Geodesic rays and stability in the cscK problem

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Uniformatization Theorem for Riemann Surfaces

Riemann surface: surface with a complex structure:

Topology	Metric	Curvature
$\mathbb{S}^2 = \mathbb{CP}^1$	spherical	1
$\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$	flat	0
$\Sigma_g = \mathbb{B}^1/\pi_1(\Sigma_g)$	hyperbolic	-1

Σ_g closed oriented surface of genus $g \geq 2$.
$\mathbb{B}^1 = \{z \in \mathbb{C}; |z| < 1\}$.

Generalization for higher dimensional complex projective manifolds?

Kähler manifolds and Kähler metrics

X: complex manifold, $\{(U_\alpha, z_1, \dots, z_n)\}$.
Kähler form: a smooth closed positive $(1,1)$-form:

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad (g_{i\bar{j}}) > 0.$$

$d\omega = 0 \Longrightarrow$ Kähler class $[\omega] \in H^2(X, \mathbb{R}) \cap H^{1,1}_{\bar{\partial}}(X, \mathbb{C})$.

Local $\partial\bar{\partial}$-Lemma: $\exists$ local potentials $\varphi_0 = \{(\varphi_0)_\alpha \in C^\infty(U_\alpha)\}$

$$\omega_0 = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_0 =: \frac{\sqrt{-1}}{2\pi} \frac{\partial^2 \varphi_0}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j = dd^c \varphi_0.$$

Global $\partial\bar{\partial}$-Lemma: any Kähler form in $[\omega]$ can be written as

$$dd^c\varphi \;\; := \;\; \omega_0 + \sqrt{-1}\partial\bar{\partial}u = \sqrt{-1} \sum_{i,j} \left((\varphi_0)_{i\bar{j}} + u_{i\bar{j}} \right) dz^i \wedge d\bar{z}^j$$

where $\varphi = \varphi_0 + u$ is locally defined, while $u = \varphi - \varphi_0$ and $dd^c\varphi$ are globally defined.

Constant scalar curvature Kähler (cscK) metrics

Ricci curvature:

$$R_{i\bar{j}} := Ric(dd^c\varphi)_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det\left(\varphi_{k\bar{l}}\right).$$

Scalar curvature:

$$\begin{aligned} S(dd^c\varphi) &= g^{i\bar{j}} R_{i\bar{j}} \\ &= -g^{i\bar{j}} \frac{\partial}{\partial z_i \partial \bar{z}_j} \log\det\left(\varphi_{k\bar{l}}\right). \end{aligned}$$

cscK equation is a 4-th order highly nonlinear equation:

$$S(dd^c\varphi) = \underline{S}.$$

$\underline{S}$ is a topological constant:

$$\underline{S} = \frac{n\langle c_1(X) \wedge [\omega]^{n-1}, X\rangle}{\langle [\omega]^n, X\rangle}.$$

Kähler metric as curvature forms

If $[\omega] \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$, then $[\omega] = c_1(L)$ for an ample holomorphic line bundle L over X and $\omega = dd^c\varphi$ for a Hermitian metric $e^{-\varphi}$ on L.

Holomorphic line bundle: transition functions $f_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$.

$$L = \left(\bigsqcup_\alpha U_\alpha \times \mathbb{C} \right) / \{s_\alpha = f_{\alpha\beta} s_\beta\}.$$

Hermitian metrics: $e^{-\varphi} := \{e^{-\varphi_\alpha}\}$ Hermitian metric on L:

$$e^{-\varphi_\alpha} = |f_{\alpha\beta}|^2 e^{-\varphi_\beta}.$$

$\partial\bar{\partial}$-lemma: Fix any reference metric $e^{-\varphi_0}$, then $\exists u \in C^\infty(X)$ s.t.

$$e^{-\varphi} = e^{-\varphi_0} e^{-u}.$$

Chern curvature

$$dd^c\varphi = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_\alpha.$$

Yau-Tian-Donaldson (YTD) conjecture

Conjecture (YTD conjecture)

(X, L) admits a cscK metric if and only if (X, L) is $\mathrm{Aut}(X, L)_0$-uniformly K-stable for test configurations.

The only if direction of this Conjecture is known to be true.

Example:

If $L = -K_X$ ample, then X is Fano and cscK=Kähler-Einstein.

In this case the above YTD conjecture is equivalent to the results of Tian, Chen-Donaldson-Sun, Berman. The existence part depends on Cheeger-Colding-Tian theory and partial C^0-estimates.

Different variational approach, based on pluripotential theory and non-Archimedean geometry, works also for singular Fano varieties and has been successfully carried out by Berman-Boucksom-Jonsson, **L.** -Tian-Wang, Hisamoto and **L.** .

Moreover the K-stability condition for Fano varieties are in many cases checkable.

Main results

Theorem (L. '20)

Let $\mathbb{G}$ be a reductive subgroup of $\mathrm{Aut}(X, L)_0$. If (X, L) is $\mathbb{G}$-uniformly K-stable for models (or for filtrations), then (X, L) admits a cscK metric.

