\left(h_\omega \text{ satisfies: } Ric(\omega) - \mu\omega = \sqrt{-1}\partial\bar{\partial}h_\omega, \text{ and } \int_X e^{h_\omega}\omega^n = \int_X \omega^n.\right)$$

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$\mu = -1$	Solvable (Aubin, Yau)	$c_1(X) < 0$
$\mu = 0$	Solvable (Yau)	$c_1(X) = 0$
$\mu = 1$	in general not solvable	$c_1(X) > 0$

Compare: Yamabe invariant in a given conformal class.

U plurisubharmonic i.e. $(U_{i\bar{j}}) \geq 0$; $B_1 = \{z \in \mathbb{C}^n ; |z| \leq 1\}$.

$$\det(U_{i\bar{j}}) = \frac{e^{-tU}}{\int_{B_1} e^{-tU} dV_{\mathbb{C}^n}} \text{ on } B_1, \quad U = 0 \text{ on } \partial B_1. \qquad (1)$$

Solution U would produce Kähler-Einstein metrics (if $(U_{i\bar{j}}) > 0$):

$$\omega = \sqrt{-1} \sum_{i,j} U_{i\bar{j}} dz^i \wedge d\bar{z}^j \quad \overset{(1)}{\Longrightarrow} \quad Ric(\omega) = t\,\omega.$$

Theorem (L. '13)

$\exists t^* = t^*(n) > 0$ *s.t.* (1) *has a regular nonpositive solution iff* $t < t^*$.

Rotationally symmetric solutions

Reduction to ODE $\rightsquigarrow$ Solutions:

$$U(t) = \frac{(n+1)}{t} \log \left[1 + \frac{(n!)^{1/n} t}{(n+1)\pi} (|z|^2 - 1) \right]$$

Range of t: $\quad -\infty < t < \dfrac{(n+1)\pi}{(n!)^{1/n}} =: t^*.$

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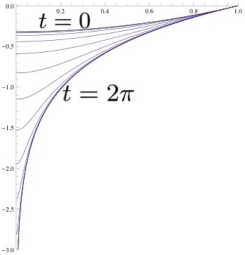
- $t^+ > 0$: $\omega = \dfrac{(n+1)}{t^+} \epsilon^* \omega_{\mathrm{FS}}$. $\quad \epsilon : z \mapsto \epsilon \cdot z, 0 < \epsilon < +\infty.$

  $t^+ = t^+(\epsilon) = \frac{(n+1)\pi}{(n!)^{1/n}(1+\epsilon^{-2})} \in (0, t^*).$

Deformation with positive curvature



(a) Potential

(b) Geometry

Curvature	Potential	Geometry
0	$\frac{1}{\pi}(\lvert z\rvert^2 - 1)$	flat disk
t	$\frac{2}{t}\log\left[1 + \frac{t}{2\pi}(\lvert z\rvert^2 - 1)\right]$	spherical cap
2π	$\frac{1}{\pi}\log\lvert z\rvert^2$	sphere

A nonexistence result

Theorem (L. '13)

$\nexists$ nonpositive solution to (1) in $C^2(\overline{B_1}) \cap C^4(B_1)$ when $t \geq t^*$.

Proof: • Pohožaev identity (compare (3)):

$$2n(n+1)\int_{B_1} \frac{e^{-tu}-1}{\int_{B_1} e^{-tu}dV} = t\int_{\partial B_1} 2^{-(n+1)}|\nabla u|^{n+1}d\sigma \qquad (2)$$

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Conjecture (Berman-Berndtsson)

All solutions to (1) are a priori radially symmetric.

A priori radially symmetric property holds for *real* Monge-Ampère equations (Gidas-Ni-Nirenberg, Delanoë).

Fano manifolds

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   1. $\dim_{\mathbb{C}} = 2$: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2 \sharp k \overline{\mathbb{P}^2}, 1 \leq k \leq 3$;
   2. $\dim_{\mathbb{C}} = 3$: 18 toric Fano threefolds.

First obstruction: KE $\Longrightarrow \mathrm{Aut}(X)$ is reductive (Matsushima).
Example: rule out $\mathbb{P}^2$ blown-up one or two points:

- $\mathrm{Aut}(\mathbb{P}^2 \sharp \overline{\mathbb{P}^2}) \cong \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{PGL}(3, \mathbb{C}) \right\}.$
- $\mathrm{Aut}(\mathbb{P}^2 \sharp 2\overline{\mathbb{P}^2}) \cong \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{PGL}(3, \mathbb{C}) \right\}.$

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In $\dim_{\mathbb{C}} X = 2$, this is the only obstruction (Tian '90).

In higher dimensions, there are other obstructions, using

Futaki invariant, energy functionals and K-stability .

Choose a closed positive (1,1)-current $\eta \in 2\pi c_1(X)$.
Twisted KE metrics:

$$Ric(\omega_{\varphi_t}) = t\omega_{\varphi_t} + (1-t)\eta$$

$$\Updownarrow \qquad\qquad (*)_t$$

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{H_{\omega,(1-t)\eta} - t\varphi_t}\omega^n$$

Twisted KE metrics and continuity methods

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- Determine set $\mathcal{S} = \{t; (*)_t \text{ can be solved }\}$.
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Two basic twistings:

♣ (Aubin) $\eta = \omega$.

