On Yau-Tian-Donaldson conjecture for singular Fano varieties

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A Fano manifold X is a projective manifold such that:
Fano: $-K_X = \wedge^n T_{\mathrm{hol}} X$ is an ample line bundle.
Equivalently: $\exists$ a Kähler metric g s.t. its Kähler form $\omega \in 2\pi c_1(-K_X)$.

$$\text{Kähler form: } \omega = g(J\cdot, \cdot) = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j = \sqrt{-1}\partial\bar{\partial}\varphi_\alpha. \tag{1}$$

Examples $\mathbb{P}^2 \sharp k \overline{\mathbb{P}^2}$, $0 \leq k \leq 8$; $\mathbb{P}^n$; $\{F(z_1, \dots, z_n) = 0\} \subset \mathbb{P}^{n-1}$ with $\deg(F) < n$.

Hermitian metric on $-K_X$: $h = e^{-\varphi} = \{e^{-\varphi_\alpha}\}$ s.t. $|\partial_{z_\alpha}|^2 e^{-\varphi_\alpha} = |\partial_{z_\beta}|^2 e^{-\varphi_\beta}$.

$$Ric(\omega) = -\sqrt{-1} \sum_{i,j} \frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j \in 2\pi c_1(-K_X) \in H^2(X, \mathbb{R}). \tag{2}$$

Kähler-Einstein (KE) Equation: $\quad Ric(\omega) = \omega$

KE equation is equivalent to a complex Monge-Ampère equation:

$$(\sqrt{-1}\partial\bar{\partial}\varphi_\alpha)^n = |\partial_{z_\alpha}|^2 e^{-\varphi_\alpha} (\sqrt{-1})^{n^2} dz_\alpha \wedge d\bar{z}_\alpha \overset{\text{locally}}{\Longleftrightarrow} \det(\varphi_{i\bar{j}}) = e^{-\varphi}. \tag{3}$$

K-stability from variational appoint of view

Theorem (Tian, Tian-Zhu, Phong-Song-Sturm-Weinkove, Darvas-Rubinstein)

X admits a KE metric if and only if Mabuchi or Ding energy (denoted by F) is proper modulo the holomorphic automorphism group $Aut(X)$.

K-stability: a Hilbert-Mumford type criterion for properness of energy:

Definition (K-stability after Tian, equivalent to Donaldson's formulation by L.-Xu)

X is K-polystable if for any special degeneration $(\mathcal{X}, \eta)$ of X, $\mathrm{Fut}(X_0, -\eta) \geq 0$ and the identity holds iff $\mathcal{X}$ is induced by a holomorphic vector field η on X.

Conjecture (Yau-Tian-Donaldson (YTD) conjecture)

X has a Kähler-Einstein metric if and only if X is K-polystable.

Necessary (needs energy properness): Tian ('97), Berman (works for any $\mathbb{Q}$-Fano)

Sufficient (partial C^0-estimate for conical KE): Chen-Donaldson-Sun, Tian

Definition

A $\mathbb{Q}$-Fano variety X is a normal projective variety satisfying 2 conditions:
Fano: $-K_X$ is ample $\mathbb{Q}$-line bundle, i.e. $-mK_X := (\wedge^n TX^{\mathrm{reg}})^{\otimes m}$ extends as an ample line bundle for some $m \in \mathbb{Z}$;
Klt (Kawamata log terminal): $\forall x \in X, \exists$ an open neighborhood U s.t. for a nowhere vanishing section $s \in \mathcal{O}_{mK_X}(U)$

$$\int_{U^{\mathrm{reg}}} (\sqrt{-1}^{mn^2} s \wedge \bar{s})^{1/m} < +\infty. \tag{4}$$

Hermitian metric on the $\mathbb{Q}$-line bundle $-K_X$: $e^{-\varphi} = \{e^{-\varphi_\alpha}\}$ s.t. $|s_\alpha^*|^2 e^{-m\varphi_\alpha} = |s_\beta^*|^2 e^{-m\varphi_\beta}$. We always assume $\{\varphi_\alpha\}$ are bounded.

$$\text{KE equation:} \quad (\sqrt{-1}\partial\bar{\partial}\varphi_\alpha)^n = |s_\alpha^*|^{2/m} e^{-\varphi_\alpha} \left(\sqrt{-1}^{mn^2} s_\alpha \wedge \bar{s}_\alpha\right)^{1/m}. \tag{5}$$

weak KE metrics: bounded solutions (in Bedford-Taylor sense) to (5).