We have implications and conjecture they are all equivalent:
$\mathrm{Aut}(X, L)_0$-uniformly K-stable for models $\Longrightarrow$ cscK
$\Longrightarrow$ $\mathrm{Aut}(X, L)_0$-uniformly K-stable for test configurations

Applications: reproving the toric YTD conjecture (without Donaldson's toric analysis):

Theorem (Donaldson, Zhou-Zhu, Chen-Li-Sheng, Hisamoto, Chen-Cheng, L.)

A polarized toric manifold (X, L) admits a cscK metric if and only if (X, L) is $(\mathbb{C}^*)^r$-uniformly K-stable.

Mabuchi functional (K-energy): Chen-Tian's formula:

$$\begin{aligned}\mathbf{M}(\varphi) &= -\int_0^1 dt \int_X \dot{\varphi}\cdot(S(\varphi(t)) - \underline{S})(dd^c\varphi(t))^n \\ &= \mathbf{H}(\varphi) - \mathbf{H}(\varphi_0) + \mathbf{E}^{-Ric(\Omega)}(\varphi) + \frac{\underline{S}}{n+1}\mathbf{E}(\varphi).\end{aligned}$$

Entropy, twisted energy and Monge-Ampère energy:

$$\begin{aligned}\mathbf{H}(\varphi) &= \int_X \log\frac{(dd^c\varphi)^n}{\Omega}(dd^c\varphi)^n. \\ \frac{d}{dt}\mathbf{E}^{-Ric(\Omega)}(\varphi) &= -n\int_X \dot{\varphi} Ric(\Omega)\wedge(dd^c\varphi)^{n-1}. \\ \frac{d}{dt}\mathbf{E}(\varphi) &= \int_X \dot{\varphi}(dd^c\varphi)^n.\end{aligned}$$

Space of Kähler metrics

Space of smooth Kähler metrics:

$$\mathcal{H} = \{\varphi = \varphi_0 + u; u \in C^\infty(X), \omega_0 + dd^c u > 0\}.$$

Finite energy metrics as Completion of $\mathcal{H}$ (Cegrell, Guedj-Zeriahi)

$$\begin{aligned} \mathcal{E}^1 &= \{\varphi \in \mathrm{PSH}(X, [\omega]); \\ &\qquad \mathbf{E}(\varphi) := \inf\{\mathbf{E}(\tilde\varphi); \tilde\varphi \geq \varphi, \tilde\varphi \in \mathcal{H}\} > -\infty\}. \end{aligned}$$

Strong topology on $\mathcal{E}^1$: $\varphi_m \to \varphi$ strongly if $\varphi_m \to \varphi$ in $L^1(\omega^n)$ and $\mathbf{E}(\varphi_m) \to \mathbf{E}(\varphi)$.

All 3-parts in **M** are defined on $\mathcal{E}^1$. There is a norm-like energy:

$$\begin{aligned} \mathbf{J}(\varphi) &= \int_X (\varphi - \varphi_0)(dd^c\varphi)^n - \mathbf{E}(\varphi) \\ &= \sum_{i=0}^{n-1} \frac{n-i}{n+1} \frac{\sqrt{-1}}{2\pi} \int_X \partial u \wedge \bar\partial u \wedge \omega_u^i \wedge \omega^{n-1-i} \geq 0. \end{aligned}$$

Definition

Given $\varphi_1, \varphi_2 \in \mathcal{E}^1$, *a geodesic segment joining* φ_1, φ_2 *is:*

$$\Phi = \sup\{\tilde{\Phi} \in \mathrm{PSH}(X \times [s_1, s_2] \times S^1, p_1^* L); \tilde{\Phi}(\cdot, s_i) \leq \varphi_i, i = 1, 2\}.$$

A geodesic ray emanating from φ_0 *is a map* $\Phi : \mathbb{R}_{\geq 0} \to \mathcal{E}^1$ *s.t.* $\forall s_1, s_2 \in \mathbb{R}_{\geq 0}$, $\Phi|_{[s_1, s_2]}$ *is the geodesic segment joining* $\varphi(s_1)$ *and* $\varphi(s_2)$, *and* $\Phi(\cdot, 0) = \varphi_0$.

- Geodesics originates from Mabuchi's L^2-metric on $\mathcal{H}$ and satisfies the Homogeneous Complex Monge-Ampère (HCMA) equation in pluripotential sense (Semmes, Donaldson):

$$(\sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} = 0.$$

- $\mathbf{E}(\varphi(s))$ is linear with respect to s.
- $\sup(\varphi(s) - \varphi_0)$ is linear with respect to s.