♠ (Donaldson) $\eta = \{D\} \longleftrightarrow$ conical KE
$D \sim_{\mathbb{Q}} -K_X$: smooth $\mathrm{codim}_{\mathbb{C}} = 1$ complex submanifold.

$$Ric(\omega_{\varphi_t}) = t\omega_{\varphi_t} + (1-t)\omega \quad (> t\omega_{\varphi_t})$$
$$\Updownarrow \qquad\qquad (**)_t$$
$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{h_\omega - t\varphi_t}\omega^n$$

Theorem (Tian)

$(**)_t$ is solvable for $0 < t \ll 1$; $\exists$ obstructions when t is near 1.

Define $R(X) = \sup\{t; (**)_t \text{ is solvable}\}$. It is independent of ω:

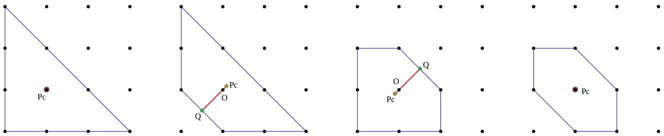
Theorem (Székelyhidi)

$R(X) = \sup\{t; \exists\, \eta \in 2\pi c_1(X)$ *s.t.* $Ric(\eta) > t\eta\}$ *(Greatest Ricci).*

Question: How to determine $R(X)$? What happens as $t \to R(X)$?

Aubin's continuity method on toric Fano manifolds

Toric manifolds: $(\mathbb{C}^*)^n$ action with dense orbits; determined by lattice polytopes. Fano $\leftrightarrow$ reflexive polytope $\rightsquigarrow O \in \triangle$.



(c) $\mathbb{P}^2$ (d) $Bl_p\mathbb{P}^2$ (e) $\mathbb{P}^2\sharp 2\overline{\mathbb{P}^2}$ (f) $\mathbb{P}^2\sharp 3\overline{\mathbb{P}^2}$

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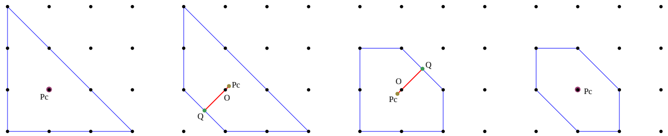

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Theorem (L. '09)

If $P_c \neq O$, then $R(X_\triangle) = \left|\overline{OQ}\right| / \left|\overline{P_cQ}\right|$, where $Q = \overrightarrow{P_cO} \cap \partial\triangle$.

Example: $R(\mathbb{P}^2\sharp\overline{\mathbb{P}^2}) = 6/7$ (Székelyhidi); $R(\mathbb{P}^2\sharp 2\overline{\mathbb{P}^2}) = 21/25$.

Proof

Revisit of the proof of Wang-Zhu's result: $P_c = O \Rightarrow \exists KE$.

Torus symmetry: $X_\triangle \setminus D \cong \mathbb{C}^n = \mathbb{R}^n \times (S^1)^n$ reduces $(**)_t$ to

Real Monge-Ampère: $\quad \det(u_{ij}) = e^{-(1-t)\tilde{u}-tu}$ on $\mathbb{R}^n$.

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Key Relation: (Compare with Pohožaev identity (2))

$$\begin{array}{ccc} \frac{1}{\mathrm{Vol}(\triangle)} \int_{\mathbb{R}^n} (D\tilde{u}) e^{-(1-t)\tilde{u}-tu} dx & = & -\frac{t}{1-t} P_c \\ \downarrow & & \downarrow \\ Q & = & -\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c. \end{array} \tag{3}$$

Theorem (L. '10)

As $t \to R(X_\triangle)$,

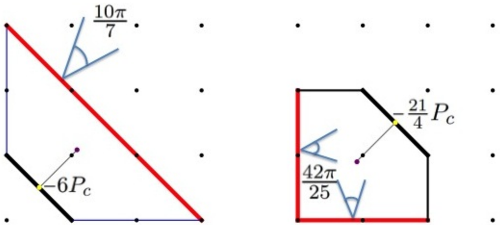
1. $\exists \sigma_{t_i} \in (\mathbb{C}^*)^n$ *s.t.* $\sigma_{t_i}^* \omega_{t_i} \to \omega_\infty = \omega + \sqrt{-1}\partial\bar{\partial}\psi_\infty$;
2. $\psi_\infty \in L^\infty(X_\triangle) \cap C^\infty(X_\triangle \backslash Bs(\mathfrak{L}_{\mathcal{F}}))$;
3. ω_∞ *satisfies a twisted KE equation* $(0 < b_\alpha < 1)$:

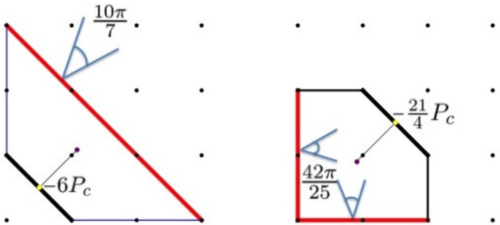
$$Ric(\omega_\infty) = R(X_\triangle)\omega_\infty + (1 - R(X_\triangle))\sqrt{-1}\partial\bar{\partial}\log\Big(\sum_{p_\alpha^{\mathcal{F}}} b_\alpha |s_\alpha|^2\Big).$$

Notations:

- $\mathcal{F}$: *the minimal face containing* Q;
- $\{p_\alpha^{\mathcal{F}}\}$: *vertex lattice points of* $\mathcal{F}$;
- $\mathfrak{L}_{\mathcal{F}}$: *the sub-linear system of* $|-K_{X_\triangle}^{-1}|$ *determined by* $\{p_\alpha^{\mathcal{F}}\}$
- $Bs(\mathfrak{L}_{\mathcal{F}})$: *base locus of* $\mathfrak{L}_{\mathcal{F}}$.