Note: Condition (4) $\Longleftrightarrow$ the right-hand-side of (5) is integrable.

Digression: Why Klt singularities

It is the biggest class of singularities for which the Yau-Tian-Donaldson conjecture is expected to hold, and for which the Minimal Model Program in birational algebraic geometry is known to work.

1. $\dim_{\mathbb{C}} X = 2$: Klt=isolated quotient singularity $\mathbb{C}^2/G$.
2. $\dim_{\mathbb{C}} X = 3$: partial classifications ($\{\text{terminal}\} \subset \{\text{canonical}\} \subset \{\text{Klt}\}$)
3. Quotient singularities and toric singularities are Klt.
4. Klt singularities are local correspondent of (log-)Fano varieties. Any Klt singularity degenerates to (orbifold-)cones over (log-)Fano varieties.

Check Klt condition: first choose a resolution of singularities $\mu : M \to X$ (M is smooth and μ is isomorphism over X^{reg}) and write:

$$\mu^*(s \wedge \bar{s})^{\frac{1}{m}} = h(z) \prod_i |z_i|^{2a_i} dz \wedge d\bar{z}, \quad (h(z) \text{ nowhere vanishing});$$

or algebraically:

$$K_M = \mu^* K_X + \sum_i a_i E_i, \quad E_i = \{z_i = 0\}.$$

Then (4) is equivalent $a_i > -1$. a_i is called the discrepancy of E_i over X.

Main Result

We say the singularities of X are admissible if $\exists$ a log resolution of singularities $\mu : M \to X$ s.t.

(A1) If $K_M = \mu^* K_X + \sum_i a_i E_i$, then $-1 < a_i \leq 0$ for any i; and

(A2) $\exists \theta_i \in \mathbb{Q}_{>0}$ s.t. $\mu^*(-K_X) - \sum_i \theta_i E_i$ is an ample $\mathbb{Q}$-line bundle on M.

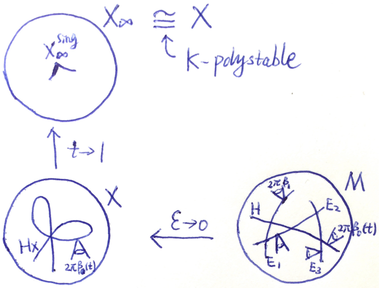
Theorem (L.-Tian-Wang '17)

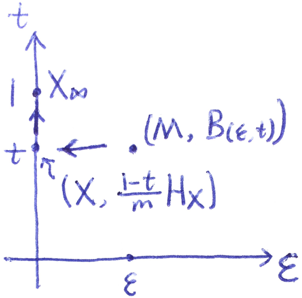
Let X be a $\mathbb{Q}$-Fano variety with admissible singularities. If X is K-polystable, then X admits a Kähler-Einstein metric.

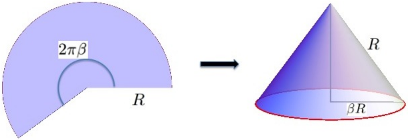
1. (A2) is always satisfied for $\mathbb{Q}$-factorial singularities. There are a lot of admissible Klt singularities including all 2-dimensional Klt singularities, $\mathbb{Q}$-factorial Klt singularities admitting crepant resolutions.
2. First existence result of YTD for "non-smoothable" Fano varieties.
3. YTD is expected to be true for any $\mathbb{Q}$-Fano variety. But not all KE Fano varieties are expected to be Gromov-Hausdorff (GH) limits of smooth Riemannian manifolds with lower Ricci bounds.