CscK metrics are minimizers of Mabuchi functional

Theorem (Chen-Tian, Berman-Berndtsson, Berman-Darvas-Lu)

$\mathbf{M}$ *is convex along geodesics in* $\mathcal{E}^1$. *It is linear if and only if the geodesic is generated by holomorphic vector fields.*

Consequences of convexity:

Theorem (Berman-Berndtsson, Berman-Darvas-Lu)

CscK metrics obtain the minimum of $\mathbf{M}$ *over* $\mathcal{E}^1$. *Moreover (smooth) cscK metrics are unique up to* $Aut(X, [\omega])_0$.

This reproves and generalizes previous results of Chen-Tian, Donaldson and Mabuchi.

Variational criterion

$\mathbb{G}$: a reductive Lie group, $\mathbb{G} = \mathbb{K}^{\mathbb{C}}$ and $\mathbb{T} \cong (\mathbb{C}^*)^r$ the center of $\mathbb{G}$.

Definition (Tian, refined by Darvas-Rubinstein and Hisamoto)

M *is $\mathbb{G}$-coercive if there exists $\gamma > 0$ such that for any $\varphi \in \mathcal{H}^{\mathbb{K}}$,*

$$\mathbf{M}(\varphi) \geq \gamma \cdot \mathbf{J}_{\mathbb{T}}(\varphi),$$

where $\mathbf{J}_{\mathbb{T}}(\varphi) := \inf_{\sigma \in \mathbb{T}} \mathbf{J}(\sigma^* \varphi)$.

We have hard results:

Theorem (Chen-Cheng, Darvas-Rubinstein, Berman-Darvas-Lu)

Tian's properness conjecture is true: there exists a cscK metric in $(X, [\omega])$ if and only if **M** *is $Aut(X, [\omega])_0$-coercive.*

Hisamoto, **L.** : $\mathrm{Aut}(X, [\omega])_0$ can be replaced by any reductive $\mathbb{G}$ that contains a maximal torus of $\mathrm{Aut}(X, [\omega])_0$.

Criterion via destabilizing geodesic rays

For a geodesic ray Φ and a functional **F** defined over $\mathcal{E}^1$, set:

$$\mathbf{F}'^{\infty}(\Phi) = \lim_{s\to+\infty} \frac{\mathbf{F}(\varphi(s))}{s}.$$

The limit exists for all $\mathbf{F} \in \{\mathbf{E}, \mathbf{E}^{-Ric(\Omega)}, \mathbf{H}, \mathbf{M}, \mathbf{J}, \mathbf{J}_{\mathbb{T}}\}$.

Based on compactness result about strong topology in Berman-Boucksom-Eyssidieux-Guedj-Zeriahi (BBEGZ), destabilizing sequence produces destabilizing a geodesic ray:

Theorem (Darvas-He, Chen-Cheng, Berman-Boucksom-Jonsson)

M *is* $\mathbb{G}$*-coercive iff there exists* $\gamma > 0$ *s.t. for any geodesic ray* Φ,

$$\mathbf{M}'^{\infty}(\Phi) \geq \gamma \cdot \mathbf{J}'^{\infty}_{\mathbb{T}}(\Phi).$$

Contact with algebraic geometry: Test configurations (Tian, Donaldson)

A test configuration (TC) $(\mathcal{X}, \mathcal{L})$ is a $\mathbb{C}^*$-equivariant degeneration of (X, L):

1. $\pi : \mathcal{X} \to \mathbb{C}$: a $\mathbb{C}^*$-equivariant family of projective varieties;
2. $\mathcal{L} \to \mathcal{X}$: a $\mathbb{C}^*$-equiv. semiample holomorphic $\mathbb{Q}$-line bundle;
3. $\eta : (\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^* \cong (X, L) \times \mathbb{C}^*$.

Trivial test configuration: $(X_{\mathbb{C}}, L_{\mathbb{C}}) := (X, L) \times \mathbb{C}$.

$(\mathcal{X}, \mathcal{L})$ is dominating if there is a $\mathbb{C}^*$-equivariant birational morphism $\rho : \mathcal{X} \to X \times \mathbb{C}$.

Under the isomorphism η, psh metrics on $\mathcal{L}|_{\pi^{-1}(\mathbb{C}^*)}$ are considered as *subgeodesic* rays on (X, L).

Geodesic rays from test configurations

For any TC $(\mathcal{X}, \mathcal{L})$, there are many smooth subgeodesic ray which extend to be a smooth psh metrics on $\mathcal{L}$.

Theorem (Phong-Sturm)

For any test configuration, there exists a unique geodesic ray Φ emanating from φ_0 s.t. Φ extends to a bounded psh metric on $\mathcal{L}$.

Φ is obtained by solving the HCMA on a resolution of $\mathcal{X}$:

$$(\mu^*(dd^c\tilde{\Phi}) + U)^{n+1} = 0; \quad U|_{X\times S^1} = 0,$$

where $\tilde{\Phi}$ is any smooth positively curved Hermitian metric on $\mathcal{L}$. In general the solution $\Phi := \tilde{\Phi} + U$ is at most $C^{1,1}$ (Phong-Sturm, Chu-Tosatti-Weinkove).

