Kähler-Einstein metrics and K-stability

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May 3, 2012

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Basic Kähler geometry

$$
\begin{aligned}
(X, J, g) &\quad\longleftrightarrow\quad (X, J, \omega_g) \\
g(\cdot,\cdot) = \omega_g(\cdot, J\cdot) &\quad\longleftrightarrow\quad \omega_g(\cdot,\cdot) = g(J\cdot,\cdot) \\
\nabla^{LC} J = 0 &\quad\Longleftrightarrow\quad d\omega_g = 0
\end{aligned}
$$

Locally,

$$
\omega_g = \sqrt{-1}\sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad g = (g_{i\bar{j}}) > 0
$$

Curvature form

Riemannian curvature

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{r\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{r\bar{l}}}{\partial \bar{z}_j}$$

Ricci curvauture:

$$Ric(\omega) = -\frac{\sqrt{-1}}{2\pi} \log \omega^n = -\frac{\sqrt{-1}}{2\pi} \sum_{i,j} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{k\bar{l}}) dz^i \wedge d\bar{z}^j$$

Chern-Weil theory implies:

$$Ric(\omega) \in c_1(X)$$

Scalar curvature

Scalar curvature

$$S(\omega) = g^{i\bar{j}} Ric(\omega)_{i\bar{j}} = \frac{nRic(\omega) \wedge \omega^{n-1}}{\omega^n}$$

Average of Scalar curvature is a topological constant:

$$\underline{S} = \frac{\langle nc_1(X)[\omega]^{n-1}, [X]\rangle}{\langle [\omega]^n, [X]\rangle}$$

Kähler-Einstein equation

$$Ric(\omega_{KE}) = \lambda\omega_{KE}$$

Generalization of uniformization theorem of Riemann surfaces.

1. $\lambda = -1$: $-c_1(X) > 0$, canonically polarized. Existence: Aubin, Yau
2. $\lambda = 0$: $c_1(X) = 0$, Calabi-Yau manifold. Existence: Yau
3. $\lambda > 0$: $c_1(X) > 0$, Fano manifold. There are obstructions.

Kähler-Einstein metric on Fano manifold

Obstructions:

1. Automorphism group is reductive (Matsushima'57)
2. Futaki invariant vanishes (Futaki'83)
3. K-stability (Tian'97, Donaldson'02)

Existence result when Futaki invariant vanishes:

1. Homogeneous Fano manifold
2. Del Pezzo surface: Tian (1990)
3. toric Fano manifold: Wang-Zhu (2000)

Complex Monge-Ampère equation

PDE of Kähler-Einstein metric:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{h_\omega - \phi}\omega^n$$

Ricci potential:

$$Ric(\omega) - \omega = \partial\bar{\partial}h_\omega, \quad \int_X e^{h_\omega}\omega^n = \int_X \omega^n$$

Space of Kähler metrics in $[\omega]$:

$$\mathcal{H}(\omega) = \{\phi \in C^\infty(X); \omega_\phi := \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

Aubin-Yau Continuity Method and invariant $R(X)$

For $0 \leq t \leq 1$,

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)^n \quad = \quad e^{h_\omega - t\phi}\omega^n$$

This equivalent to

$$Ric(\omega_{\phi_t}) = t\omega_{\phi_t} + (1-t)\omega \qquad (*)_t$$

Define

$$R(X) = \sup\{t : (*)_t \text{ is solvable } \}$$

Tian('92) studied this invariant first. He estimated $R(Bl_p\mathbb{P}^2) \leq \frac{15}{16}$.

Other previous results

1. Székelyhidi('09) showed:
   - $R(X)$ is the same as

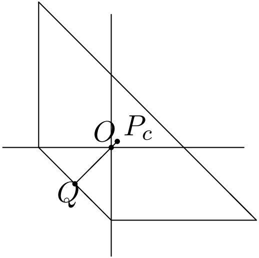
     $$\sup\{t : \exists \text{ a Kähler metric } \omega \in c_1(X) \text{ such that } Ric(\omega) > t\omega\}$$

     In particular, $R(X)$ is independent of reference metric ω.
   - $R(Bl_p\mathbb{P}^2) = \frac{6}{7}$. $R(Bl_{p,q}\mathbb{P}^2) \leq \frac{21}{25}$
2. Shi-Zhu('10) studied the limit behavior of solutions of continuity method on $Bl_p\mathbb{P}^2$.

