YTD conjecture for weighted Kähler solitons and applications

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Moment map of torus actions on projective manifolds

- X a projective manifold. $L \to X$ an ample holomorphic line bundle.

  Kähler metric $\omega = \sqrt{-1}\sum_{i,j}\omega_{i\bar{j}}dz_i \wedge d\bar{z}_j > 0$ in $2\pi \cdot c_1(L)$.

- $T \cong (S^1)^r$, with Lie algebra $\mathfrak{t}$, acts holomorphically on (X, L).

  Hamiltonian action: ω is T-invariant and there exists a moment map:

$$\mathbf{m} : X \to \mathfrak{t}^\vee \cong \mathbb{R}^r, \quad \underline{\iota_\xi\omega = d\langle \mathbf{m}, \xi\rangle} \quad \text{for any } \xi \in \mathfrak{t}.$$

- Atiyah-Guillemin-Sternberg: the image $\mathbf{m}(X) =: P$ is a convex polytope

  Duistermaat-Heckman measure: $\mathbf{m}_*(\omega^n)$ does not depend on the Kähler form in the same Kähler class. $\Rightarrow$ For any smooth function g on $\mathbb{R}^r$,

$$\mathrm{V}_g := \int_X \underset{g\circ m}{g(\mathbf{m})}\omega^n = \int_P g(x)\mathbf{m}_*(\omega^n) \quad \text{is independent of } \omega \in [\omega].$$

Fano manifolds

- X Fano: $-K_X := \wedge^n T_{hol}X$ is ample, $c_1(X) = c_1(-K_X) > 0$.
- Hermitian metric on $-K_X \longleftrightarrow$ volume form

$$h_\varphi = h_0 e^{-\varphi} \quad \longleftrightarrow \quad \Omega_\varphi = |\underset{\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}}{\partial_z}|^2_{h_\varphi}(\sqrt{-1})^{n^2} dz \wedge d\bar{z}.$$

Chern curvature:

$$Ric(\Omega_\varphi) = Ric(\Omega_0) + \sqrt{-1}\partial\bar{\partial}\varphi = \boxed{\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi = \omega_\varphi} \in 2\pi c_1(X).$$

Space of Kähler potentials:

$$\mathcal{H}(\omega_0) = \left\{\varphi \in C^\infty(X); \quad \overset{\omega_\varphi}{\overset{\parallel}{\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi}} > 0\right\}.$$

- holomorphic vector field $v \longrightarrow$ a canonical Hamiltonian function:

$$\boxed{\theta_v(\varphi) = -\frac{\mathscr{L}_v \Omega_\varphi}{\Omega_\varphi}} \quad \Longrightarrow \quad \boxed{\iota_v \omega_\varphi = \sqrt{-1}\bar{\partial}\theta_v(\varphi).}$$

If $\boxed{v_\xi := \frac{1}{2}(-J\xi - \sqrt{-1}\xi)}$, $\theta_v(\varphi)$ is real $\Longleftrightarrow$ ξ is Killing w.r.t. ω_φ.

g-soliton equations

- Moment map of $T = (S^1)^r$ action with respect to ω_φ:

$$\mathbf{m}_\varphi : X \longrightarrow P \subset \mathfrak{t}^\vee, \quad \langle \mathbf{m}_\varphi, \xi \rangle = \theta_{v_\xi}(\varphi).$$

- Let $g : P \to \mathbb{R} > 0$ be a smooth function.

g-weighted Kähler-Ricci soliton, or just g-soliton: equation for $\varphi \in \mathcal{H}(\omega_0)$:

$$\boxed{g(\mathbf{m}_\varphi)(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-\varphi}\Omega_0}$$

$$\Longleftrightarrow \quad Ric(\omega_\varphi) - \omega_\varphi = \sqrt{-1}\partial\bar{\partial}\log g(\mathbf{m}_\varphi).$$

$g \circ m_\varphi$
$\nabla \varphi$

Examples: $g = \exp(\langle x, \xi \rangle)$: Kähler-Ricci soliton; $g = 1$: Kähler-Einstein.