Mabuchi slopes along (sub)geodesic rays on TCs

For any TC $(\mathcal{X}, \mathcal{L})$, set:

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^{\cdot n} + \frac{S}{n+1} \bar{\mathcal{L}}^{\cdot n+1}$$

$$\mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \bar{\mathcal{L}} \cdot L^{\cdot n}_{\mathbb{P}^1} - \frac{\bar{\mathcal{L}}^{\cdot n+1}}{n+1}.$$

Theorem (Tian, Boucksom-Hisamoto-Jonsson)

For any smooth psh metric Φ on $\mathcal{L}$, we have the slope formula:

$$\mathbf{M}'^{\infty}(\Phi) = \mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{1}{d} \mathrm{CM}((\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}, t \mapsto t^d} \mathbb{C}).$$

Theorem (L. '20 (Xia proved $\leq$))

If Φ is the geodesic ray associated to $(\mathcal{X}, \mathcal{L})$, then:

$$\mathbf{M}'^{\infty}(\Phi) = \mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}).$$

Proposition (Hisamoto)

For any $\mathbb{G}$-equivariant test configuration $(\mathcal{X}, \mathcal{L})$,

$$\mathbf{J}'^{\infty}_{\mathbb{T}}(\Phi) = \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L}) := \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi).$$

Definition (Tian, Donaldson, Székelyhidi, Dervan, BHJ, Hisamoto)

(X, L) is $\mathbb{G}$-uniformly K-stable if there exists $\gamma > 0$ such that for any $\mathbb{G}$-equivariant test configuration $(\mathcal{X}, \mathcal{L})$,

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L}). \tag{1}$$

Proposition (Hisamoto for $\mathrm{Aut}(X, L)_0$, L. for general $\mathbb{G}$)

Assume that (X, L) admits a cscK metric. If $\mathbb{G}$ contains a maximal torus of $\mathrm{Aut}(X, L)_0$, then (X, L) is $\mathbb{G}$-uniformly K-stable.

Berkovich's analytic space

Let X be a projective variety defined over $\mathbb{C}$.

- If $\mathbb{C}$ is endowed with the standard (Archimedean) absolute valuation, then X^{an} is the usual complex analytic manifold.
- If $\mathbb{C}$ is given the trivial valuation, then $(X^{\mathrm{an}}, L^{\mathrm{an}})$ is the non-Archimedean Berkovich space. The set of divisorial valuations $X_{\mathbb{Q}}^{\mathrm{div}}$ is dense in $X^{\mathrm{NA}} := X^{\mathrm{an}}$. A metric ϕ on $L^{\mathrm{NA}} := L^{\mathrm{an}}$ is represented by the function $\phi - \phi_{\mathrm{triv}}$ on $X_{\mathbb{Q}}^{\mathrm{div}}$.

Each (dominating) TC $(\mathcal{X}, \mathcal{L})$ defines a smooth NA metric: $\forall v \in X_{\mathbb{Q}}^{\mathrm{div}}$, if $G(v) \in (X \times \mathbb{C})_{\mathbb{Q}}^{\mathrm{div}}$ is the Gauss extension (i.e. $G(v)$ is $\mathbb{C}^*$-invariant extension of v satisfying $G(v)(t) = 1$), we have

$$f_{\mathcal{L}}(v) := f_{(\mathcal{X},\mathcal{L})}(v) = G(v)(\mathcal{L} - \rho^* L_{\mathbb{C}}).$$

Smooth NA psh metrics $\Leftrightarrow$ equivalence class of test configurations

$$\mathcal{H}^{\mathrm{NA}}(L) = \{\phi_{(\mathcal{X},\mathcal{L})} := \phi_{\mathrm{triv}} + f_{\mathcal{L}}; (\mathcal{X}, \mathcal{L}) \text{ is a test configuration}\}.$$

Non-Archimedean $\mathcal{E}^{1,\mathrm{NA}}$ (by Boucksom-Favre-Jonsson)

For any $\phi = \phi_{(\mathcal{X},\mathcal{L})} \in \mathcal{H}^{\mathrm{NA}}$, set:

$$\mathbf{E}^{\mathrm{NA}}(\phi) := \frac{\bar{\mathcal{L}}^{\cdot n+1}}{n+1}.$$

Non-Archimedean version of PSH/finite energy metrics:

$$\begin{aligned}
\mathrm{PSH}^{\mathrm{NA}}(L) &= \{\phi : X^{\mathrm{div}}_{\mathbb{Q}} \to \mathbb{R} \cup \{-\infty\}; \exists \text{ a decreasing sequence} \\
&\qquad \phi_{(\mathcal{X}_m,\mathcal{L}_m)} \in \mathcal{H}^{\mathrm{NA}} \text{ such that } \phi = \lim_{m\to+\infty} \phi_{(\mathcal{X}_m,\mathcal{L}_m)}\}, \\
\mathcal{E}^{1,\mathrm{NA}} &= \{\phi \in \mathrm{PSH}^{\mathrm{NA}}; \\
&\qquad \mathbf{E}^{\mathrm{NA}}(\phi) := \inf\{\mathbf{E}^{\mathrm{NA}}(\tilde{\phi}); \tilde{\phi} \geq \phi\} > -\infty\}.
\end{aligned}$$

Strong topology: $\phi_m \to \phi$ strongly if converges pointwise and $\mathbf{E}^{\mathrm{NA}}(\phi_m) \to \mathbf{E}^{\mathrm{NA}}(\phi)$.
All Archimedean functionals before can be defined on $\mathcal{E}^{1,\mathrm{NA}}$.