Toric Fano manifolds

$\{$ toric Fano manifold $X_\triangle\} \longleftrightarrow \{$ reflexive lattice polytope $\triangle\}$

Example: $Bl_p\mathbb{P}^2$. P_c: Center of mass



$R(X)$ on any toric Fano manifold

Theorem (Li'09)

If $P_c \neq O$,

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_c Q}|}$$

Here $|\overline{OQ}|$, $|\overline{P_c Q}|$ are lengths of line segments $\overline{OQ}$ and $\overline{P_c Q}$. If $P_c = O$, then there is Kähler-Einstein metric on $X_\triangle$ and $R(X_\triangle) = 1$.

limit as $t \to R(X)$

Theorem (Li'11)

There exist biholomorphic transformations $\sigma_t : X_\triangle \to X_\triangle$, such that $\sigma_{t_i}^* \omega_{t_i}$ converge to a Kähler current $\omega_\infty = \omega + \partial\bar{\partial}\psi_\infty$, which satisfy a complex Monge-Ampère equation of the form

$$(\omega + \partial\bar{\partial}\psi_\infty)^n = e^{-R(X)\psi_\infty} \left(\sum_\alpha{}' b_\alpha \|s_\alpha\|^2 \right)^{-(1-R(X))} \Omega$$

In particular,

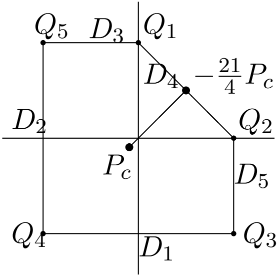
$$Ric(\omega_{\psi_\infty}) = R(X)\omega_{\psi_\infty} + (1 - R(X))\partial\bar{\partial}\log(\sum_\alpha{}' b_\alpha |s_\alpha|^2)$$

Corollaries

1. Conic type singularity for the limit metric (compatible with Cheeger-Colding's theory)
2. Partial C^0-estimate along continuity method $(*)_t$ for toric metrics. (Follows from convergence up to gauge transformation)
3. Calculate multiplier ideal sheaf for toric Fano manifold with large symmetries.

Singularity from the polytope

The nonzero terms in $\sum'$ can be determined by the geometry of the polytope!
Example: $R(Bl_{p,q}\mathbb{P}^2) = \frac{21}{25}$



Idea of the proof

Aubin-Yau Continuity method:

$$(\omega+\sqrt{-1}\partial\bar{\partial}\phi_t)^n=e^{h_\omega-t\phi}\omega^n$$

Solvability for $t\in[0,t_0]$ $\Longleftrightarrow$ uniform C^0-estimate on ϕ_t for $t\in[0,t_0]$.

toric invariant Kähler metrics

$$X_\triangle \backslash D \cong (\mathbb{C}^*)^n = \mathbb{R}^n \times (S^1)^n$$
$$z_i = (\log |z_i|^2, \frac{z_i}{|z_i|})$$

$(S^1)^n$-invariant Kähler metric $\longleftrightarrow$ convex function $u = u(x)$ on $\mathbb{R}^n$. Potential of reference $\omega = \omega_{FS}$:

$$\tilde{u} = \log \sum_\alpha |s_\alpha|^2 = \log \left(\sum_\alpha e^{\langle p_\alpha, x \rangle} \right).$$

Assume potential of $\omega_\phi = \omega + \partial\bar{\partial}\phi$ is a convex function u.
$\tilde{u}$, u are proper convex functions on $\mathbb{R}^n$ satisfying

$$D\tilde{u}(\mathbb{R}^n) = Du(\mathbb{R}^n) = Dw(\mathbb{R}^n) = \triangle$$

Real Monge-Ampère by torus symmetry

$$\det(u_{ij}) = e^{-(1-t)\tilde{u}-tu} \qquad (**)_t$$

$$u = \tilde{u} + \phi$$

Combined potential:

$$w_t(x) = tu(x) + (1-t)\tilde{u}$$

Define

$$m_t = inf\{w_t(x) : x \in \mathbb{R}^n\} = w_t(x_t)$$

Basic estimates

Proposition (Wang-Zhu)

1. *there exists a constant C, independent of $t < R(X_\triangle)$, such that*
$$|m_t| < C$$
2. *There exists $\kappa > 0$ and a constant C, both independent of $t < R(X_\triangle)$, such that*
$$w_t \geq \kappa |x - x_t| - C \quad (1)$$

Equivalent criterion for C^0 estimate

Proposition (Wang-Zhu)

Uniform bound of $|x_t|$ *for any* $0 \leq t \leq t_0 \iff C^0$*-estimates for the solution* ϕ_t *in* $(*)_t$ *for* $t \in [0, t_0]$.