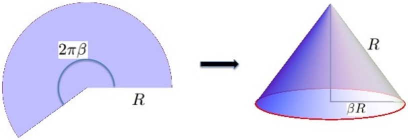
Example and Proof





1. Construct σ_t using Wang-Zhu's C^0-estimate;
2. Apply σ_t to get regularized equation and the limit equation;
3. C^0-estimate for regularized equation, Harnack inequality;
4. Partial C^2-estimate (Chern-Lu's inequality);
5. Partial $C^{2,\alpha}$-estimate (Evans-Krylov's estimate).

Conical Kähler metrics: complex 1-dimensional case



$$\frac{|dz|^2}{|z|^{2(1-\beta)}} \xlongequal{r=\frac{|z|^\beta}{\beta}} dr^2 + \beta^2 r^2 d\theta^2$$

Kähler form: $\hat{\omega} = \sqrt{-1}\dfrac{dz \wedge d\bar{z}}{|z|^{2(1-\beta)}} = \sqrt{-1}\partial\bar{\partial}\left(\beta^{-2}|z|^{2\beta}\right)$.

Higher dimensional conical Kähler metrics

D: a smooth complex submanifold $\mathrm{codim}_{\mathbb{C}} D = 1$ (a smooth divisor).

Locally $D = \{z^1 = 0\} \rightsquigarrow$ local conical model metric:

$$\hat{\omega} \quad = \quad \sqrt{-1}\left(\frac{dz^1 \wedge d\bar{z}^1}{|z^1|^{2(1-\beta)}} + \sum_{i=2}^{n} dz^i \wedge d\bar{z}^i\right).$$

$$Ric(\hat{\omega}) = \sqrt{-1}\partial\bar{\partial}\log|z^1|^{2(1-\beta)} = (1-\beta)2\pi\delta_{\{z^1=0\}}dx^1 \wedge dy^1.$$

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Definition (Conical KE on $(X, (1-\beta)D)$ with cone angle $2\pi\beta$)

$$Ric(\omega) = \mu\,\omega + (1-\beta)2\pi\{D\}.$$

Cone angle$= 2\pi\beta$.

Fix a smooth divisor $D \sim -\lambda K_X$ with $0 < \lambda \in \mathbb{Q}$.

$$\begin{aligned} Ric(\omega_\varphi) &= t(\beta)\omega_\varphi + (1-\beta)2\pi\{D\} \\ &\Updownarrow \\ (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n &= e^{h_\omega - t(\beta)\varphi}\frac{\omega^n}{\|s\|^{2(1-\beta)}} \end{aligned} \qquad (**)_\beta$$

Conical continuity method

Fix a smooth divisor $D \sim -\lambda K_X$ with $0 < \lambda \in \mathbb{Q}$.

$$
\begin{array}{c}
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\Updownarrow \\
(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_\omega - t(\beta)\varphi} \frac{\omega^n}{\|s\|^{2(1-\beta)}} \\
\Updownarrow \\
(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{H_{\hat{\omega},(1-\beta)D} - t(\beta)\phi} \hat{\omega}^n.
\end{array}
\qquad (**)_\beta
$$

Notations:

- $t(\beta) = 1 - (1-\beta)\lambda$. (cohomological condition)
- $\hat{\omega} = \omega + \epsilon\sqrt{-1}\partial\bar{\partial}\|s\|^{2\beta}$ $(0 < \epsilon \ll 1)$.

Existence and obstruction of conical KE

Existence:

- Linear theory by Donaldson: $C^{2,\alpha,\beta}(X, D)$ space and openness
- (log-K-energy is proper) Apriori estimates and closedness
  - C^0-estimate: Berman, Jeffres-Mazzeo-Rubinstein (JMR)
  - C^2-estimate: JMR-L. ('11), CDS
  - $C^{2,\alpha,\beta}$-estimate: JMR-Tian, CDS.

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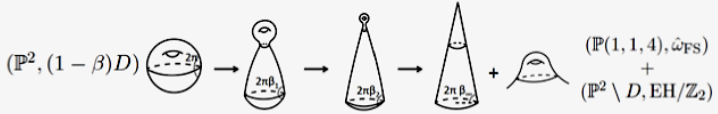
Obstruction:

1. log-Futaki invariant vanishes: Donaldson
2. log-K-stability: Berman, L. ('11), L.-Sun ('12)
3. $\mathrm{Aut}(X, D)$ is reductive: CDS (Berndtsson and BBEGZ)

$$X = \mathbb{P}^2,\ D = \{Z_0^2 + Z_1^2 + Z_2^2 = 0\}.$$

Theorem (L.-Sun, '12)

$\exists$ a conical KE on $(\mathbb{P}^2, (1-\beta)D)$ if and only if $2\pi\beta \in (\pi/2, 2\pi]$.