Theorem (Boucksom-Favre-Jonsson, Boucksom-Jonsson)

$\exists$ operator $\mathrm{MA}^{\mathrm{NA}} : \mathcal{E}^1 \to \mathcal{M}^{1,\mathrm{NA}}$ (finite energy radon measures):

1. *For any TC $(\mathcal{X}, \mathcal{L})$, one recovers Chambert-Loir's measure:*

$$\mathrm{MA}^{\mathrm{NA}}(\phi_{(\mathcal{X},\mathcal{L})}) = \sum_j b_j \left(\mathcal{L}|_{E_j}\right)^{\cdot n} \delta_{x_j}, \qquad (2)$$

   where $x_j = b_j^{-1} r(\mathrm{ord}_{E_j}) \in X_{\mathbb{Q}}^{\mathrm{div}}$ with $\mathcal{X}_0 = \sum_j b_j E_j$.

2. *The Monge-Ampère operator defines a homeomorphism*

$$\mathrm{MA}^{\mathrm{NA}} : \mathcal{E}^{1,\mathrm{NA}}(L)/\mathbb{R} \to \mathcal{M}^{1,\mathrm{NA}} \qquad (3)$$

   w.r.t. the strong topology. Moreover, if ν is a Radon measure supported on a dual complex $\Delta_{\mathcal{X}}$ for a SNC model $\mathcal{X}$, then $(\mathrm{MA}^{\mathrm{NA}})^{-1}(\nu)$ is continuous.

Non-Archimedean metrics from geodesic rays

A subgeodesic ray $\Phi = \{\varphi(s)\}_{s\geq 0}$ is of linear growth if

$$\sup_{s>0} \frac{\sup(\varphi(s) - \varphi_0)}{s} < +\infty.$$

Subgeodesic rays of linear growth define non-Archimedean metrics:

$$\Phi^{\mathrm{NA}}(v) = -G(v)(\Phi), \quad \forall v \in X_{\mathbb{Q}}^{\mathrm{div}}.$$

$\Phi^{\mathrm{NA}} \in \mathcal{E}^{1,\mathrm{NA}}$ as a decreasing limit of $\phi_m \in \mathcal{H}^{\mathrm{NA}}$:

1. Consider the multiplier ideal sheaf (MIS) over $X \times \mathbb{C}$:

$$\mathcal{J}(m\Phi)(\mathcal{U}) = \left\{ f \in \mathcal{O}(\mathcal{U}); \int_{\mathcal{U}} |f|^2 e^{-m\Phi} < +\infty \right\}.$$

2. $\mu_m : \mathcal{X}_m = Bl_{\mathcal{J}(m\Phi)} X_{\mathbb{C}} \to X_{\mathbb{C}}$, $\mathcal{L}_m = \mu_m^* L_{\mathbb{C}} - \frac{1}{m+m_0} E_m$.
   - Using the Nadel vanishing and global generation property of MIS, $(\mathcal{X}_m, \mathcal{L}_m)$ is a test configuration of (X, L)
   - Using valuative description of MIS (Boucksom-Favre-Jonsson), $\phi_m := \phi_{(\mathcal{X}_m, \mathcal{L}_m)}$ decreases to ϕ.

Maximal geodesic rays

Definition (Berman-Boucksom-Jonsson (BBJ))

A geodesic ray Φ *is maximal if for any subgeodesic ray* $\tilde{\Phi}$ *satisfying* $\tilde{\Phi}_{\mathrm{NA}} \leq \Phi_{\mathrm{NA}}$, *we have* $\tilde{\Phi} \leq \Phi$.

Theorem (Berman-Boucksom-Jonsson)

There is a one-to-one correspondence between $\mathcal{E}^{1,\mathrm{NA}}$ *and the set of maximal geodesic rays. For any maximal geodesic ray* Φ, *we have:*

$$\mathbf{E}'^{\infty}(\Phi) = \mathbf{E}^{\mathrm{NA}}(\Phi_{\mathrm{NA}}).$$

- Not every geodesic ray is maximal (examples of Darvas, BBJ).
- Maximal geodesic rays are exactly those that are algebraically approximable, i.e. approximable by geodesic rays associated to test configurations. Moreover for such approximations:

$$\lim_{m \to +\infty} \mathbf{E}'^{\infty}(\Phi_m) = \mathbf{E}'^{\infty}(\Phi).$$