Corollary

1. *If* $R(X_\triangle) < 1$, *then* $\lim_{t_i \to R(X_\triangle)} |x_{t_i}| = +\infty$.
2. *If* $R(X_\triangle) < 1$, *then there exists* $y_\infty \in \partial\triangle$, *such that*

$$\lim_{t_i \to R(X_\triangle)} D\tilde{u}_0(x_{t_i}) = y_\infty \tag{2}$$

Key identity

$$\frac{1}{Vol(\triangle)}\int_{\mathbb{R}^n} D\tilde{u}_0 e^{-w} dx = -\frac{t}{1-t}P_c \tag{3}$$

Remark

General formula: for any holomorphic vector field v*,*

$$-\int_X div_\Omega(v)\omega_t^n = \frac{t}{1-t}F(K_X^{-1}, v)$$

key observation

Assume the reflexive polytope $\triangle$ is defined by

$$\langle \lambda_r, x\rangle \geq -1, r = 1, \dots, N$$

Proposition (Li)

If $P_c \neq O$,

$$Q := -\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c \in \partial\triangle$$

Precisely,

$$\lambda_r(Q) \geq -1$$

Equality holds if and only if $\lambda(y_\infty) = -1$. So Q and y_∞ lie on the same faces.

Transformation

$\sigma_t : X_\triangle \longrightarrow X_\triangle$ in toric coordinate:

$$\sigma_t : x \longrightarrow x + x_t$$

In complex coordinate:

$$\sigma_t : z_i \longrightarrow e^{x_t^i/2} z_i$$

Transformation on the potential:

$$U(x) = \sigma_t^* u(x) - u(x_t) = u(x + x_t) - u(x_t)$$

$$\tilde{U}_t(x) = \sigma_t^* \tilde{u}_0(x) - \tilde{u}_0(x_t) = \tilde{u}_0(x + x_t) - \tilde{u}_0(x_t)$$

Regularization of singular Monge-Ampère

$$\det(U_{ij}) = e^{-tU-(1-t)\tilde{U}-w(x_t)} \Longleftrightarrow$$

$$(\omega+\partial\bar{\partial}\psi)^n = e^{-t\psi}\left(\sum_{\alpha} b(p_\alpha,t)\|s_\alpha\|^2\right)^{-(1-t)} e^{h_\omega-w(x_t)}\omega^n \qquad (**)'_t$$

$$\det(U_{ij}) = e^{-tU-(1-t)\tilde{U}_\infty-w(x_t)} \Longleftrightarrow$$

$$(\omega+\partial\bar{\partial}\psi)^n = e^{-R(X)\psi}\left(\sum_{\alpha}{}' b_\alpha\|s_\alpha\|^2\right)^{-(1-R(X))} e^{h_\omega-c}\omega^n \qquad (**)'_\infty$$

Uniform a priori estimate

1. C^0-estimate. The point is that: the gauge transformation σ_t offset the blow up of $\|\phi\|_\infty$ so that the transformed potential ψ has uniformly bounded $\|\psi\|_\infty$.
   Proof: Prove Harnack estimate:
   $$\sup_X(-\psi) \leq n \sup_X \psi + C(n)t^{-1}.$$
2. Partial C^2-estimate: Uniform C^2-estimate on compact set away from the blow up set.
   Proof: Adapt Yau's proof of C^2-estimate.
3. $C^{2,\alpha}$-estimate: Evans-Krylov estimate. This estimate is purely local.

My work on other continuity methods I

- (Conic Kähler-Einstein metric) Fix a smooth divisor $D \in |-K_X|$, construct Kähler-Einstein metric with conic singularity along the divisor D of cone angle $2\pi\beta$. This corresponds to solving the following equation:

$$Ric(\omega_{KE,\beta}) = \beta\omega_{KE,\beta} + (1-\beta)\{D\}$$

  My work in this direction:

  1. Construction of Kähler-Einstein metric with cone angle in $(\pi, 2\pi)$. (Joint with Yanir Rubinstein).
  2. Formulate log-K-stability (obstruction to the existence).