$g(x) = 1 + \langle x - \bar{x}, \xi \rangle$: Mabuchi soliton

$\boxed{g = (c + \langle x, \xi \rangle)^{-n-2}:}$ Ricci-flat Kähler cone metric (Apostolov et al.)

Futaki invariant and Matsushima type result

- Set $g_\varphi := g(\mathbf{m}_\varphi)$. For any holomorphic vector field v

$$\mathrm{Fut}_g(v) \quad := \quad -\int_X \theta_v(\varphi) g_\varphi \omega_\varphi^n.$$

Fact: Fut_g does not depend on the choice of $\omega_\varphi \in 2\pi c_1(-K_X)$.

$$\exists \;\; g\text{-soliton in } c_1(X) \Longrightarrow \quad \mathrm{Fut}_\mathrm{g} \equiv 0.$$

- There is a generalized Matsushima reductivity result:

Theorem

If $(X, \mathbb{T})$ admits a g-soliton, then the following group is reductive:

$$\mathrm{Aut}(X, \mathbb{T}) := \mathrm{Aut}(X, L, \mathbb{T}) = \{\sigma \in \mathrm{Aut}(X, L); t \circ \sigma = \sigma \circ t \, \forall t \in \mathbb{T}\}. \qquad (1)$$

Energy functionals and coercivity

g-weighted functionals generalizing the unweighted case:

$$\mathbf{E}_g(\varphi) = \frac{1}{\mathrm{V}_g}\int_0^1 dt \int_X \varphi g_{t\varphi}\omega_{t\varphi}^n, \quad \mathbf{\Lambda}_g(\varphi) = \frac{1}{\mathrm{V}_g}\int_X \varphi g_{\varphi_0}\omega_0^n$$

$$\mathbf{I}_g(\varphi) = \frac{1}{\mathrm{V}_g}\int_X \varphi(g_{\varphi_0}\omega_{\varphi_0}^n - g_\varphi\omega_\varphi^n), \quad \mathbf{J}_g(\varphi) = \mathbf{\Lambda}_g(\varphi) - \mathbf{E}_g(\varphi)$$

$$\mathbf{L}(\varphi) = -\log\left(\frac{1}{\mathrm{V}_g}\int_X e^{-\varphi}\Omega_0\right)$$

$$\boxed{\begin{aligned}\mathbf{D}_g(\varphi) &= -\mathbf{E}_g(\varphi) + \mathbf{L}(\varphi)\\ \mathbf{M}_g(\varphi) &= \frac{1}{\mathrm{V}_g}\int_X \log\frac{g_\varphi\omega_\varphi^n}{\Omega_0}g_\varphi\omega_\varphi^n - (\mathbf{I}_g - \mathbf{J}_g)(\varphi).\end{aligned}}$$

critical pts
"
g-soliton

Analytic criterion

- Automorphisms: $\mathrm{Aut}(X, \mathbb{T}) = \{\sigma \in \mathrm{Aut}(X); \sigma \cdot t = t \cdot \sigma, \forall t \in \mathbb{R}\}$
  $\tilde{\mathbb{T}}$: maximal torus of $\mathrm{Aut}(X, \mathbb{T})$; $\tilde{T}$: maximal compact torus of $\tilde{\mathbb{T}}$.
- $\mathbf{F}_g$ is reduced coercive if $\exists \gamma, C > 0$ s.t. $\forall \tilde{T}$-invariant $\varphi \in \mathcal{H}(\omega_0)^{\tilde{T}}$

coercive
modulo
automorphism

$$\boxed{\mathbf{F}_g(\varphi) \geq \gamma \cdot \inf_{\sigma \in \tilde{\mathbb{T}}} \mathbf{J}_g(\sigma^* \omega_\varphi) - C.}$$

Theorem (Han-**L.** , generalize Tian, Tian-Zhu, Phong-Song-Sturm-Weinkove, Berman-Witt-Nyström, Darvas-Rubinstein, . . .)