Idea/sketch of proofs

- Take an admissible resolution $\mu : M \to X$ and solve for appropriate (edge) conical Kähler-Einstein metric on M.
- Prove that the (edge) conical KE metrics on M converge to a Kähler-Einstein metric on X under the assumption of K-polystability.







$$g = \frac{|dz|^2}{|z|^{2(1-\beta)}} = dr^2 + \beta^2 r^2 d\theta^2, \quad \omega = \sqrt{-1}\frac{dz\wedge d\bar{z}}{|z|^{2(1-\beta)}} = \sqrt{-1}\partial\bar{\partial}\left(\beta^{-2}|z|^{2\beta}\right).$$

Higher dim: If $D = \sum_{i=1}^m D_i = \{z_1 z_2 \cdots z_m = 0\}$ is a SNC divisor (i.e. D_i are smooth and intersections are transversal), then local (edge) conical model:

$$\omega = \sqrt{-1}\left(\sum_{i=1}^m \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}} + \sum_{j=m+1}^n dz_j \wedge d\bar{z}_j\right).$$

Its Ricci curvature has a current term:

$$Ric(\omega) = \sqrt{-1}\partial\bar{\partial}\sum_{i=1}^m \log|z_i|^{2(1-\beta_i)} = \sum_{i=1}^m (1-\beta_i)2\pi\delta_{\{z_i=0\}} dx_i \wedge dy_i. \qquad (6)$$

Sketch of proof of Theorem 2

We divide the proof of the main result into 6 steps:

- **Step 0** (A2) $\Rightarrow$ can assume $L_\epsilon := \mu^*(-K_X) - \epsilon \sum_i \theta_i E_i$ is positive for $0 < \epsilon \leq 1 \Rightarrow$ there is a very ample divisor $H = \{s_H = 0\} \in |mL_1|$ (with $m \gg 1$) s.t. $H + \sum_i E_i$ has SNC.
- **Step 1** Prove $(M, \frac{1-t}{m} H + \sum_{i \geq 1}(1-\beta_i)E_i)$ is uniformly-K-stable for appropriate cone anges $2\pi\beta_i$. This part is purely algebraic.
- **Step 2** Use version of YTD for log smooth pairs to construct 2-parameter family of (edge) conical KE metrics $\omega_{(\epsilon,t)} \in 2\pi c_1(L_\epsilon)$ on M with edge conical singularities along $H \cup \sum_i E_i$.
- **Step 3** For fixed t, prove $(M, \omega_{(\epsilon,t)}, d_{(\epsilon,t)}) \stackrel{\epsilon_i \to 0}{\longrightarrow} (X, \omega_{(0,t)}, d_{(0,t)})$ (in both puripotential and GH senses). $\omega_{(0,t)}$ is a weak KE on $(X, \frac{1-t}{m} H_X)$ for some $H_X \in |-mK_X|$ and $(X, d_{(0,t)}) = \overline{(X^{\mathrm{reg}}, \omega_{(0,t)}|_{X^{\mathrm{reg}}})}$ (metric completion).
- **Step 4** As $t \to 1$, $(X, \omega_{(0,t)})$ subsequentially converges in Gromov-Hausdorff topology to X_∞ equipped with a weak Kähler-Einstein metric $\omega_{(0,1)}$.
- **Step 5** Construct a special degeneration of X to X_∞ with zero Futaki invariant. K-polystability of X forces $X_\infty \cong X$.

The last two steps were essentially done by CDS and Tian (although X is smooth in their case).

Step 0: set-up (edge) conical KE equation on M

Decomposition of $\mathbb{Q}$-divisor:

$$\begin{aligned} -K_M &= \mu^*(-K_X) + \sum_{i\geq 1}(-a_i)E_i = t(\mu^*(-K_X) - \epsilon \sum_i \theta_i E_i) \\ &+ \frac{1-t}{m} H + \sum_{i\geq 1}(-a_i + t\epsilon\theta_i + (1-t)\theta_i)E_i =: B_{(\epsilon,t)}. \end{aligned}$$

Correspondingly, solve the following KE equation ($E_0 = H$ for simplicity):

$$Ric(\omega_{(\epsilon,t)}) = t\omega_{(\epsilon,t)} + 2\pi\{B_{(\epsilon,t)}\} \Leftrightarrow (\sqrt{-1}\partial\bar{\partial}\varphi)^n = \frac{e^{-t\varphi}}{\prod_{i\geq 0}|s_{E_i}|^{2(1-\beta_i)}} \quad ((*)_{(\epsilon,t)})$$

