Generalized YTD conjecture on Fano varieties

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Workshop on Geometric Analysis, September 21, 2020
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Riemann surface: surface with a complex structure:

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Riemannian metric:
\[ g = E|dz|^2 = \frac{\partial^2 \varphi}{\partial z \partial \overline{z}} |dz|^2 = \frac{1}{4} \Delta \varphi |dz|^2. \]

Constant Gauss/Ricci curvature equation
\[ Ric(\omega) = \lambda \omega \iff -\Delta \log \Delta \varphi = \lambda \Delta \varphi \iff \Delta \varphi = e^{-\lambda \varphi}. \]
Kähler manifolds and Kähler metrics

X: complex manifold; J: $TX \to TX$ integrable complex structure; g: Kähler metric, $g(J \cdot, J \cdot) = g(\cdot, \cdot)$ and $d\omega = 0$.

$$\omega = g(\cdot, J \cdot) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} \omega_{ij} dz^i \wedge d\bar{z}^j, \quad (\omega_{ij}) > 0.$$  

Kähler class $[\omega] \in H^2(X, \mathbb{R})$.

Fact ($\partial \bar{\partial}$-Lemma): Set $dd^c = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial}$. Any $\omega' \in [\omega]$ is of the form

$$\omega_u := \omega + dd^c u := \omega + \frac{\sqrt{-1}}{2\pi} \sum_{i,j} \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} dz^i \wedge d\bar{z}^j.$$
Kähler metric as curvature forms

$L \to X$: a $\mathbb{C}$-line bundle with holomorphic transition $\{f_{\alpha\beta}\}$.

$e^{-\varphi} := \{e^{-\varphi_\alpha}\}$ Hermitian metric on $L$:

$$e^{-\varphi_\alpha} = |f_{\alpha\beta}|^2 e^{-\varphi_\beta}.$$  \hspace{1cm} (1)

**Definition:** $L$ is positive (=ample) if $\exists e^{-\varphi} = \{e^{-\varphi_\alpha}\}$ on $L$ s.t.

$$\omega + dd^c u = dd^c \varphi := dd^c \varphi_\alpha > 0.$$  \hspace{1cm} (2)

**Anticanonical line bundle:** $-K_X = \wedge^n T_{\text{hol}} X$, $K_X = \wedge^n T^*_{\text{hol}} X$.

**Fact:** \{smooth volume forms\} = \{Hermitian metrics on $-K_X$ \}

$$c_1(X) \ni Ric(\omega) = -dd^c \log \omega^n$$

$$= -\frac{\sqrt{-1}}{2\pi} \sum_{i,j} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(\omega_{k\bar{l}}) dz^i \wedge d\bar{z}^j.$$
Kähler-Einstein metric and Monge-Ampère equation

KE equation:

\[ \text{Ric}(\omega_u) = \lambda \omega_u \iff (\omega + dd^c u)^n = e^{h\omega - \lambda u} \omega^n \]

\[ \iff \det \left( \omega_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \right) = e^{h\omega - \lambda u} \det(\omega_{ij}). \]

\[ \lambda = -1 \quad \text{Solvable (Aubin, Yau)} \quad c_1(X) < 0 \]
\[ \lambda = 0 \quad \text{Solvable (Yau)} \quad c_1(X) = 0 \]
\[ \lambda = 1 \quad \exists \text{ obstructions} \quad c_1(X) > 0 \]
\( \text{X Fano: } c_1(X) > 0 \iff \exists \text{ Kähler metric } \omega \text{ with } \text{Ric}(\omega) > 0. \)

1. \( \dim_{\mathbb{C}} = 1: \mathbb{P}^1 = S^2. \)
2. \( \dim_{\mathbb{C}} = 2: \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2 \# k\overline{\mathbb{P}}^2, 1 \leq k \leq 8 \text{ (del Pezzo).} \)
3. \( \dim_{\mathbb{C}} = 3: \) 105 deformation families (Iskovskikh, Mori-Mukai)
4. Smooth hypersurface in \( \mathbb{P}^n \) of degree \( < n + 1; \)
5. Toric Fano manifolds

**Fact:** there are finitely many deformation family in each dimension (Campana, Kollár-Miyaoka-Mori, Nadel '90).
Obstructions and uniqueness

1. \( \exists \text{KE} \implies \text{Aut}(X) \) is reductive: \( \text{Aut}(X)_0 \) is the complexification of a compact Lie group (Matsushima)

2. Futaki invariant: \( \forall \) holomorphic vector field \( \xi \),

\[
\exists \text{KE} \implies \text{Fut}(\xi) := \int_X \xi(h_\omega)\omega^n = 0. \tag{4}
\]

3. Energy coerciveness (Tian, Tian-Zhu, Phong-Song-Sturm-Weinkove, Darvas-Rubinstein, Hisamoto)

4. K-stability (Tian, Donaldson)
   Ding stability (Berman, Boucksom-Jonsson)
   all equivalent (L.-Xu, Berman-Boucksom-Jonsson, Fujita).