My work on other continuity methods II

- (Kähler-Ricci flow) My work: Construction of rotationally symmetric Kähler-Ricci solitons.
- (Complex structure continuity method) My work: still need to be done.

Obstruction to existence: Calabi-Futaki invariant

For any holomorphic vector field v, define $\theta_v = \mathcal{L}_v - \nabla_v$ on $K_X^{-1} = \wedge^n TX$, s.t. $\sqrt{-1}\bar{\partial}\theta_v = i_v\omega$.

$$F(K_X^{-1}, v) = -\int_X (S(\omega) - \underline{S})\theta_v \frac{\omega^n}{n!}$$

Theorem (Futaki)

$F(K_X^{-1}, v)$ does not depend on the choice of ω in $c_1(X)$.

Mabuchi energy and Ding-energy

Definition

1. *(Mabuchi-energy)*

$$\nu_\omega(\omega_\phi) = -\int_0^1 dt \int_X (S(\omega_{\phi_t}) - \underline{S})\dot{\phi}_t \omega_\phi^n / n!$$

2. *(Ding-energy)*

$$F_\omega(\phi) = F_\omega^0(\phi) - V \log\left(\frac{1}{V}\int_X e^{h_\omega - \phi}\omega^n/n!\right)$$

Tian's analytic criterion

Theorem (Tian'97)

If $Aut(X, J)$ is discrete. There exists a Kähler-Einstein metric on X if and only if either $F_\omega(\omega_\phi)$ or $\nu_\omega(\omega_\phi)$ is proper.

Definition

[Tian'97] A functional $F : \mathcal{H}(\omega) \to \mathbb{R}$ proper:

$$F(\omega_\phi) \geq f\left(J_\omega(\omega_\phi)\right), \text{ for any } \omega_\phi \in \mathcal{H}$$

where $f(t) : \mathbb{R}_+ \to \mathbb{R}$ is some monotone increasing function satisfying $\lim_{t\to+\infty} f(t) = +\infty$. $J_\omega(\omega_\phi)$ is a positive energy norm defined on the space $\mathcal{H}_\omega$.

Finite dimensional approximation I

1.

$$\begin{aligned}\mathcal{H}_k &= \{\text{ inner products on the vector space } H^0(X, L^{\otimes k})\} \\ &\cong GL(N_k, \mathbb{C})/U(N_k, C)\end{aligned}$$

2.

$$\mathcal{B}_k := \left\{ \frac{1}{k} \log \sum_{\alpha=1}^{N_k} |s_\alpha^{(k)}|^2_{h^{\otimes k}}; \{s_\alpha^{(k)}\} \text{ is a basis of } H^0(X, L^k) \right\}$$

Note that $\mathcal{B}_k \subset \mathcal{H}_\omega = \{\phi \in C^\infty(X); \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$.

Finite dimensional approximation II

Define two maps between $\mathcal{H}_k$ and $\mathcal{H}$.

$$
\begin{aligned}
\mathrm{Hilb}_k : \mathcal{H} &\longrightarrow \mathcal{H}_k \\
h &\mapsto \langle s_1, s_2 \rangle_{\mathrm{Hilb}_k(h)} = \int_X (s_1, s_2)_{h^{\otimes k}} \frac{\omega_h^n}{n!}, \\
\mathrm{FS}_k : \mathcal{H}_k &\longrightarrow \mathcal{B}_k \subset \mathcal{H} \\
H_k &\mapsto |s|^2_{\mathrm{FS}_k(H_k)} = \frac{|s|^2}{\left(\sum_{\alpha=1}^{N_k} \left| s_\alpha^{(k)} \right|^2\right)^{1/k}}, \quad \forall s \in L.
\end{aligned}
$$

In the above definition, $\{s_\alpha^{(k)}; 1 \le \alpha \le N_k\}$ is an orthonormal basis of the Hermitian complex vector space $(H^0(X, L^k), H_k)$.