Geometrically, $\omega_{(\epsilon,t)}$ is a (edge) conical KE metric which is smooth on $M \setminus \mathrm{Supp}(B_{(\epsilon,t)})$ and has cone singularities along E_i with cone angle $2\pi\beta_i$ where

$$\beta_i = \begin{cases} 1 - \frac{1-t}{m}; & \text{for } i = 0 \text{ i.e. } E_0 = H \\ 1 + a_i - t\epsilon_i - (1-t)\theta_i & \text{for } i \geq 1. \end{cases}$$

Why Admissible: $a_i \in (-1, 0] \Rightarrow \beta_i \in (0, 1]$ for $0 \leq \max\{\epsilon, 1-t\} \ll 1$

Step 1: Uniform K-stability of $(M, B_{(\epsilon,t)})$

Proposition

X K-polystable $\Rightarrow (M, B_{(\epsilon,t)})$ uniformly K-stable if $0 < \max\{\epsilon, 1-t\} \ll 1$.

The proof uses the valuative criterion of K-stability developed by Fujita and L.. For any divisorial valuation ord_F over M, define:

$$\Phi_{(M,B)}(F) := \frac{A_{(M,B)}(E)(-K_M - B)^n}{\int_0^{+\infty} \mathrm{vol}_M(-K_M - B - xE)dx}, \quad \tilde{\delta}(M,B) := \inf_F \Phi_{(M,B)}(F). \quad (7)$$

Theorem (Fujita, L.)

1. *(M, B) is K-semstable iff $\tilde{\delta}(M, B) \geq 1$.*
2. *(M, B) is uniform K-stable iff $\tilde{\delta}(M, B) > 1$.*

Why this helps: because $\mathbb{C}(M) \cong \mathbb{C}(X)$ and the set of valuations do not change. On the other hand, the set of special degenerations change!

Need a logarithmic version of YTD for the pair (M, B) with smooth ambient space:

Theorem (L.-Tian-Wang, Tian-Wang)

(M, B) is uniformly K-stable $\Longrightarrow$ energy is proper $\Longrightarrow \exists$ solution to $(*)_{(\epsilon,t)}$.

Two props for energy properness:

1. Generalize Berman-Boucksom-Jonsson's argument to the logarithmic setting:

$$
\begin{aligned}
&\text{Uniformly K-stable}\\
\overset{def}{\Longleftrightarrow}\quad & F^{\mathrm{NA}} \geq \delta J^{\mathrm{NA}} \text{ on smooth non-Archimedean metrics}\\
\Longleftrightarrow\quad & F^{\mathrm{NA}} \geq \delta J^{\mathrm{NA}} \text{ on finite energy non-Archimedean metrics}\\
\Longleftrightarrow\quad & F \geq \delta J - C \text{ on the space of (smooth or finite energy) Kähler metrics.}
\end{aligned}
$$

2. Generalize CDS-Tian's argument to the logarithmic setting (need the conical version of Cheeger-Colding-Tian's theory developed recently by Tian-Wang).

The Euler-Lagrange equation of F is $(*)_{(\epsilon,t)}$:

$$F := F_{B_{(\epsilon,t)}}(\varphi) = -E_{\psi_\epsilon}(\varphi) - \frac{1}{t}\log\left(\int_M \frac{e^{-t\varphi}}{|s_B|^2}\right). \tag{8}$$

J-energy measures the distance between two potentials:

$$J := J_{\psi_\epsilon}(\varphi) = -E_{\psi_\epsilon}(\varphi) + \frac{1}{(L_\epsilon^{\cdot n})}\int_M (\varphi - \psi_\epsilon)(\sqrt{-1}\partial\bar\partial\psi_\epsilon)^n. \tag{9}$$

Step 3: take limit of $(M, \omega_{(\epsilon,t)})$ as $\epsilon \to 0^+$: potential Part

Proposition (weak uniform properness)