**Uniqueness:** KE metrics are unique up to \( \text{Aut}(X)_0 \) (Bando-Mabuchi, Berndtsson)
Theorem (Tian, Chen-Donaldson-Sun, Berman)

A Fano manifold $X$ admits a KE metric if and only if $X$ is K-stable ($\text{Aut}(X)$ is discrete), or $X$ K-polystable ($\text{Aut}(X)$ is continuous).

We extend this theorem in two directions (with different proofs with those of the above):

1. Any (singular) $\mathbb{Q}$-Fano variety $X$ (L.-Tian-Wang, L.)
2. Kähler-Ricci $g$-soliton (Han-L.)

Remark: In particular, we recover the above theorem with K-polystability replaced by appropriate ($\mathbb{G}$-)uniform K-stability. Conjecturally, uniform K-stability is equivalent to K-stability, which is reduced to an algebraic geometric problem.
Definition

$\mathbb{Q}$-Fano variety $X$ is a normal projective variety satisfying:

1. **Fano**: $\mathbb{Q}$-line bundle $-K_X$ is ample.
2. **klt (Kawamata log terminal)**: $\forall s_\alpha^* \sim dz^1 \wedge \ldots dz^n \in \mathcal{O}_{K_X}(U_\alpha)$

$$\int_{U_{\text{reg}}} (\sqrt{-1}^n s_\alpha^* \wedge \bar{s}_\alpha^*) < +\infty. \quad (5)$$

Let $\mu : Y \to X$ be a resolution of singularities (Hironaka)

$$K_Y = \mu^* K_X + \sum_i (A_X(E_i) - 1)E_i. \quad (6)$$

The condition $(5) \iff \text{mld} := \min_i A_X(E_i) > 0.$

**Fact**: (Birkar ’16) $\epsilon$-klt (i.e. $\text{mld} \geq \epsilon > 0$) Fanos are bounded.

**Fact**: KE equation (only) makes sense on all $\mathbb{Q}$-Fano varieties.
Klt singularities: Examples

1. Smooth points, Orbifold points = (normal) Quotient singularities

2. $X = \{ F(z_1, \ldots, z_{n+1}) = 0 \} \subset \mathbb{C}^{n+1}$ with $F$ homogenenous \[ \text{deg}(F) < n + 1 \] s.t. $X$ has an isolated singularity at 0.

3. Orbifold cones over log-Fano varieties. weighted homogeneous examples:

$$z_1^2 + z_2^2 + z_2^2 + z_3^{2k} = C \left( (\mathbb{P}^1 \times \mathbb{P}^1, (1 - \frac{1}{k}) \Delta), H \right)$$

$$z_1^2 + z_2^2 + z_2^2 + z_3^{2k+1} = C \left( (\mathbb{P}^2, (1 - \frac{1}{2k+1}) D), H \right)$$

4. $(\mathbb{Q}$-Gorenstein) deformation of Klt singularities are also Klt singularities.
Hermitian metric on the \((\mathbb{Q})\)-line bundle \(-K_X\): 
\[ e^{-\varphi} = \{ e^{-\varphi_\alpha} \} \text{ s.t.} \]
\[ e^{-\varphi_\alpha} = |s_\alpha|^2 e^{-\varphi}, \quad \{ s_\alpha \} \text{ trivializing sections of } -K_X \]

Kähler-Einstein equation on Fano varieties:

\[ (dd^c \varphi)^n = e^{-\varphi} := |s_\alpha|^2 e^{-\varphi} \left( \sqrt{-1} n^2 s^{*}_\alpha \wedge \bar{s}^{*}_\alpha \right). \quad (7) \]

\[ \omega = dd^c \psi \Rightarrow u = \varphi - \psi \text{ is globally defined. Then } (7) \iff (3). \]

(weak) KE potential: generalized solutions in pluripotential sense.