An example

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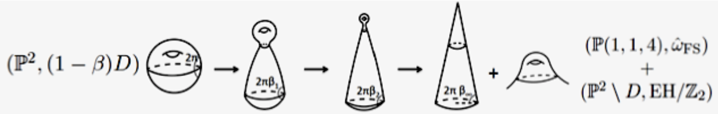
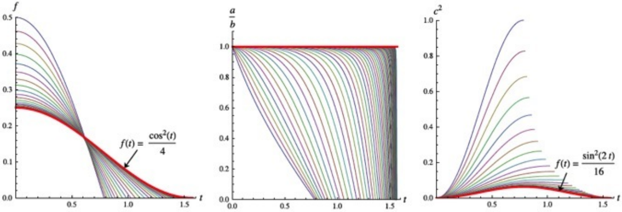


Corollary (Question by Gauntlett-Martelli-Sparks-Yau, $\beta = 1/3$)

$\exists$ complete conical Calabi-Yau metric on 3-dimensional $\mathbb{A}_2$ singularity: $\{z_0^2 + z_1^2 + z_2^2 + z_3^3 = 0\} \subset \mathbb{C}^4$.

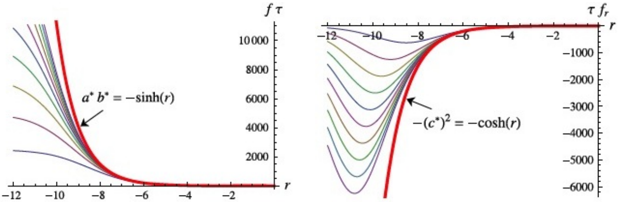
A_{k-1} singularity: $\{z_0^2 + z_1^2 + z_2^2 + z_3^k = 0\} \longleftrightarrow \beta = 1/k$

So $\nexists$ such metric on 3-dim A_{k-1} if $k \geq 4 \quad \longleftrightarrow \beta \leq 1/4$.



(a) $(t, f = ab)$ (b) $(t, a/b)$ (c) (t, c^2)

Fig. 5 Convergence of data



(a) $(r, \tau f)$ (b) $(r, \tau f_r)$

Fig. 6 Bubbling

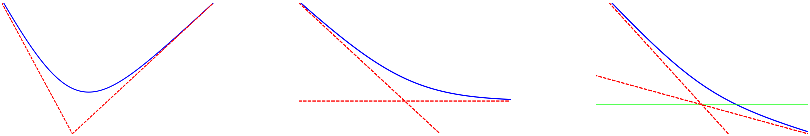
Proof I: Variational point of view

Existence of conical Kähler-Einstein metric

$\Updownarrow$

log-Ding-energy is proper

(strong Moser-Trudinger-Onofri inequality)



(k) Proper (l) Bounded $\geq -C$ (m) Unbounded

Figure : Stability from functionals

Conformal case: Normalized Einstein-Hilbert (Sobolev inequality) log-Ding-energy (recall that $t(\beta) = 1 - (1-\beta)\lambda$):

$$F_\beta(\omega_\varphi) := F^0_\omega(\varphi) - \log\left(\frac{1}{V}\int_X \frac{e^{h_\omega - t(\beta)\varphi}\omega^n}{||s||^{2(1-\beta)}}\right).$$

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Monge-Ampère energy: $\qquad \delta F^0_\omega(\varphi) = -\int_X (\delta\varphi)\omega^n_\varphi.$

Proof II: Interpolation

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Observation: Concave in β (by Hölder's inequality):

$$\beta_t = t\beta_1 + (1-t)\beta_2 \implies F_{\beta_t} \geq tF_{\beta_1} + (1-t)F_{\beta_2}$$

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Consequence: "F_β proper at $\beta_1 = 2\pi$" + "bounded at $\beta_2 = 2\pi/4$"
$\implies$ proper for $2\pi\beta \in (\pi/2, 2\pi]$.

Proof III: Degeneration

Theorem (L.-Sun '12, L. '13)

- $(X, D) \overset{\mathbb{C}^*}{\rightsquigarrow} (\mathcal{X}_0, \mathcal{D}_0)$ *over* $B_1(0)$
- $(\mathcal{X}_0, (1-\beta)\mathcal{D}_0)$ *conical KE*

$\Longrightarrow F_\beta$ *bounded from below*

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- $(\mathcal{X}_0, (1-\beta)\mathcal{D}_0)$ *conical KE*

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1. Embed into $\mathbb{P}^N$ & use Fubini-Study as reference metrics
2. Construct geodesic ray starting φ_t from φ inside $\mathcal{K}_\omega(X)$.
3. Subharmonicity of $F_\beta^{\mathcal{X}_t}(\varphi_t)$ as function of $t \in B_1(0)$
   - Subharmonicity away from 0 (Berndtsson)
   - Continuity of energy under degeneration ([Li '13])
4. Boundedness on the central fibre (Ding-Tian, BBEGZ)

Nonexistence when $\beta \leq 1/4$: Two ways:

1. Lichnerowicz obstruction (Gauntlett-Martelli-Sparks-Yau)
2. log-slope-stability (Ross-Thomas, L.-Sun ('12))

Completion of proof and generalization

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Generalization: X Fano manifold, $D \sim \lambda K_X^{-1}$ with $0 < \lambda < 1$.

$$\text{Adjunction}: \quad K_D^{-1} = (1-\lambda) K_X^{-1}|_D \Longrightarrow D \text{ is also Fano.}$$

Denote $\beta_{\inf} = \frac{\lambda^{-1}-1}{n}$ (example $\beta_{\inf} = \frac{((2/3)^{-1}-1}{2} = 1/4$).