Fix $t \in (0,1)$ there exist $\epsilon^* = \epsilon^*(t)$, $\delta^* = \delta^*(t)$ and $C > 0$ s.t. for any $\epsilon \in (0, \epsilon^*]$ and any $\varphi \in PSH(L_\epsilon)$, the following inequality holds:

$$F_{B_{(\epsilon,t)}}(t, \varphi) \geq \delta^* J_{\psi_\epsilon}(\varphi) - C\epsilon^* \|\varphi - \varphi_\epsilon\|_\infty - C. \tag{10}$$

We proved this estimate by using the properness of $F_{B_{(\epsilon^*,t)}}$ and comparing the energy functional for parameter ϵ and ϵ^* by a rescaling map:

$$\begin{aligned} P_\epsilon : PSH(L_\epsilon) &\to PSH(L_{\epsilon^*}) \\ \varphi &\mapsto \psi_{\epsilon^*} + \frac{1}{1 + 2(\epsilon^* - \epsilon)}(\varphi - \psi_\epsilon) \end{aligned}$$

From weak properness to $\|\varphi_{(\epsilon,t)} - \psi_\epsilon\|_{L^\infty}$, need 2 facts:

1. Uniform bound on Sobolev constants for (edge) conical KE's $(M, \omega_{(\epsilon,t)})$. This allows us to bound $\|\cdot\|_{L_\infty}$ in terms of $J_{\psi_\epsilon}(\varphi)$ by Moser iteration.
2. $F_{(B_{\epsilon,t})}(\varphi_{(\epsilon,t)})$ ($=$ infimum of $F_{B_{(\epsilon,t)}}$) is uniformly bounded from above.

Combining these facts, we get uniform L^∞ estimates:

Uniform estimates for the potential functions of $\omega_{(\epsilon,t)}$

Proposition (uniform L^∞ estimate)

There exists a constant $C = C(X,t) > 0$ s.t. the solution $\varphi_{(\epsilon,t)}$ to $(*)_{(\epsilon,t)}$ satisfies the uniform L^∞ estimate: $\|\varphi_{(\epsilon,t)} - \psi_\epsilon\|_{L^\infty} < C$.

Derive higher-order estimate away from singular set from the uniform L^∞ estimate

Proposition (higher order estimates)

For any $V \Subset M \setminus (\cup_{i\geq 1} E_i)$ and any $\alpha < (1 - \frac{1-t}{m})^{-1} - 1$, there exists a constant $C = C(M,V,t,\alpha) > 0$ s.t. $\|\omega_{(\epsilon,t)}\|_{C^{\alpha,\beta_0}(V)} \leq C$.

As $\epsilon \to 0^+$, $\omega_{(\epsilon,t)}$ converges to a solution to weak Kähler-Einstein metric $\omega_{(0,t)}$ on (X, H_X) where $\mu^* H_X = H + m\sum_i \theta_i E_i$. So we get:

Theorem (weak version of YTD)

- *If an admissible $\mathbb{Q}$-Fano X is uniformly K-stable, then X ha a KE metric.*
- *If X is K-semistable, then there exists KE $\omega_{(0,t)}$ on $(X, \frac{1-t}{m} H_X)$ for $0 < 1 - t \ll 1$.*

Theorem (Tian-Wang)

Let $(X_{(0,t)}, d_{(0,t)})$ be a GH limit of a sequence $(M, \omega_{(\epsilon_i,t)})$ as $\epsilon_i \to 0$. Then there is a decomposition $X_{(0,t)} = \mathcal{R} \cup \mathcal{S}$ satisfying:

1. *$\mathcal{R}$ is open in $X_{(0,t)}$ and has a smooth manifold structure equipped with a smooth KE metric.*
2. *The singular set has a decomposition $\mathcal{S} = \cup_{k=1}^{n} \mathcal{S}_{2n-2k}$ where $\mathcal{S}_{2n-2k}$ consists of the points whose metric tangent cones do not split $\mathbb{R}^{2n-2k+1}$-factor. $\mathcal{S}_{2n-2k}$ satisfies $\mathrm{codim}_{\mathbb{R}}(\mathcal{S}_{2n-2k}) \geq 2k$.*

Main problem: Prove $X_{(0,t)} = X$. In particular, $X_{(0,t)}$ is an algebraic variety.