Non-Archimedean metrics from Models

In the definition of a test configuration $(\mathcal{X}, \mathcal{L})$, if we don't require $\mathcal{L}$ to be semiample, then we say that $(\mathcal{X}, \mathcal{L})$ is a model of (X, L). Let $\mathfrak{b}_m$ be the relative base ideal of $m\mathcal{L}$ and set

$$\mathcal{X}_m = \mathrm{Bl}_{\mathfrak{b}_m}\mathcal{X} \stackrel{\mu_m}{\to} \mathcal{X}, \quad \mathcal{L}_m = \mu_m^*\mathcal{L} - \frac{1}{m}E_m.$$

We associate a model psh metric:

$$\phi_{\mathcal{L}} := \phi_{(\mathcal{X},\mathcal{L})} := \lim_{m\to+\infty} \phi_{(\mathcal{X}_m,\mathcal{L}_m)}.$$

Theorem-Definition (Movable Intersection Formula, L. '20)

For $\phi = \phi_{(\mathcal{X},\mathcal{L})}$, with $\mathcal{L}_c = \mathcal{L} + c\mathcal{X}_0, c \gg 1$,

$$\mathbf{M}^{\mathrm{NA}}(\phi) := \langle \bar{\mathcal{L}}_c^n \rangle \cdot \left(K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} + \frac{\underline{S}}{n+1}\bar{\mathcal{L}}_c \right)$$

where $\langle \cdot \rangle$ is the movable intersection product of big line bundles studied in Boucksom-Demailly-Păun-Peternell.

K-stability for models

Model psh metric by using associated filtration $\mathcal{F}R_\bullet = \{\mathcal{F}^\lambda R_m\}$:

$$\mathcal{F}^\lambda H^0(X, mL) = \{s \in H^0(X, mL); t^{-\lceil\lambda\rceil}\bar{s} \in H^0(\mathcal{X}, m\mathcal{L})\}.$$

To any filtration $\mathcal{F}R_\bullet$, one can associate a maximal geodesic ray (Ross-WittNyström) and a lower regularizable NA psh metric (Boucksom-Jonsson, Székelyhidi).
$\phi_{\mathcal{L}}$ is also a non-Archimedean envelope which is always continuous:

$$\phi_{\mathcal{L}} = \sup\{\phi \in \mathrm{PSH}^{\mathrm{NA}}(L); \phi - \phi_{\mathrm{triv}} \leq f_{\mathcal{L}}\}.$$

Definition (L.)

(X, L) is $\mathbb{G}$-uniformly K-stable for models if $\exists\gamma > 0$ such that for any model $(\mathcal{X}, \mathcal{L})$,

$$\mathbf{M}^{\mathrm{NA}}(\phi_{(\mathcal{X},\mathcal{L})}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\phi_{(\mathcal{X},\mathcal{L})}).$$

Key result: destabilizing geodesic rays are maximal

Theorem (Thm A, L. , '20)

A geodesic ray Φ *satisfies* $\mathbf{M}'^{\infty}(\Phi) < +\infty$ *is necessarily maximal.*

The proof uses two key ingredients: equisingularity of multiplier approximation (via valuative description of MIS) and Jensen's inequality (motivated by Tian's α-type estimate): for any $\alpha > 0$,

$$\begin{aligned} C(\alpha) &> \log \int_{X\times\mathbb{D}} e^{\alpha(\hat{\Phi}-\Phi)}\Omega\sqrt{-1}dt\wedge d\bar{t} \\ &\geq \alpha\int_X (\hat{\varphi}(s)-\varphi(s))(dd^c\varphi(s))^n - \mathbf{H}_{\Omega}(\varphi(s)) - s \\ &\geq C\alpha\cdot(\mathbf{E}(\hat{\varphi}(s)) - \mathbf{E}(\varphi(s))) - \mathbf{H}(\varphi(s)) - s. \end{aligned}$$

Divide both sides by s and letting $s \to +\infty$ to get $\mathbf{E}'^{\infty}(\hat{\Phi}) = \mathbf{E}'^{\infty}(\Phi)$, which by linearity of $\mathbf{E}$ implies $\mathbf{E}(\hat{\varphi}(s)) \equiv \mathbf{E}(\varphi)$ and consequently by Dinew's domination principle gives $\hat{\varphi} \equiv \varphi$.