The following are equivalent:

1. $(X, \mathbb{T})$ *admits a g-soliton.*
2. $\mathbf{D}_g$ *is reduced coercive.*
3. $\mathbf{M}_g$ *is reduced coercive.*

g-Monge-Ampère measure for singular potentials

- $\varphi \in L^1(\omega_0^n)$ is ω_0-psh if it is u.s.c. and $\psi + \varphi$ is psh ($\omega_0 = \sqrt{-1}\partial\bar{\partial}\psi$).
$$\mathrm{PSH}(\omega_0) = \{\varphi; \varphi \text{ is } \omega_0\text{-psh}\}.$$
Non-pluripolar product:
$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \lim_{j\to+\infty} (\omega_0 + \sqrt{-1}\partial\bar{\partial}\max\{\varphi, -j\})^n.$$
- $g = \prod_\kappa (x_\kappa + c)^{d_\kappa}$ such that $P + c(1, \dots, 1) \in \mathbb{R}^r_{>0}$, then
$$\int_X f g_\varphi (\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_{X^{[\vec{d}]}} (f^T)^{[\vec{d}]} \Big(\sum_\kappa \theta_\kappa(\varphi)\omega_{\mathrm{FS},\kappa} + \omega_\varphi\Big)^{n+d}.$$
- General g, find polynomials $g_j \to g$ uniformly and set:
$$\boxed{\int_X f g_\varphi \omega_\varphi^n = \lim_{j\to+\infty} \int_X f (g_j)_\varphi \omega_\varphi^n.}$$

Finite energy space:
$$\mathcal{E}^1_g = \left\{\varphi \in \mathrm{PSH}(\omega_0)^T; \int_X |\varphi| g_\varphi \omega_\varphi^n < +\infty\right\}.$$

Fibration construction

- $P \to B$: hol. line bundle with Hermitian metric h_P; $\bar{P}$ circle bundle.
  $L \to X$: hol. line bundle with $\mathbb{C}^*$-action, S^1-invariant Hermitian metric h_L
  $(Y, F) = (P^* \times (X, L))/\mathbb{C}^* \to B$: associated holomorphic fibre bundle.
  Induced Hermitian metric h_F on F whose Chern curvature at $\bar{P}$:

$$\sqrt{-1}\bar{\partial}\partial \log h_F = \pi^*(\theta_v \sqrt{-1}\bar{\partial}\partial \log h_P) + \sqrt{-1}\bar{\partial}\partial \log h_L.$$

  The right-hand-side is considered as an equivariantly closed form on X.
- Example:
  $\mathbb{S}^{[\vec{d}]} = S^{2d_1+1} \times \cdots \times S^{2d_r+1} \to \mathbb{P}^{[\vec{d}]} = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_r}$.

$$(X^{[\vec{d}]}, L^{[\vec{d}]}) = \mathbb{S}^{[\vec{d}]} \times_{(S^1)^r} (X, L) \to \mathbb{P}^{[\vec{d}]}.$$

  Curvature: $\sum_{\kappa=1}^r \theta_\kappa(\varphi)\omega_{\mathrm{FS},\mathbb{P}^{d_\kappa}} + \omega_\varphi$.

Test configurations

- Let $X \to \mathbb{P} = \mathbb{P}^{N_m-1}$ be the $\mathbb{T}$-equivariant Kodaira embedding via $|mL|$.

  $\sigma(s) = \exp(s\zeta), s \in [0, +\infty)$: one parameter subgroup of $GL(N_m, \mathbb{C})$.

  Limit scheme: $[\mathcal{X}_0] = \lim_{s\to+\infty} \sigma(s) \circ [X]$ and induced test configuration:

$$\begin{aligned} \mathcal{X} &= \{(z,t) \in \mathbb{P}^{N_m-1} \times \mathbb{C}; z \in \sigma(-\log|t|^2) \circ X\} \\ \mathcal{L} &= \left(p_1^* \mathcal{O}_{\mathbb{P}}(1)\right)^{1/m}. \end{aligned}$$

  Path in $\mathcal{H}(\omega_0)$: $\Phi = \{\varphi(s)\}$ with $\varphi(s) = \dfrac{1}{m} \log \dfrac{|\sigma(s) \cdot Z|^2}{|Z|^2}$.