Difficulty: $\omega_{(\epsilon,t)}$ is curvature form of varying line bundle L_ϵ. The usual partial C^0-estimate technique does not apply directly.

Fortunately, similar problems have been considered in the study of Kähler-Ricci flow and other continuity method by J. Song and Tian-Zhang.

$X_{(0,t)} \cong X$: Gauge fixing

Using L^∞ and higher order estimate of $\varphi_{(\epsilon,t)}$, we get a gauge fixing result:

Proposition (Proposition A: gauge fixing motivated by Rong-Zhang)

For $0 < t < 1$, $(X_{(0,t)}, d_{(0,t)})$ is the metric completion of $(X^{\mathrm{reg}}, \omega_{(0,t)}|_{X^{\mathrm{reg}}})$. Moreover, $\mathrm{id} : (X^{\mathrm{reg}} = M \setminus E, d_{g(0,t)}) \to (M, \omega_{(\epsilon,t)})$ gives a GH approximation for the convergence $(M, \omega_{(\epsilon,t)}) \xrightarrow{\epsilon \to 0} (X_{(0,t)}, d_t)$.

The next result says $X_{(0,t)}$ coincides with the algebraic variety X:

Proposition (Proposition B)

$X_{(0,t)}$ is homeomorphic to X. As a consequence, $(X, \frac{1-t}{m} H_X)$ admits a weak Kähler-Einstein metric $\omega_{(0,t)}$ such that the $(X, d_{(0,t)})$ is the metric completion of the geodesically convex subset $(X^{\mathrm{reg}}, \omega_{(0,t)}|_{X^{\mathrm{reg}}})$.

$X_{(0,t)} \cong X$: construction of map

Let $L = \mu^*(-K_X)$ and $\Phi^\ell : M \to \mathbb{P}^N$ be the morphism defined by an o.n.b. of $(H^0(M, L^{m\ell}), h_{\mathrm{FS}}^{m\ell}, \omega_{\mathrm{FS}})$. Then

$$\Phi^\ell_{(\epsilon,t)} = \Phi^\ell : (M, \omega_{(\epsilon,t)}) \to (\Phi^\ell(M) \cong X, \omega_{\mathrm{FS}}) \tag{11}$$

is uniformly Lipschitz (by using Chern-Lu's inequality):

$$\omega_{\mathrm{FS}} \leq C \cdot \omega_{(\epsilon,t)} \text{ with uniform } C.$$

As $\epsilon \to 0$ with t fixed, $\Phi_{(\epsilon_i,t)}$ subsequentially converges to a Lipschitz map:

$$\Phi^\ell_{(0,t)} : (X_{(0,t)}, d_{(0,t)}) \to (\Phi^\ell(M) \cong X, \omega_{\mathrm{FS}}). \tag{12}$$

Recall: $X_{(0,t)}$ is the metric completion of $(X^{\mathrm{reg}}, \omega_{(0,t)})$ and $\Phi_{(0,t)}|_{X^{\mathrm{reg}}}$ is an isometry.

Proposition

$\Phi^{\ell^*}_{(0,t)}$ is injective for some $\ell^* \gg 1$. As a consequence, $X_{(0,t)}$ is homeomorphic to X. Hence $(M, \omega_{(\epsilon,t)})$ Gromov-Hausdorff converges to $(X, d_{(0,t)})$ which is the metric completion of $(X^{\mathrm{reg}}, \omega_{(0,t)})$.

1. Need a lot of peak sections in $(H^0(X^{\mathrm{reg}}_{(0,t)}, -K^{mk}_{X^{\mathrm{reg}}_{0,t}}), \|\cdot\|_{L^2(h^k_{(0,t)}, \omega_{(0,t)})})$.
2. Need gradient estimate of $|\nabla^{h^k_{(0,t)}}\zeta|_{h^k_{(0,t)}\otimes\omega_{(0,t)}}$ for $\zeta \in H^0(X^{\mathrm{reg}}_{(0,t)}, L^k)$.