Finite Approximation III

Theorem (Tian)

$$\mathcal{H} = \overline{\bigcup_k \mathcal{B}_k}$$

More precisely, for any $h \in \mathcal{H}$,

$$FS_k(\mathrm{Hilb}_k(h)) \to h$$

In terms of Kähler form:

$$\frac{1}{k}\sqrt{-1}\partial\bar{\partial}\log\sum_{\alpha=1}^{N_k}|s_\alpha^{(k)}|^2 \to \omega_h$$

One application: Quantization of Mabuchi-energy

Theorem (Donaldson'05)

1. *The k-th Chow-norm functionals approximate Mabuchi-energy as $k \to +\infty$.*
2. *If $Aut(X, L)$ is discrete, then constant scalar curvature Kähler metric in $c_1(L)$ obtains the minimum of Mabuchi-energy.*

Theorem (Li10)

Without assuming $Aut(X, L)$ is discrete, CSCK metric in $c_1(L)$ always obtains the minimum of Mabuchi-energy.

Remark

The more general case is proved by Chen-Tian('08).

Tian's conjecture and partial C^0-estimate

Conjecture (Tian)

There is a Kähler-Einstein metric on X if and only if for sufficiently large k, ν_ω is proper on $\mathcal{B}_k$.

This will follow from the following conjecture by Tian.

Conjecture (Partial C^0-estimate)

If $\omega_t = \omega_{h_t}$ are solutions in a 'continuity method' to KE problem, then when $k \gg 1$, $\exists C_k > 0$ independent of parameter t, such that

$$\rho_k(\omega_t) = \sum_{\alpha=1}^{N_k} |s_\alpha^{(k)}|^2_{h_t} \geq C > 0.$$

here $\{s_\alpha^{(k)}\}$ is orthonormal basis of $(H^0(X, K_X^{-\otimes k}), \mathrm{Hilb}_k(h_t))$.

Properness on $\mathcal{B}_k$: K-stability

Assume these conjectures. Then to see whether there is a Kähler-Einstein metric, we need to test whether the Mabuchi energy is proper on $\mathcal{B}_k$.

- Tian ('97) introduced algebraic condition: K-stability for testing such properness.
- Donaldson ('02) reformulated it for the more general setting.

Combine the above discussion, we have the following conjecture due to Tian, which is also a specialization of Yau-Tian-Donaldson conjecture to the Fano case:

Conjecture (Tian)

There exists a Kähler-Einstein metric on Fano manifold (X, J) if and only if $(X, -K_X)$ is K-polystable.

Test configuration

A test configuration is a direction in $\mathcal{B}_k$.

Definition

Let X be a Fano manifold. A $\mathbb{Q}$-test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, K_X^{-\otimes k})$ consists of

- *a variety $\mathcal{X}$ with a $\mathbb{C}^*$-action,*
- *a $\mathbb{C}^*$-equivariant ample line bundle $\mathcal{L} \to \mathcal{X}$,*
- *a flat $\mathbb{C}^*$-equivariant map $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication in the standard way $(t, a) \to ta$,*

such that for any $t \neq 0$, $(\mathcal{X}_t = \pi^{-1}(t), \mathcal{L}|_{\mathcal{X}_t})$ is isomorphic to $(X, K_X^{-\otimes k})$.

Algebraic definition of Futaki invariant

$$d_k = \dim H^0(X, \mathcal{O}_X(-rkK_X)) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

$\mathbb{C}^*$-weight of $H^0(X, \mathcal{O}_X(-rkK_X))$:

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

Definition

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = -\frac{F_1}{a_0} = \frac{a_1 b_0 - a_0 b_1}{a_0^2} \tag{4}$$

This is asymptotic slope in the direction determined by $(\mathcal{X}, \mathcal{L})$.

K-stability

Definition

(X, L) is K-semistable along $(\mathcal{X}, \mathcal{L})$ if $F(\mathcal{X}, \mathcal{L}) \geq 0$. Otherwise, it's unstable.
(X, L) is K-polystable for any test configuration $(\mathcal{X}, \mathcal{L})$ if $F(\mathcal{X}, \mathcal{L}) > 0$, or $F(\mathcal{X}, \mathcal{L}) = 0$ and the normalization $(\mathcal{X}^\nu, \mathcal{L}^\nu)$ is a product test configuration.
(X, L) is K-semistable (resp. K-polystable) if, for any integer $k > 0$, (X, L^k) is K-semistable (K-polystable) along any test configuration of (X, L^k).

Tian's original definition using special degeneration

Definition (Special Degeneration)

Special degeneration is a test configuration $(\mathcal{X}, \mathcal{L})$ *of a Fano manifold* $(X, K_X^{-\otimes k})$ *such that* $\mathcal{X}_0$ *is a* $\mathbb{Q}$*-Fano variety and* $\mathcal{L} = K_{\mathcal{X}/\mathbb{C}}^{-\otimes k}$.