- For $\xi \in \mathfrak{t}$, twist $(\mathcal{X}, \mathcal{L}) \to (\mathcal{X}_\xi, \mathcal{L}_\xi) \longleftrightarrow \sigma \cdot \exp(s\xi)$.
- σ commutes with $\mathbb{T} \Longrightarrow \mathbb{T} \times \mathbb{C}^*$ acts on $(\mathcal{X}_0, \mathcal{L}_0)$ and hence on $V_m = H^0(\mathcal{X}_0, m\mathcal{L}_0)$ with weight decomposition:

$$V_m = \bigoplus_{\alpha} V_{m,\alpha} = \bigoplus_{\alpha \in \mathbb{Z}^r} \bigoplus_{i \in \mathbb{Z}} V_{m,\alpha}(\lambda_i^{(m,\alpha)}).$$

Non-Archimedean functionals

$$E(\varphi) \Leftrightarrow \mathbf{E}_g^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \frac{1}{\mathrm{V}_g} \lim_{m\to+\infty} \frac{n!}{m^n} \sum_{\alpha,i} \frac{\lambda_i^{(m,\alpha)}}{m} g\left(\frac{\alpha}{m}\right)$$

$$= \frac{1}{\mathrm{V}_g} \int_{\mathcal{X}_0} \theta_\zeta g_\varphi \omega_\varphi^n$$

$$\mathbf{\Lambda}_g^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \lim_{m\to+\infty} \max_{\alpha,i} \frac{\lambda_i^{(m,\alpha)}}{m}$$

$$\mathbf{J}_g^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \mathbf{\Lambda}_g^{\mathrm{NA}} - \mathbf{E}_g^{\mathrm{NA}}$$

$$\mathbf{L}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \mathrm{lct}\,(\mathcal{X}, -(K_{\mathcal{X}}+\mathcal{L}); \mathcal{X}_0) - 1$$

$$\mathbf{D}_g^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = -\mathbf{E}_g^{\mathrm{NA}} + \mathbf{L}^{\mathrm{NA}}.$$

Theorem (Generalized slope formula, proof uses fibration construction)

For each $\mathbf{F} \in \{\mathbf{E}_g, \mathbf{J}_g, \mathbf{D}_g\}$, *we have the identity:*

$$\boxed{\mathbf{F}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \lim_{s\to+\infty} \frac{\mathbf{F}(\varphi(s))}{s}.}$$

G-Stability (after Hisamoto, which generalizes Tian, Donaldson, Székelyhidi, ...)

- $(X, \mathbb{T})$ is reduced uniformly g-weighted Ding-stable if there exists $\gamma > 0$ such that for any $\tilde{\mathbb{T}}$-equivariant test configuration $(\mathcal{X}, \mathcal{L})$:

$$\boxed{\mathbf{D}_g^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \mathbf{J}_{g,\tilde{\mathbb{T}}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})}$$

  where

$$\mathbf{J}_{g,\tilde{\mathbb{T}}}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \inf_{\xi \in \tilde{N}_{\mathbb{R}}} \mathbf{J}_g^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi).$$

- It is g-weighted (Ding)-polystable if $\mathbf{D}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq 0$ and equality holds iff it is induced by a holomorphic vector field.

Algebraic study of weighted K-stability

Theorem (Han-L. , Generalizing K. Fujita and L.)

$(X, \mathbb{T})$ *is reduced uniformly g-Ding stable if and only if it is so for special test configurations (the central fibre is a normal Fano variety). Moreover there is a valuative criterion for uniform g-Ding-stability.*

The proof for the first statement uses the Minimal Model Program first used by **L.** -Xu, and the fibration construction.
The last statement is obtained by expressing the non-Archimedean functionals by using invariants of the valuation $v = r(\mathrm{ord}_{\mathcal{X}_0})$.

Theorem (Liu-Xu-Zhuang, Blum-Liu-Xu-Zhuang)

$(X, \mathbb{T})$ *is reduced uniformly g-Ding-stable if and only if it is polystable with respect to* $\tilde{\mathbb{T}}$.

log Fano case

- X: projective normal variety; D: an effective Weil divisor satisfying: $-(K_X + D)$ is $\mathbb{Q}$-Cartier and ample.