$X_{(0,t)} \cong X$: construct L^2 section using $\hat{h}_\epsilon$

Singular metric on $\mu^*(-K_X) = L_\epsilon + \sum_i \theta_i E_i$: (write ϵ for (ϵ, t))

$$\hat{h}_\epsilon := e^{-\hat{\varphi}_\epsilon} = \frac{e^{-\varphi_\epsilon}}{\prod_i |s_i|^{2\epsilon\theta_i}}. \tag{13}$$

satisfies (recall $1 - \beta_i = -a_i + t\epsilon\theta_i + (1-t)\theta_i \in [0, 1)$):

$$e^{-k\hat{\varphi}_\epsilon}\omega_\epsilon^n = \frac{e^{-(k+t)\varphi_\epsilon}}{|s_H|^{2\frac{1-t}{m}} \prod_i |s_i|^{2(k\epsilon\theta_i + 1 - \beta_i)}} \quad \text{and} \quad \Theta(\hat{h}_\epsilon^k) + Ric(\omega_\epsilon) \geq (k+t)\omega_\epsilon.$$

Weitzenböch formula together with Hörmander's L^2-estimates give:

Proposition (solve $\bar{\partial}$-equation with L^2-estimate)

Assume $k\epsilon\theta_i + 1 - \beta_i < 1$. Then $\exists C > 0$ independent of ϵ s.t. for any $\xi \in \Gamma(T^{*(0,1)}M \otimes L^k)$ with $\bar{\partial}\xi = 0$, we can find a solution to $\bar{\partial}\zeta = \xi$ which satisfies:

$$\int_M |\zeta|^2_{\hat{h}_\epsilon^k} \omega_\epsilon^n \leq \frac{C}{k} \int_M |\xi|^2_{\hat{h}_\epsilon^k \otimes \omega_\epsilon} \omega_\epsilon^n. \tag{14}$$

$X_{(0,t)} \cong X$: construct L^2-sections using $h_{(0,t)}$

Proposition (convergence away from B)

Assume $k\epsilon\theta_i + 1 - \beta_i < 1$. Let ζ_j be a sequence of holomorphic sections of L^k, $k \geq 1$, satisfying $\int_M |\zeta_j|^2_{\hat{h}^k_{\epsilon_j}} \omega^n_{\epsilon_j} \leq 1$. Then as $\epsilon_j \to 0$, ζ_j subsequentially converges to a locally bounded holomorphic section ζ_∞ of L^k over $\mathcal{R} = M \setminus \mathrm{supp}(B)$.

Need boundedness of $|\zeta|_{h^k_{(0,t)}}$ and $|\nabla^{h^k_{(0,t)}} \zeta_\infty|_{h^k_{(0,t)} \otimes \omega_{(0,t)}}$ on the regular part X^{reg}. Let h_{FS} be the pull back of Fubini-Study metric $\Phi^* h_{\mathrm{FS}}$ on $L = \mu^*(-K_X)$. Use Bochner formula and Moser iteration to get:

Proposition (uniform estimates w.r.t. h_{FS})

There exists $C > 0$ independent of ϵ, s.t. for any $\zeta \in H^0(M, L^k)$ we have:

$$\sup_M |\zeta|^2_{h^k_{\mathrm{FS}}} \leq Ck^n \int_M |\zeta|^2_{h^k_{\mathrm{FS}}} \omega^n_\epsilon; \tag{15}$$

$$|\nabla^{h_{\mathrm{FS}}} \zeta|^2 \leq Ck^{n+1} \int_M |\zeta|^2_{h^k_{\mathrm{FS}}} \omega^n_\epsilon. \tag{16}$$

To transfer the estimates to estimates for $h_{(0,t)}$, we need gradient estimate for $\varphi_{(0,t)}$ on X^{reg}.