Definition (Tian'97)

$(X, -K_X)$ *is K-semistable (resp. K-polystable) if* $(X, K_X^{-\otimes k})$ *is K-semistable (resp. K-polystable) along any special degeneration.*

Apply MMP to the problem of K-stability

The following result is from joint work with Dr. Chenyang Xu.
The result roughly says:

Theorem (Li-Xu)

For any test configuration, we can modify it using Minimal Model Program to get a special test configuration with smaller Donaldson-Futaki invariant.

K-stability and special test configuration

Theorem

(Li-Xu) For any test configuration $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}^1$*, we can construct a special test configuration* $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ *and a positive integer* m*, such that*

$$m\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \mathrm{DF}(\mathcal{X}^s, -rK_{\mathcal{X}^s}).$$

Furthermore, if we assume $\mathcal{X}$ *is normal, then the equality holds only when* $(\mathcal{X}, \mathcal{X}_0)$ *itself is a special test configuration.*

Tian's conjecture on K-stability of Fano manifolds

As a corollary of the above construction, we prove

Theorem

(Li-Xu) If X is destablized by a test configuration, then X is indeed destablized by a special test configuration. More precisely, the following statements true.

1. *(unstable) If $(X, -kK_X)$ is not K-semi-stable, then there exists a special test configuration $(\mathcal{X}^s, -kK_{\mathcal{X}^s})$ with a negative Futaki invariant $\mathrm{DF}(\mathcal{X}^s, -kK_{\mathcal{X}^s}) < 0$.*
2. *(semistable\stable) Let X be a K-semistable variety. If $(X, -kK_X)$ is not K-polystable, then there exists a special test configuration $(\mathcal{X}^{st}, -kK_{\mathcal{X}^s})$ with Donaldson-Futaki invariant 0 such that $\mathcal{X}^s$ is not isomorphic to $X \times \mathbb{C}$.*

Intersection formula of Futaki invariant

Proposition

(Tian, Paul-Tian, Zhang, Donaldson, Wang, Odaka, Li-Xu)
Assume $\mathcal{X}$ is normal, then

$$\mathrm{DF}(\mathcal{X},\mathcal{L}) = \frac{1}{(n+1)!2a_0}\left(\frac{2a_1}{a_0}\bar{\mathcal{L}}^{n+1} + (n+1)K_{\bar{\mathcal{X}}/\mathbb{P}^1}\cdot\bar{\mathcal{L}}^n\right). \quad (5)$$

MMP: Variation of polarization in the direction of K_X

Assume $\mathcal{L}|_{\mathcal{X}_t} \sim_{\mathbb{C}} -K_X$. We vary the polarization in the direction of K_X:

$$\bar{\mathcal{M}}_s = \frac{\bar{\mathcal{L}} + sK_{\bar{\mathcal{X}}}}{1-s}, \quad \bar{\mathcal{M}}_s|_{\bar{\mathcal{X}}_t} \sim -K_X$$

$$\frac{d}{ds}\bar{\mathcal{M}}_s = \frac{1}{(1-s)^2}(\bar{\mathcal{L}} + K_{\bar{\mathcal{X}}}) \overset{\tilde{s}=1/(1-s)}{\Longleftrightarrow} \frac{d}{d\tilde{s}}\bar{\mathcal{M}}_{\tilde{s}} = K_{\bar{\mathcal{X}}} + \bar{\mathcal{L}}$$

$$\left(\leftrightsquigarrow \text{ normlized Kähler-Ricci flow: } \frac{\partial \omega}{\partial \tilde{s}} = -Ric(\omega_{\tilde{s}}) + \omega_{\tilde{s}} \right)$$

MMP with scaling

Assume $\mathcal{X}$ has mild singularities (log canonical), we can run $(K_{\bar{\mathcal{X}}} + \bar{\mathcal{L}})$-MMP with scaling $\bar{\mathcal{L}}$:

$$\bar{\mathcal{M}}_\lambda = \frac{(K_{\bar{\mathcal{X}}} + \bar{\mathcal{L}}) + \lambda \bar{\mathcal{L}}}{\lambda}, \quad \bar{\mathcal{M}}_\infty = \bar{\mathcal{L}}.$$