  (X, D) has klt singularities if for any $v = \mathrm{ord}_E \in X_{\mathbb{Q}}^{\mathrm{div}}$, $A_{(X,D)}(v) = \mathrm{ord}_D(K_{X'/X}) + 1 > 0$ where E is an ordinary divisor on $X' \to X$.

- Example 1: Orbifold (X, D), for any $p \in X$, $\exists$ a neighborhood U s.t.

$$U \cong D/\Gamma, \quad D = \sum_i (1 - d_i^{-1}) D_i.$$

- Example 2: Regular Fano cone singularities S is Fano variety, $-K_S = \gamma L$ with $\gamma > 0$, then the cone singularity $X = C(S, L) = \mathrm{Spec}(\bigoplus_m H^0(S, mL))$ is klt.

YTD conjecture for the log Fano case

polystability

Theorem (Han-L.)

$(X, D, \mathbb{T})$ *admits a g-soliton iff it is reduced uniformly g-Ding stable.*

The proof uses an approach based on non-Archimedean approximations, first proposed by Berman-Boucksom-Jonsson for smooth Fano manifolds with discrete automorphism groups, extended by Hisamoto and **L.** to the case of continuous automorphism groups.

Idea to overcome the difficulty caused by singularities is based perturbation approach developed by **L.** -Tian-Wang. We work on log resolutions, perturb the semi-positive class and then take limits. This depends crucially on the valuative criterion, and new uniform Archimedean/non-Archimedean estimates.

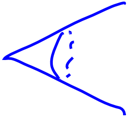
Kähler cone metric

- $Y = \mathrm{Spec}(R)$: affine variety isolated singularity $o \in Y$.

  $\hat{\mathbb{T}} \cong (\mathbb{C}^*)^{r+1}$-action $\Rightarrow$ Weight decomposition: $R = \sum_{\hat{\alpha} \in \mathbb{Z}^{r+1}} R_{\hat{\alpha}}$.

- $\hat{T}$-invariant Radius function: $r : Y \to \mathbb{R}_{\geq 0}$, $S = \{r = 1\}$.

  Kähler cone metric: $\hat{\omega} = \sqrt{-1}\partial\bar{\partial}r^2$, $\hat{\omega}(\cdot, J\cdot) = dr^2 + r^2 g_S$.



- Reeb vector field: $J(r\partial_r) = \hat{\xi} \in \hat{\mathfrak{t}}^+ = \left\{\xi \in \hat{\mathfrak{t}}; \langle \hat{\alpha}, \hat{\xi} \rangle > 0\right\}$ (Reeb cone).

  Quasi-regular: $\hat{\xi} \in \hat{\mathfrak{t}}_{\mathbb{Q}}$. $\langle v_{\hat{\xi}} \rangle \cong \mathbb{C}^*$ and $Y/\langle v_{\hat{\xi}} \rangle = (X, D)$ is an orbifold.

  Regular: if $\langle v_{\hat{\xi}} \rangle$ acts freely on Y^*. $Y/\langle v_{\hat{\xi}} \rangle$ is a projective manifold.

  Irregular: $\hat{\xi} \notin \hat{\mathfrak{t}}^+$. $\langle v_{\xi} \rangle \cong (\mathbb{C}^*)^d$ for $d > 1$ (main interest).

Ricci-flat Kähler cone (Martelli-Sparks-Yau)

- Assume Y $\mathbb{Q}$-Gorenstein. s: $\hat{\mathbb{T}}$-equivariant generating section of $|mK_Y|$.

  Volume form: $dV_Y = \left((\sqrt{-1})^{m(n+1)^2} s \wedge \bar{s}\right)^{1/m}$.