Proposition (L. '13)

Assume D and X both KE. $(X, (1-\beta)D)$ conical KE iff $\beta \in (\beta_{\inf}, 1]$.

Convergence and bubbling

$$(X,(1-\beta)D) \xrightarrow{\beta\to\beta_{\inf}} \overline{C}(D,N_D) + \ \{\text{Tian-Yau metric on } X\setminus D\}.$$

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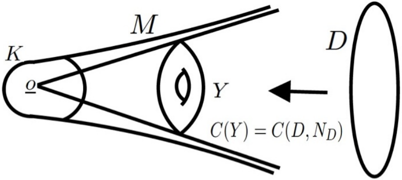

Figure : Asymptotically conical Kähler manifold and compactification

Tian-Yau metric $\leftrightarrow$ Asymptotically Conical Calabi-Yau

- Asymptotical rate (Cheeger-Tian, Conlon-Hein, L. '14)
- Compactification (L. 14)

$\{(X_i, \omega_i)\}$: Fano Kähler manifolds. $Ric(\omega_i) \geq t\omega_i$ $(t > 0)$. Then:

- $\mathrm{Diam}(X_i, \omega_i) \leq D = D(n, t)$ (Myers Theorem)
- $\mathrm{Vol}(B_r(x))/\mathrm{Vol}(B_r(\underline{0})) \searrow$ as $r \nearrow$ (Bishop-Gromov)

Gromov compactness $\Longrightarrow (X_i, \omega_i) \xrightarrow{GH} (X_\infty, \omega_\infty)$.

Algebraic structure on Gromov-Hausdorff limit

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Proposition (L. '12)

Tian's Partial C^0-estimate $\Longrightarrow$ ring of holomorphic sections with uniform L^2-norms is effectively finitely generated.

Proof: Skoda's theorem on finite generation and Siu's global version.

Corollary (L. '12)

Tian's Partial C^0-estimate $\Longrightarrow$ X_∞ has an algebraic structure.

There are applications of partial C^0-estimate to moduli problem.

Conjecture (Yau-Tian-Donaldson)

Fano manifold X has KE $\Longleftrightarrow$ $(X, -K_X)$ is K-polystable.

Recently completed by Chen-Donaldson-Sun, Tian independently.
Idea of Proof:

- Introduce the divisor $D \sim -mK_X$. Consider conical KE on $(X, (1-\beta)D)$.
- Varying β to show $\hat{\omega}_\beta$ changes continuously. β^*: critical angle
- Proving log version of partial C^0-estimates to show that the limit X_∞ (as $\beta \to \beta^*$) is a $\mathbb{Q}$-Fano variety.
- If $X_\infty \neq X$, then construct some special degeneration to contradict K-polystability.

$\mathbb{Q}$-Fano degenerations

$(\mathcal{X}, \mathcal{L}) \to C$: flat family of polarized projective varieties.
$(\mathcal{X}^*, \mathcal{L}^*) \cong (\mathcal{X}^*, K^{-1}_{\mathcal{X}^*/C^*}) \to C^*$: family of smooth Fano manifolds.
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Theorem (L.-Xu, '12)

There exists a $\mathbb{Q}$-Fano filling after base change:

$$\begin{tikzcd}
\mathcal{X}^s \arrow[r, dashed] \arrow[d] & \mathcal{X} \times_C C' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}^* \arrow[l, hook'] \arrow[d] \\
C' \arrow[r, equal] & C' \arrow[r, "\phi(=z^m)"] & C & C^* \arrow[l, hook']
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Moreover, $\mathrm{CM}(\mathcal{X}^s/C', -K_{\mathcal{X}^s}) \le \deg(\phi) \cdot \mathrm{CM}(\mathcal{X}/C, \mathcal{L})$.

- Use Minimal Model Program to simplify the family
- Keep track of the *CM*-degree in the process

Application: Confirmation of Tian's conjecture on the test of K-polystability.

Uniqueness of Filling

Compare 2 flat families of $\mathbb{Q}$-Fano with isomorphic generic fibres:

$$
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- Smooth $\dim_{\mathbb{C}} = 3$: Mukai-Umemura's example.
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Theorem (L.-Wang-Xu, '14)

Separatedness holds for families of smoothable K-polystable $\mathbb{Q}$-Fano varieties

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$\mathcal{M}$: moduli space of K-polystable smooth Fano manifolds.

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- Openness: L.-Wang-Xu ('14)
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Theorem (L.-Wang-Xu, '14)

$\exists$ *proper algebraic moduli space* $\overline{\mathcal{M}}$ *of K-polystable, smoothable, Fano varieties.*

- Locally K-polystable slice $=$ GIT moduli
- Gluing: $\overline{\mathcal{M}} = \bigcup_{i=1}^{I} (\mathcal{U}_{z_i} /\!\!/ G_{z_i})$.

Projectivity of $\mathcal{M}$

Theorem (L.-Wang-Xu, '15)

$\mathcal{M}$ is quasi-projective.

Proof.

- Existence (extension) of Weil-Petersson positive (1,1)-current with *continuous* potentials (partial C^0 & explicit calculations).
- Existence of continuous metric on CM-line bundle over $\mathcal{U}_{z_i}^{\mathrm{kps}}$.
- Descend CM line bundle and metrics to $\mathcal{U}_{z_i} /\!\!/ G_{z_i}$ and glue
- Quasi-projective criterion ($\longleftrightarrow$ Schumacher-Tsuji).