Note that

$$K_{\bar{\mathcal{X}}} + \bar{\mathcal{M}}_\lambda = \frac{\lambda+1}{\lambda}(K_{\bar{\mathcal{X}}} + \bar{\mathcal{L}})$$

$$\frac{d}{d\lambda}\bar{\mathcal{M}}_\lambda = -\frac{1}{\lambda^2}(K_{\bar{\mathcal{X}}} + \bar{\mathcal{L}})$$

As λ decreases from $+\infty$ to 0, we get a sequence of critical points of λ and a sequence of models:

End product of MMP with scaling

$$\begin{array}{ccccccccc} +\infty & \geq & \lambda_1 & \geq & \dots & \geq & \lambda_k & > & \lambda_{k+1}=0 \\ \mathcal{X}^0 & \dashrightarrow & \mathcal{X}^1 & \dashrightarrow & \dots & \dashrightarrow & \mathcal{X}^k & \to & \mathbb{C} \end{array}$$

The $\mathcal{X}^k$ in the above sequence has very good properties:

1. $\mathcal{L}^k \sim_{\mathbb{C}} -K_{\mathcal{X}^k}$ is semiample.
2. Assume $\mathcal{X}_0 = \sum_{i=1}^N a_i \mathcal{X}_{0,i} = E \geq 0$, then $\mathcal{X} \dashrightarrow \mathcal{X}^0$ contracts precisely $Supp(E)$.

By property 1 above, we can define:

$$\mathcal{X}^{an} = Proj(\mathcal{X}^k, -K_{\mathcal{X}^k/C})$$

so that $-K_{\mathcal{X}^{an}}$ is ample.

Observation 1: Decreasing of Futaki invariant on a fixed model

Derivative of Donaldson-Futaki invariant along $\mathcal{M}_\lambda$:

$$\frac{d}{d\lambda}\mathrm{DF}(\mathcal{X}, \mathcal{M}_\lambda) = -C(n,\lambda)\bar{\mathcal{M}}_\lambda^{n-1} \cdot (\bar{\mathcal{L}} + K_{\bar{\mathcal{X}}/\mathbb{P}^1})^2 \geq 0$$

by the Hodge index theorem, because

$$\bar{\mathcal{L}} + K_{\bar{\mathcal{X}}/\mathbb{P}^1} = \sum_i a_i \mathcal{X}_{0,i}$$

only supports on $\mathcal{X}_0$. This means that DF invariant decreases as λ decreases.

Observation 2: Invariance of Futaki invariant at transition point

Two kinds of transitions:

1. Divisorial contraction
2. Flipping

Invariance of Donaldson-Futaki invariant comes from projection formula for intersection product.

A weaker statement: unstable case

Theorem

Given any test configuration $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}^1$*, for any* $\epsilon \ll 1$*, we can construct a special test configuration* $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ *and a positive integer* m*, such that*

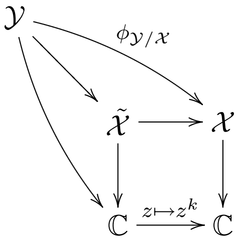
$$m(\epsilon + \mathrm{DF}(\mathcal{X}, \mathcal{L})) \geq \mathrm{DF}(\mathcal{X}^s, -rK_{\mathcal{X}^s})$$

Theorem

Tian's conjecture in the un-stable case holds.

unstable case I

Step 1: Equivariant semistable reduction:



such that

- $\mathcal{Y}$ is smooth
- $\mathcal{Y}_0 = \sum_{i=1}^{N_1} \mathcal{Y}_{0,i}$ is simple normal crossing.

unstable case II

Step 2: perturb the pull back polarization

$$\mathcal{L}_{\mathcal{Y}} = \epsilon A + \phi_{\mathcal{Y}/\mathcal{X}}^*(\mathcal{L})$$

by an ample divisor A ($\epsilon \ll 1$) such that

- $\mathcal{L}_{\mathcal{Y}}$ is still ample
- For some $a \in \mathbb{Q}$

$$\mathcal{L}_{\mathcal{Y}} + K_{\mathcal{Y}} + a\mathcal{Y}_0 = \sum_{i=\mathbf{2}}^{N} a_i \mathcal{Y}_{0,i}$$

with $a_i > 0$ for any $i \geq 2$.

unstable case III

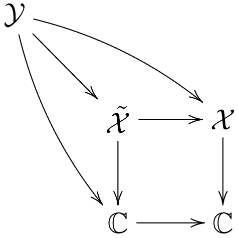
Step 3: Run $(K_{\mathcal{Y}} + \mathcal{L}_{\mathcal{Y}})$-MMP over $\mathbb{C}$. The end product is a special test configuration with $\mathbb{Q}$-Fano variety which is the strict transform of $\mathcal{Y}_{0,1}$.