Slopes of twisted energy

Theorem (Thm B, L. , Berman-Boucksom-Jonsson)

If a maximal geodesic ray Φ is approximated by $\{\Phi_m\}$ associated to test configurations, then

$$\lim_{m\to+\infty}(\mathbf{E}^{-Ric(\Omega)})'^{\infty}(\Phi_m) = (\mathbf{E}^{-Ric})'^{\infty}(\Phi).$$

As a consequence, we have:

$$(\mathbf{E}^{-Ric(\Omega)})'^{\infty}(\Phi) = (\mathbf{E}^{K_X})^{\mathrm{NA}}(\Phi_{\mathrm{NA}}).$$

The same statement holds for $\mathbf{J}$ *and* $\mathbf{J}_{\mathbb{T}}$*.*

The proof uses the following estimate from BBEGZ:

$$\int_X (\varphi_2 - \varphi_1)((dd^c\varphi_3)^n - (dd^c\varphi_4)^n)$$
$$\leq \mathbf{I}(\varphi_1,\varphi_2)^{1/2^n}\cdot \mathbf{I}(\varphi_3,\varphi_4)^{1/2^n}\max\{\mathbf{I}(\varphi_i)\}^{1-2^{1-n}}.$$

For any $\phi \in \mathcal{E}^{1,\mathrm{NA}}$, define:

$$\mathbf{H}^{\mathrm{NA}}(\phi) = \int_{X^{\mathrm{NA}}} A_X(v)\mathrm{MA}^{\mathrm{NA}}(\phi).$$

If $\phi = \phi_{(\mathcal{X},\mathcal{L})}$, then $\mathbf{H}^{\mathrm{NA}}(\phi) = K^{\log}_{\mathcal{X}/X_{\mathbb{C}}} \cdot \bar{\mathcal{L}}^{\cdot n}$.

Theorem (Thm C, L. , '20)

For any (maximal) geodesic ray Φ, we have:

$$\mathbf{H}'^{\infty}(\Phi) \geq \mathbf{H}^{\mathrm{NA}}(\Phi_{\mathrm{NA}}), \quad \mathbf{M}'^{\infty}(\Phi) \geq \mathbf{M}^{\mathrm{NA}}(\Phi_{\mathrm{NA}}).$$

The key is to use the non-Archimedean identity for entropy:

$$\mathbf{H}^{\mathrm{NA}}(\phi) = \sup\left\{\int_{X^{\mathrm{NA}}} f_{K^{\log}_{\mathcal{Y}/X_{\mathbb{C}}}} \mathrm{MA}^{\mathrm{NA}}(\phi); \mathcal{Y} \text{ an SNC model}\right\},$$

Jensen's inequality and an asymptotic lemma of Boucksom-Hisamoto-Jonsson.

Conjecture (L. , '20)

If Φ is maximal, then $\mathbf{H}'^{\infty}(\Phi) = \mathbf{H}^{\mathrm{NA}}(\Phi_{\mathrm{NA}})$.

This is implied by

Conjecture (Boucksom-Jonsson)

For any $\phi \in \mathcal{E}^{1,\mathrm{NA}}$, there exist $\phi_m \in \mathcal{H}^{\mathrm{NA}}$ s.t. ϕ_m converges to ϕ in the strong topology and

$$\mathbf{H}^{\mathrm{NA}}(\phi) = \lim_{m \to +\infty} \mathbf{H}^{\mathrm{NA}}(\phi_m).$$

Difficulty: As in the Archimedean case, $\mathbf{H}^{\mathrm{NA}}$ is only lower-semi-continuous, not continuous, under the strong convergence. One needs some nice smoothing process that preserves the non-Archimedean entropy. We give some partial smoothing in the following theorem.

Approximation of non-Archimedean entropy

Theorem (Thm D, L.)

For any $\phi \in \mathcal{E}^{1,\mathrm{NA}}$, there exist models $(\mathcal{X}_m, \mathcal{L}_m)$ such that $\phi_m = \phi_{(\mathcal{X}_m,\mathcal{L}_m)}$ converges to ϕ in the strong topology and

$$\mathbf{M}^{\mathrm{NA}}(\phi) = \lim_{m \to +\infty} \mathbf{M}^{\mathrm{NA}}(\phi_m).$$

Step 1: $\forall \phi \in \mathcal{E}^{1,\mathrm{NA}}$, $\exists \phi_m \in \mathcal{E}^{1,\mathrm{NA}} \cap C^0(L^{\mathrm{NA}})$ s.t. $\phi_m \xrightarrow{\text{strongly}} \phi$, $\mathbf{M}^{\mathrm{NA}}(\phi_m) \to \mathbf{M}^{\mathrm{NA}}(\phi)$ and $\mathrm{MA}^{\mathrm{NA}}(\phi_m)$ is supported on a dual complex $\Delta_{\mathcal{Y}}$ of an SNC model $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ of (X, L).

Step 2: $\forall \phi \in \mathcal{E}^{1,\mathrm{NA}}$ with $\mathrm{MA}^{\mathrm{NA}}(\phi)$ supported on $\Delta_{\mathcal{Y}}$, $\exists \phi_k \in \mathcal{E}^{1,\mathrm{NA}} \cap C^0(L^{\mathrm{NA}})$ s.t. $\phi_k \xrightarrow{\textit{strongly}} \phi$, $\mathbf{M}^{\mathrm{NA}}(\phi_k) \to \mathbf{M}^{\mathrm{NA}}(\phi)$ and $\mathbf{M}^{\mathrm{NA}}(\phi_k)$ is a Dirac-type measure supported on $\Delta_{\mathcal{Y}}$.