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Still open: Projectivity of $\overline{\mathcal{M}} \longleftrightarrow$ *CM*-line bundle ample.

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Still open: Projectivity of $\overline{\mathcal{M}} \longleftrightarrow CM$-line bundle ample.
Partial answer: CM-line bundle is nef and big over $\overline{\mathcal{M}}$.

Bergman kernel

X: Fano manifold; ω: Kähler metric in $2\pi c_1(X)$;
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$H^0(X, K_X^{-k})$: vector space of all holomorphic sections of K_X^{-k}. $\dim H^0(X, K_X^{-k}) = N_k$. Orthonormal basis $\{s_i\}_{i=1}^{N_k}$ under the L^2-inner product: $\langle s, s'\rangle_{L^2} = \int_X \langle s, s'\rangle_{h^{\otimes k}} \omega^n$.

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X: Fano manifold; ω: Kähler metric in $2\pi c_1(X)$;
h: Hermitian metric on K_X^{-1} satisfying $-\sqrt{-1}\partial\bar{\partial}\log h = \omega$.

$H^0(X, K_X^{-k})$: vector space of all holomorphic sections of K_X^{-k}. $\dim H^0(X, K_X^{-k}) = N_k$. Orthonormal basis $\{s_i\}_{i=1}^{N_k}$ under the L^2-inner product: $\langle s, s'\rangle_{L^2} = \int_X \langle s, s'\rangle_{h^{\otimes k}}\omega^n.$

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- $\rho_k = \rho_k(\omega)$ depends only on ω.
- $\rho_k(\omega) = \frac{k^n}{n!} + \frac{R(\omega)}{2(n-1)!}k^{n-1} + O(k^{n-2})$. (Tian, Zelditch, Lu)
  where $R(\omega) = g^{i\bar{j}}R_{i\bar{j}}$: scalar curvature of ω.

Tian's Partial C^0-estimate

Conjecture (Tian's partial C^0-estimates)

$\forall t > 0$, $\exists k = k(n,t)$ *and* $\delta = \delta(n,t) > 0$ *s.t. if* $Ric(\omega) \geq t\omega$, *then* $\rho_k \geq \delta$.

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Tian's partial C^0-estimates in various settings recently were proved by groups of people (Donaldson-Sun, Tian, Chen-Donaldson-Sun, Tian-Zhang, Jiang, Székelyhidi, Chen-Wang) It plays a central role in the recent resolution (by Chen-Donaldson-Sun and Tian) of the following conjecture:

Conjecture-Theorem (Yau-Tian-Donaldson conjecture (YTD))

X admits a KE if and only if (X, K_X^{-1}) is K-polystable.

" only if " part: by Tian, and Berman.

Effective finite generation

Theorem (Li '12)

If $\rho_k > 0$, then for any $m \geq (n+2)k$, if we let $l = \lfloor \frac{m}{k} \rfloor - n - 1$, then

$$u = \sum_{\alpha_1,\ldots,\alpha_l=1}^{N_k} u^{(l)}_{\alpha_1,\ldots,\alpha_l} s_{\alpha_1} \cdots s_{\alpha_l}$$

with $u^{(l)}_{\alpha_1,\ldots,\alpha_l} \in H^0(X,(m-lk)L)$ and

$$\|u^{(l)}_{\alpha_1,\ldots,\alpha_l}\|^2_{L^2} \leq \frac{(n+l)!}{l!n!} \frac{(\sup \rho_k)^{n+1}}{(\inf \rho_k)^{n+l+1}} \|u\|^2_{L^2}.$$

Tian's partial C^0-estimate $\Longrightarrow$ $\quad X_\infty = \mathrm{Proj} \bigoplus_{m=0}^{+\infty} H^0_{L^2}(X_\infty, L^m_\infty)$.

Uniformization for marked sphere

Global setting: Σ Riemann surface; Euler number $\chi(\Sigma) = 2 - 2g(\Sigma)$.

Singularities: $\{p_i\}_{i=1}^r$. $\hat{\omega}$ conical metric with cone angle $2\pi\beta_i$ at p_i.

Gauss-Bonnet: $\chi(\Sigma) + \sum_{i=1}^{r}(1 - \beta_i) = \frac{1}{2\pi}\int_{\Sigma} S(\hat{\omega})\mathrm{dvol}_{\hat{\omega}}$.

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$\Sigma = \mathbb{S}^2$. $D = \sum_{i=1}^r \alpha_i p_i$, $\alpha_i = 1-\beta_i$ (cone defect).

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- $\chi(\mathbb{S}^2, D) > 0$: Troyanov, McOwen, Luo-Tian:

$$\exists\, \omega \text{ s.t. } S(\omega) \equiv 1 \iff \alpha_i < \sum_{j\neq i}\alpha_j, \forall i = 1, \ldots, r;$$

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$$\begin{aligned} \exists\, \omega \text{ s.t. } S(\omega) \equiv 1 \quad &\Longleftrightarrow \quad \alpha_i < \sum_{j\neq i}\alpha_j, \forall i = 1, \dots, r; \\ &\Longleftrightarrow \quad (\mathbb{S}^2, D) \text{ is log-K-stable.} \end{aligned}$$

Futaki invariant

v: a holomorphic vector field. Recall: $Ric(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h_\omega$.