Fact: Obstructions/uniqueness continue to hold for \( \mathbb{Q} \)-Fano case.

Face: Solutions are smooth on \( X^{\text{reg}} \).

Fact: Aubin and Yau’s theorems hold on projective varieties with Klt singularities (Eyssidieux-Guedj-Zeriahi based on Kołodziej).
Theorem (L.-Tian-Wang, L. ’19)

A $\mathbb{Q}$-Fano variety $X$ has a KE potential if (and only if) $X$ is $\text{Aut}(X)_0$-uniformly $K$/Ding-stable.

1. $X$ Smooth (Chen-Donaldson-Sun, Tian, Datar-Székelyhidi).
2. $\mathbb{Q}$-Gorenstein smoothable (Spotti-Sun-Yao, L.-Wang-Xu);
3. Good (e.g. crepant) resolution of singularities (L.-Tian-Wang).

Proofs in above special cases depend on compactness/regularity theory in metric geometry and do NOT generalize to the general singular case.

4. $X$ smooth & $\text{Aut}(X)$ discrete: Berman-Boucksom-Jonsson (BBJ) in 2015 proposed an approach using pluripotential theory/non-Archimedean analysis. Our work greatly extends their work by removing the two assumptions.
Consider energy functionals on a pluripotential version of Sobolev space, denoted by $\mathcal{E}^1(X, -K_X)$ (Cegrell, Guedj-Zeriahi)

There is a distance-like energy:

$$J(\varphi) = \Lambda(\varphi) - E(\varphi) \sim \sup(\varphi - \psi) - E(\varphi) > 0.$$  \hfill (8)

$E$ is the primitive of complex Monge-Ampère operator:

$$E(\varphi) = \frac{1}{V} \int_0^1 dt \int_X \varphi(d\bar{\varphi})^n, \quad \frac{d}{dt} E(\varphi) = \frac{1}{V} \int_X \varphi(d\bar{\varphi})^n.$$  \hfill (9)
Analytic criterion for KE potentials

Energy functional with KE as critical points:

\[ L(\varphi) = -\log \left( \int_X e^{-\varphi} \right), \quad D = -E + L. \] (10)

The Euler-Lagrangian equation is just the KE equation:

\[ \delta D(\delta \varphi) = \frac{1}{V} \int_X (\delta \varphi) \left( -(dd^c \varphi)^n + C \cdot e^{-\varphi} \right). \] (11)

Analytic criterion (generalizing Tian-Zhu, Phong-Song-Sturm-Weinkove):

Theorem (Darvas-Rubinstein, Darvas, Di-Nezza-Guedj, Hisamoto, based on the compactness by BBEGZ and uniqueness by Berndtsson):

A \( \mathbb{Q} \)-Fano variety \( X \) admits a KE potential if and only if

1. \( \text{Aut}(X)_0 \) is reductive (with center \( T \cong (\mathbb{C}^*)^r \))
2. Moser-Trudinger type inequality: there exist \( \gamma > 0 \) and \( C > 0 \) s.t.
   \[ \forall \varphi \in \mathcal{E}^1(X, -K_X)_\mathbb{K}, \]
   \[ D(\varphi) \geq \gamma \cdot \inf_{\sigma \in T} J(\sigma^* \varphi) - C. \) (\( \text{Aut}(X)_0 \)-coercive)
A test configuration (TC) \((\mathcal{X}, \mathcal{L}, \eta)\) of \((X, -K_X)\) consists of:

1. \(\pi : \mathcal{X} \rightarrow \mathbb{C}\): a \(\mathbb{C}^*\)-equivariant family of projective varieties;
2. \(\mathcal{L} \rightarrow \mathcal{X}\): a \(\mathbb{C}^*\)-equivariant semiample holomorphic \(\mathbb{Q}\)-line bundle;
3. \(\eta : (\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^* \cong (X, -K_X) \times \mathbb{C}^*\).

Any test configuration is generated by a one-parameter subgroup of \(GL(N_m)\) (with \(m \gg 1\)) under the Kodaira embedding

\[
X \longrightarrow \mathbb{P}(H^0(X, -mK_X)^*) \cong \mathbb{P}^{N_m-1}.
\]

A test configuration is called special if the central fibre \(\mathcal{X}_0\) is a \(\mathbb{Q}\)-Fano variety.
Under the isomorphism $\eta$, any smooth psh metric on $\mathcal{L} \to \mathcal{X}$ induces a family of smooth psh metrics $\Phi = \{\varphi(t)\}$ on $(\mathcal{X}, -K_{\mathcal{X}})$.