Step 3: Boucksom-Favre-Jonsson showed that solution $(\mathrm{MA}^{\mathrm{NA}})^{-1}(\nu)$ for Dirac type ν is $\phi_{(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})}$ for some $\mathbb{R}$-line bundle $\mathcal{L}_{\mathcal{Y}}$. A perturbation makes $\mathcal{L}_{\mathcal{Y}}$ a $\mathbb{Q}$-line bundle.

Proof by contradiction.

Step 1: If $\mathbf{M}$ is not $\mathbb{G}$-coercive, then $\exists$ destabilizing ray Φ s.t.

$$\mathbf{M}'^{\infty}(\Phi) \leq 0, \quad \mathbf{J}_{\mathbb{T}}'^{\infty}(\Phi) = 1.$$

Step 2: By Thm A, Φ is maximal. By Thm B, with $\phi = \Phi_{\mathrm{NA}}$,

$$\mathbf{E}'^{\infty}(\Phi) = \mathbf{E}^{\mathrm{NA}}(\phi), \quad (\mathbf{E}^{-Ric(\Omega)})'^{\infty}(\Phi) = (\mathbf{E}^{K_X})^{\mathrm{NA}}(\phi),$$

Step 3: By Thm C, $\mathbf{H}'^{\infty}(\Phi) \geq \mathbf{H}^{\mathrm{NA}}(\phi)$.

Step 4: By Thm D, there exist models $(\mathcal{X}_m, \mathcal{L}_m)$:

$$\lim_{m \to +\infty} \mathbf{M}^{\mathrm{NA}}(\phi_m) = \mathbf{M}^{\mathrm{NA}}(\phi), \text{ with } \phi_m = \phi_{(\mathcal{X}_m, \mathcal{L}_m)}.$$

Step 5: Contradiction:

$$\begin{aligned} 0 \quad &\geq \quad \mathbf{M}'^{\infty}(\Phi) \geq \mathbf{M}^{\mathrm{NA}}(\phi) = \lim_{m \to +\infty} \mathbf{M}^{\mathrm{NA}}(\phi_m) \\ &\geq_{\text{stability}} \quad \lim_{m \to +\infty} \mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\phi_m) = \mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\phi) = 1. \end{aligned}$$

Toric case

A toric manifold X^n is a projective manifold with an effective $\mathbb{T} \cong (\mathbb{C}^*)^r$ action with an open dense orbit.

Ample toric line bundle $\Longleftrightarrow$ lattice (moment) polytope $\Delta \subset \mathbb{Z}^n$.

$(\mathbb{C}^*)^r$-equivariant test configurations $\Longleftrightarrow$ convex piecewise linear rational functions on Δ.

$(\mathbb{C}^*)^r$-equivariant models $\Longleftrightarrow$ piecewise linear rational functions $f_{\mathcal{L}}$ on Δ, and

$\phi_{\mathcal{L}} =$ lower convex envelope of $f_{\mathcal{L}}$, and is convex piecewise linear rational and hence comes from a test configuration.
This corresponds to the algebraic fact: toric divisors on toric varieties admit Zariski decomposition.

So we get the toric YTD conjecture for all polarized toric manifolds.

Proof by contradiction.

Step 1: If $\mathbf{M}$ and $\mathbf{D}$ are not $\mathbb{G}$-coercive, then $\exists$ geodesic Φ s.t.

$$\mathbf{D}'^{\infty}(\Phi) \leq 0, \quad \mathbf{J}'^{\infty}_{\mathbb{T}}(\Phi) = 1.$$

Step 2: By Thm A, Φ is maximal and hence with $\phi = \Phi_{\mathrm{NA}}$,

$$\mathbf{E}'^{\infty}(\Phi) = \mathbf{E}^{\mathrm{NA}}(\phi).$$

Step 3: Berman-Boucksom-Jonsson showed $\mathbf{L}'^{\infty}(\Phi) = \mathbf{L}^{\mathrm{NA}}(\phi)$.

Step 4: By Multiplier Approximation, there exist TCs $(\mathcal{X}_m, \mathcal{L}_m)$:

$$\lim_{m\to+\infty} \mathbf{D}^{\mathrm{NA}}(\phi_m) = \mathbf{D}^{\mathrm{NA}}(\phi), \text{ with } \phi_m = \phi_{(\mathcal{X}_m, \mathcal{L}_m)}.$$

Step 5: Contradiction:

$$\begin{aligned} 0 \quad &\geq \quad \mathbf{D}'^{\infty}(\Phi) = \mathbf{D}^{\mathrm{NA}}(\phi) = \lim_{m\to+\infty} \mathbf{D}^{\mathrm{NA}}(\phi_m) \\ &\geq_{\text{stability}} \quad \lim_{m\to+\infty} \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\phi_m) = \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\phi) = 1. \end{aligned}$$

Thanks for your attention!