Assume the $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$ then

1. negativity is preserved under small perturbation
2. along MMP, Futaki invariant is always decreasing on a fixed model and does not change at the point of transition to a new model.

Stable case I

How to eliminate the perturbation? More steps:
Step 1: Equivariant semistable reduction.



- $\mathcal{Y}$ is smooth
- $\mathcal{Y}_0 = \sum_{i=1}^{N_1} \mathcal{Y}_{0,i}$ is simple normal crossing.

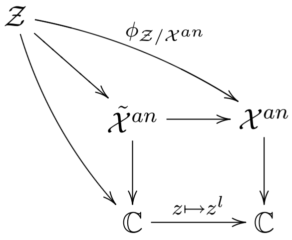
Step 2: log canonical modification: Run $(K_{\mathcal{Y}} + \mathcal{Y}_0)$-MMP over $\tilde{\mathcal{X}}$ to get $\mathcal{X}^{lc}$ such that.

- $\mathcal{X}^{lc}$ is not too singular(log canonical) so that we can run MMP
- $K_{\mathcal{X}^{lc}}$ is relatively ample so that we can perturb $\phi^*_{\mathcal{X}^{lc}}\tilde{\mathcal{L}}$ along the direction of $K_{\mathcal{X}^{lc}}$ to get $\mathcal{L}^{lc}$.

$\mathcal{X}^{lc}$ is a kind of canonical partial resolution of $\tilde{\mathcal{X}}$.
Step 3: Run $(K_{\mathcal{X}^{lc}} + \mathcal{L}^{lc})$-MMP over $\mathbb{C}$ to get $\mathcal{X}^{an}$ such that $\mathcal{L}^{an} \sim -K_{\mathcal{X}^{an}}$.

Stable case II

Step 4: Take equivariant semistable reduction of $\mathcal{X}^{an}$:



- $\mathcal{Z}$ is smooth
- $\mathcal{Z}_0 = \sum_{i=1}^{N_2} \mathcal{Z}_{0,i}$ is simple normal crossing.

Then choose ample divisor A to perturb such that

- There exists $a \in \mathbb{Q}$, and $a_i > 0$ for $i = 2, \ldots, N_2$

$$\phi^*_{\mathcal{Z}/\mathcal{X}}(\mathcal{L}^{an}) + \epsilon A + K_{\mathcal{Z}} + a\mathcal{Z}_0 = \sum_{i=2}^{N_2} a_i \mathcal{Z}_{0,i}$$

- $a(E_1, \tilde{\mathcal{X}}^{an}) = 0$.

Now one extract E_1 on $\tilde{\mathcal{X}}^{an}$ to get $\phi_{\mathcal{X}'/\tilde{\mathcal{X}}^{an}} : \mathcal{X}' \to \tilde{\mathcal{X}}^{an}$ such that $\phi^*_{\mathcal{X}'/\tilde{\mathcal{X}}^{an}} K_{\tilde{\mathcal{X}}^{an}} = K_{\mathcal{X}'}$.

Stable Case III

Step 5: Run $(K_{\mathcal{X}'}+\mathcal{L}')$-MMP over $\mathbb{C}$ to get $\mathcal{X}^s$. Note that

$$K_{\mathcal{X}'}+\mathcal{L}'=\sum_{i=2}^{N_3}a_i\mathcal{X}'_{0,i}$$

with $a_i>0$ for $i=2,\dots,N_3$. So $\mathcal{X}^s_0$ is the strict transform of $\mathcal{Z}_{0,1}$ (or $\mathcal{X}'_{0,1}$).

Steps of modification

1. (step 0): normalization
2. (step 1): equivariant semistable reduction
3. (step 2): log-canonical modification
4. (step 3): MMP with scaling
5. (step 4): base change and crepant blow up
6. (step 5): contracting extra components

Final remark: log (pair) case

The above results can be generalized to the pair case (X, D). At least when D is smooth, we have the following connection: conic Kähler-Einstein metric on $(X, D) \longleftrightarrow$ log-K-stability of the pair (X, D).

Thank you!

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