**Theorem (Ding-Tian, Paul-Tian, Phong-Sturm-Ross, Berman, Boucksom-Hisamoto-Jonsson)**

For any $F \in \{E, J, L, D\}$,

$$F'_{\infty}(\Phi) := \lim_{t \to +\infty} \frac{F(\varphi(t))}{- \log |t|^2}$$

exists, and is equal to an algebraic invariant $F^{NA}(\mathcal{X}, \mathcal{L})$.

Coercivity (2) $\implies$ $D'_{\infty}(\Phi) \geq \gamma \cdot \inf_{\xi \in \mathbb{N}_R} J'_{\infty}(\sigma_\xi(t)^*\Phi)$.
Boucksom-Jonsson: Test configuration defines a non-Archimedean metric, represented by a function on the space of valuations:

\[ \phi(v) = \phi(x, L)(v) = -G(v)(\Phi), \quad v \in X_Q^{\text{div}}. \tag{13} \]

\(G(v)(\Phi)\): the generic Lelong number of \(\Phi\) with respect to \(G(v)\).

Non-Archimedean functionals:

\[
\begin{align*}
\mathbf{E}^{\text{NA}}(\phi) &= \frac{1}{V} \frac{\bar{L} \cdot (n+1)}{n+1} = \int_{x_0} \theta_\eta(\varphi)(dd^c \varphi)^n, \\
\Lambda^{\text{NA}}(\phi) &= \frac{1}{V} \bar{L} \cdot p_1^*(-K_X)^n = \sup_{x_0} (\theta_\eta(\varphi)), \\
J^{\text{NA}}(\phi) &= \Lambda^{\text{NA}}(\phi) - \mathbf{E}^{\text{NA}}(\phi), \\
L^{\text{NA}}(\phi) &= \inf_{v \in X_Q^{\text{div}}} (A_X(v) + \phi(v)).
\end{align*}
\]

\(X_Q^{\text{div}}\): space of divisorial valuations; \(G(v)\): Gauss extension.
K-stability and Ding-stability

**Definition-Theorem (Berman, Hisamoto, Boucksom-Hisamoto-Jonsson)**

$X$ KE implies that it is $\text{Aut}(X)_0$-uniformly Ding-stable: $\exists \gamma > 0$ (slope) such that for all $\text{Aut}(X)_0$-equivariant test configurations

$$D_{NA}(X, L) \geq \gamma \cdot \inf_{\xi \in \mathbb{N}_R} J_{NA}(X, L, \eta + \xi). \quad (14)$$

When $\text{Aut}(X)_0$ is discrete: (14) can be written as

$$L_{NA}(\phi) \geq (1 - \gamma)E_{NA}(\phi)(+\gamma \Lambda_{NA}(\phi)).$$

Based on Minimal Model Program (MMP) devised in [L.-Xu ’12]:

1. Equivalent to K-stability of Tian and Donaldson (BBJ, Fujita).
2. valuative criterions (Fujita, L., Boucksom-Jonsson).
3. algebraically checkable for (singular) Fano surfaces, and Fano varieties with large symmetries (e.g. all toric Fano varieties)
Examples: toric Fano manifolds

Toric manifolds $\leftrightarrow$ lattice polytopes. Fano $\leftrightarrow$ reflexive polytope.

Set $\beta(X) = \sup\{t; \exists \omega \in 2\pi c_1(X) \text{ s.t. } \text{Ric}(\omega) > t\omega\} \in (0, 1]$.

$\beta(X) = 1 \iff \text{KE} \overset{\text{Wang-Zhu}}{\iff} P_c = O \iff \text{Aut}(X)_0$ uniformly K-stable.

**Theorem (L. '09)**

If $P_c \neq O$, then $\beta(X_\triangle) = |OQ| / |P_cQ|$, where $Q = \overrightarrow{P_cO} \cap \partial \triangle$.

Example: $\beta(\mathbb{P}^2 \# \mathbb{P}^2) = 6/7$ (Székelyhidi), $\beta(\mathbb{P}^2 \# 2\mathbb{P}^2) = 21/25$. 