Futaki invariant: $\mathrm{Fut}_X(v) = \int_X v(h_\omega)\omega^n.$

Theorem (Futaki)

- $\mathrm{Fut}_X(v)$ *is independent of* $\omega \in 2\pi c_1(X)$.
- $KE \implies \mathrm{Fut}_X \equiv 0$.

Interpretation: $(\bar{\partial} - \sqrt{-1}i_v)(\omega + \mathrm{div}_\Omega(v)) = 0 \quad (\Omega = e^{h_\omega}\omega^n)$

$$\mathrm{Fut}_X(v) = \frac{1}{n+1}\int_X (\omega + \mathrm{div}_\Omega v)^{n+1}.$$

Equivariant cohomology $\Rightarrow$ localization formula.

Test configuration (TC): $\mathbb{C}^*$-equivariant degeneration of Fano manifolds over $\mathbb{C}$:

$$\begin{tikzcd}
(X \times \mathbb{C}^*, -K_X) \cong (\mathcal{X}^*, \mathcal{L}|_{\mathcal{X}^*}) \arrow[r, hook] \arrow[d] & (\mathcal{X}, \mathcal{L}) \arrow[d] & (\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0}) \arrow[l, hook'] \arrow[d] \\
\mathbb{C}^* \arrow[r] & \mathbb{C} & \{0\} \arrow[l]
\end{tikzcd}$$

Special test configuration (STC) $\mathcal{X}_0$ irreducible normal Fano variety, $\mathbb{Q}$-factorial, Kawamata-log-terminal (klt); $\mathcal{L} \sim_{\mathbb{C}} -K_{\mathcal{X}}$.

$\mathbb{C}^*$-action $\rightsquigarrow$ holomorphic vector field v on $\mathcal{X}_0$. For STC, define:

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = -\mathrm{Fut}_{\mathcal{X}_0}(v)$$

Futaki invariant for general TC as CM weight:

Definition (CM line bundle, [Tian, Fujiki-Schumacher])

$$L_{CM} = \det(\pi_![n(\mathcal{L}-\mathcal{L}^{-1})^{n+1} - (n+1)(K_{\mathcal{X}}^{-1} - K_{\mathcal{X}}) \cdot (\mathcal{L}-\mathcal{L}^{-1})^n]).$$

CM weight: $\mathrm{CM}(\mathcal{X}, \mathcal{L}) = n\bar{\mathcal{L}}^{n+1} + (n+1)K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n.$

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Definition-Theorem (Donaldson, Paul-Tian, Wang)

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For any test configuration $\mathcal{X}$, $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \mathrm{CM}(\mathcal{X}, \mathcal{L})$.

Special test configuration $\rightsquigarrow$ simplifications:

$$L_{CM} = \det(\pi_! [-(K_{\mathcal{X}}^{-1} - K_{\mathcal{X}})^{n+1}]), \quad \mathrm{CM}(\mathcal{X}, \mathcal{L}) = -(K_{\bar{\mathcal{X}}/\mathbb{P}^1}^{-1})^{n+1}.$$

$$\det\left(\pi_*(K_{\mathcal{X}/\mathbb{C}}^{-k})\right) = -L_{CM} \frac{k^{n+1}}{(n+1)!} + O(k^n).$$

Definition (K-polystability, Tian '97)

$\mathrm{Fut}(\mathcal{X}, K_{\mathcal{X}}^{-1}) \geq 0 \; \forall \; \mathrm{STC} \; \mathcal{X}$ *of* X, *with equality holds iff* $\mathcal{X} \cong X \times \mathbb{C}$.

Two versions K-polystability

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Definition (K-polystability, Donaldson '02)

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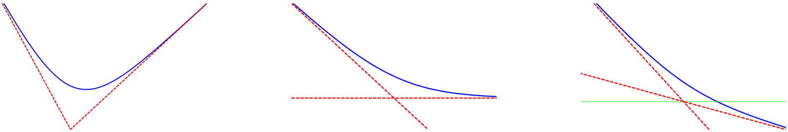
Imitating Hilbert-Mumford Numerical criterion in GIT:

$$\text{Slope at infinity} \longleftrightarrow \mathrm{Fut}(\mathcal{X}, \mathcal{L}).$$



(g) Stable (h) Semistable (i) Unstable

Tian's conjecture

Compactness in Riemannian geometry (partial C^0-estimate) $\rightsquigarrow$

Conjecture (Tian)

For K-polystability of Fano, special degenerations are enough.

Theorem (Li-Xu '12)

$(\mathcal{X}, \mathcal{L})$: any TC. Then $\exists$ an integer $k > 0$, an STC $(\mathcal{X}^{\mathrm{s}}, -K_{\mathcal{X}^{\mathrm{s}}})$ and a birational map $\mathcal{X}^{\mathrm{s}} \dashrightarrow \mathcal{X} \times_{z^k} \mathbb{C}$ inducing an isomorphism

$$(\mathcal{X}^{\mathrm{s}}, \mathcal{L}^{\mathrm{s}})|_{\mathbb{C}^*} \cong (\mathcal{X}, \mathcal{L})|_{\mathbb{C}^*} \times_{z^k} \mathbb{C}^*,$$

such that $$\mathrm{Fut}(\mathcal{X}^{\mathrm{s}}, \mathrm{K}_{\mathcal{X}^{\mathrm{s}}}^{-1}) \le \mathrm{k} \cdot \mathrm{Fut}(\mathcal{X}, \mathcal{L}),$$

and the equality holds iff $(\mathcal{X}, \mathcal{L}) \to C$ is itself a STC.
As a consequence, Tian's conjecture holds.