- Ricci-flat Kähler cone equation:

$$\boxed{(\sqrt{-1}\partial\bar{\partial}r^2)^{n+1} = dV_Y \quad \Longleftrightarrow \quad Ric(\hat{\omega}) = 0.}$$

  Normalization of Reeb vector fields:

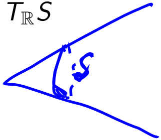
$$\mathfrak{L}_{r\partial_r} dV_Y = 2(n+1) dV_Y \quad \Longleftrightarrow \quad \mathfrak{L}_{v_{\hat{\xi}}} s = (n+1)s$$

- Regular example:
  X a Fano manifold. h_{KE}: a KE Hermitian metric on $-K_X$.
  Then $r = h_{\mathrm{KE}}^{\frac{1}{\gamma(n+1)}}$ is a radius function of Ricci-flat Kähler cone on $Y = C(X, L)$ for any $L = \gamma^{-1}(-K_X)$ with $\gamma > 0$.
  For example, $X = \mathbb{P}^n$, $L = \frac{1}{n+1}(-K_X)$, $Y = \mathbb{C}^{n+1}$.

Sasaki-Einstein manifold

- Sasaki manifold $S = \{r = 1\}$. CR structure $\mathfrak{D} = JT_{\mathbb{R}}S \cap T_{\mathbb{R}}S$

  Contact form: $\eta = -Jd \log r$ determined by:



$$\eta(\hat{\xi}) = \eta(J(r\partial_r)) = 1, \quad \eta|_{\mathfrak{D}} = 0.$$

  Kähler form: $\hat{\omega} = \sqrt{-1}\partial\bar{\partial}r^2 = d(r^2\eta) = r^2 d\eta + 2rdr \wedge \eta$.

  $\hat{\omega}^{n+1} = 2(n+1)r^{2n+1}dr \wedge (d\eta)^n \wedge \eta$.

  $dV_S = dV_S^{\hat{\xi}} = \iota_{\partial_r} dV_Y \longrightarrow dV_Y = r^{2n+1}dr \wedge dV_S$.

- Rewrite the Ricci-flat equation:

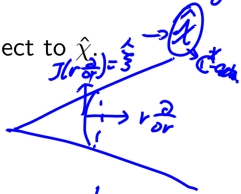
$$\boxed{(\sqrt{-1}\partial\bar{\partial}r^2)^{n+1} = dV_Y \quad \Longleftrightarrow \quad (d\eta)^n \wedge \eta = dV_S.}$$

  Y is Ricci-flat Kähler cone $\Longleftrightarrow$ S is Sasaki-Einstein.

Deformation of Reeb vector fields

- For $\hat{\xi} \in \hat{\mathfrak{t}}^+$, set $\mathcal{R}^{\hat{\xi}} = \left\{ \text{radius functions w.r.t. } \hat{\xi} \right\}$.
- Fix a reference $\hat{\chi} \in \hat{\mathfrak{t}}^+$ and a radius function r_0 with respect to $\hat{\chi}$.

grading.



$$\hat{\xi} \in \hat{\mathfrak{t}}^+ \longrightarrow r := r_0^{\hat{\xi}} \in \mathcal{R}^{\hat{\xi}}$$

satisfying:

$$J(r\partial_r) = \hat{\xi}, \quad S = \{r = 1\} = \{r_0 = 1\}.$$

$Y/\langle v_{\hat{\chi}} \rangle = (X, D)$

log Fano

Transformation of Reeb vector fields and contact forms:

$$\hat{\xi} = \eta(\hat{\xi})\hat{\chi} + \xi^h \quad \Longrightarrow \quad \begin{cases} \eta = \eta_r = \eta_0(\hat{\xi})^{-1}\eta_0 \\ \\ r\partial_r = \eta(\hat{\xi}) r_0 \partial_{r_0} + J(\xi^h). \end{cases}$$

- Ricci-flat equation to g-soliton (Apostolov et al. via Tanaka-Webster):

$$\boxed{(d\eta)^n \wedge \eta = dV_S} \quad \Leftrightarrow \quad \eta_0(\hat{\xi})^{-n-1}(d\eta_0)^n \wedge \eta_0 = \eta_0(\hat{\xi}) dV_S^{\hat{\chi}}$$

$$\Leftrightarrow \quad \boxed{\eta_0(\hat{\xi})^{-n-2}(d\eta_0)^n \wedge \eta_0 = dV_S^{\hat{\chi}}.}$$