Write $\omega_{(0,t)} = \chi_0 + \sqrt{-1}u_{(0,t)}$ over X. Then $u_{(0,t)}$ is defined up to a constant and satisfies $\Delta_{(0,t)}u_{(0,t)} = -\mathrm{tr}_{\omega_{(0,t)}}\chi_0 + n$. We approximate $u_{(0,t)}$ by functions on M as follows. Choose $p \in X^{\mathrm{reg}}$ and let $U = \mu^{-1}(X \setminus B_{\omega_{(0,t)}}(p, 2r))$. Solve:

$$\text{Dirichlet problem:} \quad \begin{cases} \Delta_\epsilon v_\epsilon = -\mathrm{tr}_{\omega_{(\epsilon,t)}}\chi_0 + n & \text{on } U; \\ v_\epsilon = u_{(0,t)} & \text{on } \partial U. \end{cases} \tag{17}$$

Proposition

There exist constant $C > 0$ independent of ϵ s.t. $|v_\epsilon| + |\nabla_\epsilon v_\epsilon| \leq C$ over U.

As $\epsilon \to 0$, $(U, \omega_{(\epsilon,t)}) \to (\hat{U}, d_{(0,t)})$. $v_\epsilon \to v$ satisfies equation:

$$\begin{cases} \Delta_\epsilon v = -\mathrm{tr}_{\omega_{\epsilon,t)}}\chi_0 + n & \text{on } \hat{U} \cap \mathcal{R}; \\ v = u_{(0,t)} & \text{on } \partial\hat{U}. \end{cases} \tag{18}$$

Then one can show that $v = u_{(0,t)}$ over $\hat{U} \cap \mathcal{R}$ and hence $|\nabla_{\omega_{(0,t)}} u_{(0,t)}|$ is indeed bounded.

$\Phi_{(0,t)}^{\ell^*}$ is homeomorphism for some $\ell^* \gg 1$

1. $\forall p \in X^{\mathrm{reg}}$, construct a local approximate holomorphic section on a small open set containing p: transfer constant section on the metric tangent cone $\mathcal{C}_p$ to small open set containing p by using a good cut function and a gauge fixing diffeomorphism $(V(p,\epsilon) \subset \mathcal{C}_p^{\mathrm{reg}}, \mathcal{C}_p \times \mathbb{C}) \to (X_{(0,t)}^{\mathrm{reg}}, L^{k_p})$.
2. For any $p \in X$, construct holomorphic peak section (almost) centered at p. This is obtained by solving $\bar{\partial}$-equation to adjust approximate holomorphic section to become a genuine holomorphic section. The gradient estimate of $\varphi_{(0,t)}$ allows to extend the uniform estimates "across the singularity".
3. Prove the $\Phi_{(0,t)}^{\ell^*}$ is injective for some $\ell^* \gg 1$.
   For $p, q \in X_{(0,t)}$, construct two peak sections in $L^{m\ell_{p,q}}$ almost centered at p and q. Prove that $\Phi^{l_{p,q}}$ is injective.
   Then use the effective finite generation of section rings to prove there exists an ℓ^* that works for all pairs p, q.

Step 4 & 5: Completion of Proof of Main Theorem

1. As $t \to 1$, $(X, d_{(0,t)})$ GH converges to (X_∞, d_∞). Tian-F.Wang's compactness applies because $(X, d_{(0,t)})$ are GH limits of strong (edge) conical KE metrics with positive Ricci curvature.
2. Use the technique of partial C^0-estimates to show that X_∞ is a normal $\mathbb{Q}$-Fano variety and admits a weak KE metric with Lipschitz potentials. Moreover, X_∞ and X can be embedded by L^2-sections into a common projective space $\mathbb{P}^N$ such that $\mathrm{Hilb}(X_\infty)$ is in the orbit closure of $\mathrm{Hilb}(X)$ under $PGL(N+1, \mathbb{C})$.
3. Prove generalized Matsushima type result: $Aut(X_\infty)$ is reductive. As a consequence and by using Luna slice theorem, there is a one parameter subgroup in the Hilbert scheme such that

$$\lambda(t) \cdot \mathrm{Hilb}(X) \to \mathrm{Hilb}(X_\infty) \text{ as } t \to 0.$$

   This gives a special degeneration of X with central fibre X_∞.
4. X being KE implies $\mathrm{Fut}(X_\infty, -v) = 0$ where v is the generator of $\lambda(t)$. The K-polystability of X forces $X_\infty \cong X$.

Thanks for your attention!