Proof I: Relative Minimal model program

Vary the polarization in the direction of $K_{\mathcal{X}}$ and fix the volume:

$$\mathcal{L}_s := \frac{\mathcal{L} + sK_{\mathcal{X}}}{1-s}, \quad \mathcal{L}_s|_{\mathcal{X}_t} \sim -K_X.$$

Denote $(\mathcal{X}^{(0)}, \mathcal{L}^{(0)})$ = a semistable reduction of $(\mathcal{X}, \mathcal{L})$.

Increase s: $s \nearrow \lambda_{i+1} < 1$: $\mathcal{L}^i_s$ stops being ample

$\rightsquigarrow$ extremal ray $[R] \in \mathrm{NE}(\mathcal{X}/\mathbb{C})$

$\rightsquigarrow$ birational contraction $\begin{cases} \text{divisorial} \\ \text{small} \rightsquigarrow \text{flip} \end{cases}$

$\rightsquigarrow$ $(\mathcal{X}^{i+1}, \mathcal{L}^{i+1}_{\lambda_i})$, $\mathcal{L}^{i+1}_{\lambda_{i+1}}$ ample

$\rightsquigarrow$ go to **Increase s**

More technical issues: log canonical modifications, Fano extension.

Proof IIa: Decrease of Futaki invariant on a fixed model

$$\mathrm{Fut}(\mathcal{X},\mathcal{L}) = n\bar{\mathcal{L}}^{n+1} + (n+1)K_{\bar{\mathcal{X}}/\mathbb{P}^1}\cdot\bar{\mathcal{L}}^n; \quad \mathcal{L}_s = \frac{\mathcal{L}+sK_{\mathcal{X}}}{1-s}.$$

$$\begin{aligned}
\frac{d}{ds}\mathrm{Fut}(\mathcal{X},\mathcal{L}_s) &= n(n+1)\bar{\mathcal{L}}_s^{n-1}\left(\bar{\mathcal{L}}_s + K_{\bar{\mathcal{X}}/\mathbb{P}^1}\right)\cdot\left(\frac{d}{ds}\bar{\mathcal{L}}_s\right)\\
&= n(n+1)\bar{\mathcal{L}}_s^{n-1}\cdot(\bar{\mathcal{L}}_s + K_{\bar{\mathcal{X}}/\mathbb{P}^1})^2/(1-s)^2\\
&\le 0. \quad \text{(Zariski Lemma)}
\end{aligned}$$

$\Longrightarrow \mathrm{Fut}(\mathcal{X},\mathcal{L}_s)$ is a decreasing function of s.

Proof IIb: Invariance at birational surgery

Divisorial contraction	Flip
$\begin{array}{ccc} \mathcal{X}^i & \xrightarrow{f^i} & \mathcal{X}^{i+1} \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^1 & \end{array}$	$\begin{array}{ccc} \mathcal{X}^i & \overset{\phi^i}{\dashrightarrow} & \mathcal{X}^{i+1} \\ & {\scriptstyle f^i}\searrow \quad \swarrow{\scriptstyle f^{i+}} & \\ & \mathcal{Y}^i & \end{array}$
$\mathcal{L}^i+\lambda_{i+1}K_{\mathcal{X}^i}=(f^i)^*(\mathcal{L}^{i+1}+\lambda_{i+1}K_{\mathcal{X}^{i+1}})$	$\mathcal{L}^i+\lambda_{i+1}K_{\mathcal{X}^i}=(f^i)^*(D_{\mathcal{Y}^i})$ $\mathcal{L}^{i+1}+\lambda_{i+1}K_{\mathcal{X}^{i+1}}=(f^{i+})^*(D_{\mathcal{Y}^i})$

Projection formula for intersection numbers $\Rightarrow$ invariance of Futaki

More technical issues: normalization, base change, log canonical modifications and Fano extensions

Projectivity/non-projectivity of moduli spaces

$\mathcal{M}^-$: moduli space of canonically polarized manifolds
$\overline{\mathcal{M}^-}$: Kollár-Shepherd-Barron-Alexeev compactification
(properness + boundedness + openness + separatedness)

- Viehweg: $\mathcal{M}^-$ is quasi-projective (nef K_X is enough)
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- *Cons*: Kollár: moduli space of polarized uniruled manifolds in general is not quasi-projective
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Constraint: Use of canonical metrics $\longleftrightarrow$ Weil-Petersson geometry

Three basic types of manifolds as building blocks:

Property	$c_1(X) < 0$	$c_1(X) = 0$	$c_1(X) > 0$
Example	$\mathbb{B}^n/\Gamma$	$\mathbb{C}^n/\Lambda$	$\mathbb{CP}^n$
KE	✓ (Aubin, Yau)	✓ (Yau)	YTD conjecture
Aut(X)	finite	reductive	can be non-reductive
Moduli	KSBA	good moduli	non-separatedness
Rational Curv.	few (conj.)	few (conj.)	rationally-connected
$\cdots$	$\cdots$	$\cdots$	$\cdots$

Reductive: Complexification of compact Lie group.
Example: $\mathbb{C}^* = (S^1)^{\mathbb{C}}$, $SL(N+1, \mathbb{C}) = SU(N+1)^{\mathbb{C}}$.

Thanks for your attention!