Reduce to g-soliton equation on Fano orbifolds

- $(X, D) = Y/\langle v_{\hat{\chi}} \rangle$: Fano orbifold and an orbifold l.b. $L \to X$ satisfying:

$$-(K_X + D) = \gamma L.$$

The no-vanishing section of $|K_Y|$ and dV_Y:

$$s = dz \wedge dw^{\gamma}, \quad dV_Y = (\sqrt{-1})^{n^2+1} dz \wedge d\bar{z} \wedge dw^{\gamma} \wedge d\bar{w}^{\gamma}.$$

- $r = r_0 e^{\varphi/2}$: radius function w.r.t. $\hat{\chi} \leftrightarrow$ orbifold metric $h = h_0 e^{-\varphi}$ on $-K_X$.
$\eta = (\partial - \bar{\partial}) \log h$: connection form

$$(d\eta)^n \wedge \eta = \omega^n \wedge d\psi, \quad dV_S^{\hat{\chi}} = \iota_{\partial_r} dV_Y|_S = 2d\psi \wedge \Omega_{\varphi}$$

$$\eta(\hat{\xi}) = 1 + \frac{\theta_{v_\xi}(\varphi)}{n+1}.$$

Sasaki-Einstein metric on $S \Longleftrightarrow$ g-soliton equation on (X, D):

$$\boxed{\eta(\hat{\xi})^{-n-2} (d\eta)^n \wedge \eta = dV_S^{\hat{\chi}} \quad \Longleftrightarrow \quad (n + 1 + \theta_{v_\xi})^{-n-2} \omega_{\varphi}^n = e^{-\varphi} \Omega_0.}$$

Stability of affine cones vs. weighted stability

- $\mathcal{Y}$: a special test configuration of Y.
  $(\mathcal{X}, \mathcal{D}) := \mathcal{Y}_0 / \langle v_{\hat{\chi}} \rangle$ a special TC of (X, D)
  Volume of $\hat{\xi} = \frac{1}{n+1}\hat{\chi} + \xi \in \hat{\mathfrak{t}}^+(\mathcal{Y}_0)$:

$$\mathrm{vol}(\hat{\xi}) = \mathrm{vol}(S, (d\eta)^n \wedge \eta) = \int_{\mathcal{X}_0} \frac{\omega^n}{(1+\theta_{v_\xi})^{n+1}}$$

- With $g = (1 + \langle x, \xi \rangle)^{-n-2}$ and $\mathrm{Fut}_\xi \equiv 0$, we have:

$$\begin{aligned} \mathbf{D}^{\mathrm{NA}}(\mathcal{Y}) &= \frac{1}{\mathrm{V}_g} D_\zeta \mathrm{vol}(\hat{\xi}) = \frac{1}{\mathrm{V}_g} \int_{\mathcal{X}_0} \frac{-\theta_\zeta \omega^n}{(1+\theta_{v_\xi})^{n+2}} \\ &= \mathbf{D}_g^{\mathrm{NA}}(\mathcal{X}, \mathcal{D}, \mathcal{L}) \qquad (\text{ see } (2)) \end{aligned}$$

(poly)Stability of Y (Collins-Székelyhidi) $\longleftrightarrow$ g-Ding (poly)stability of (X, D)

YTD conjecture for Ricci-flat Kähler cones

The following theorem recovers a result of Collins-Székelyhidi. The argument works for general Fano cones (not necessarily with isolated singularities):

Theorem

A Fano cone Y admits a Ricci-flat Kähler cone metric if and only if Y is K-polystable.

Proof: use the following diagram and the YTD conjecture for (X, D):

$$\begin{tikzcd}
\text{stability for } Y \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, dashed] & g\text{-weighted stability for } (X, D) \arrow[d, leftrightarrow, "\textit{Han}-\textbf{L}."', "\textit{BLXZ}"] \\
\text{Ricci-flat Kähler cone metric on } Y \arrow[r, leftrightarrow] & g\text{-soliton on } (X, D)
\end{tikzcd}$$

For example, the YTD conjecture for toric g-solitons implies

Theorem (Futaki-Ono-Wang, Berman)

Any $\mathbb{Q}$-Gorenstein toric affine variety admits a Ricci-flat Kähler cone.

Thanks for your attention!