**Ideas and Proofs**
Valuative criterion

Any divisor $E$ on $Y$ ($\mu \rightarrow X$) defines a valuation $\nu := \text{ord}_E$.

$$\text{vol}(\mathcal{F}_v^{(x)}) = \lim_{m \rightarrow +\infty} \frac{h^0(X, -m\mu^* K_X - \lceil mx \rceil E)}{m^n/n!}$$

$$S(\nu) = \frac{1}{V} \int_0^{+\infty} \text{vol}(\mathcal{F}_v^{(x)}) dx = \frac{1}{V} \int_0^{+\infty} x(-d\text{vol}(\mathcal{F}^{(x)})) .$$

$S(\nu)$ is some average vanishing order of holomorphic sections.

**Theorem (Fujita, L.)**

1. $(X, -K_X)$ is uniformly Ding/K-stable if and only if $\exists \delta > 1$ such that for any $\nu \in X_{\mathbb{Q}}^{\text{div}}$, $A_X(\nu) \geq \delta S(\nu)$.

2. $(X, -K_X)$ is $\text{Aut}(X)_0$-uniformly stable if $\text{Aut}(X)_0$ is reductive, $\text{Fut} \equiv 0$ and $\exists \delta > 1$ such that for any $\nu \in X_{\mathbb{Q}}^{\text{div}}$, there exists $\xi \in N_{\mathbb{R}}$ s.t. $A_X(\nu_\xi) \geq \delta S(\nu_\xi)$. 

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Ideas and Proofs
Proof of a special singular case (L.-Tian-Wang ’17)

1. Take a resolution of singularities $\mu : Y \to X$:

$$-K_Y = \left( \mu^*(-K_X) - \epsilon \sum_i \theta_i E_i \right) + \sum_i \left( (1 - A(E_i) + \epsilon \theta_i) E_i \right).$$

2. Prove that $(Y, B_\epsilon = \sum_i (1 - \beta_{i, \epsilon}) E_i)$ is uniformly K-stable when $0 < \epsilon \ll 1$, by using the valuative criterion.

3. If the cone angle $0 < 2\pi A(E_i) \leq 2\pi$, then $(Y, B_\epsilon)$ is a log Fano pair and we can construct KE metrics $\omega_\epsilon$ on $Y$ with edge cone singularities along $E_i$.

4. Prove that as $\epsilon \to 0$, $\omega_\epsilon$ converges to a KE metric in potential=metric=algebraic sense. Techniques include: pluripotential theory, Cheeger-Colding-Tian theory for edge cone Kähler-Einstein metrics, partial $C^0$-estimates.
Serious difficulties when cone angle is bigger than $2\pi$:

$$\beta_i, \epsilon = A(E_i) - \epsilon \theta_i > 1 \iff B_\epsilon = B_\epsilon^+ - B_\epsilon^- \text{ non-effective.}$$

Fortunately, a different strategy initiated by Berman-Boucksom-Jonsson. **Key observation:** the valuative/non-Archimedean side works for in-effective pairs.
BBJ's proof in case $X$ smooth and $\text{Aut}(X)$ discrete

Proof by contradiction: Assume $D$ (and $M$) not coercive.

1. construct a destabilizing geodesic ray $\Phi$ in $E^1(-K_X)$ such that

$$0 \geq D'^\infty(\Phi) = -E'^\infty(\Phi) + L'^\infty(\Phi), \quad E'^\infty(\Phi) = -1.$$  

2. $\phi_m := (\text{Bl}_{J(m\Phi)}(X \times \mathbb{C}), L_m = \pi^*_m p^*_1(-K_X) - \frac{1}{m+m_0} E_m)$. Just need to show that the TC $\Phi_m, m \gg 1$ is destabilizing.

3. Comparison of slopes:

$$\Phi_m \geq \Phi - C \implies E^{NA}(\phi_m) \geq E'^\infty(\Phi) \quad (\text{FAILS when } X \text{ is singular!})$$

$$\lim_{m \to +\infty} L^{NA}(\phi_m) = L^{NA}(\phi) = L'^\infty(\Phi).$$

4. Contradiction to uniform stability:

$$-1 = E'^\infty(\Phi) \geq L'^\infty(\Phi) = L^{NA}(\phi) \leftarrow L^{NA}(\phi_m)$$

$$\geq \text{Stability} \quad (1 - \gamma)E^{NA}(\phi_m) \geq (1 - \gamma)E'^\infty(\Phi) = \gamma - 1.$$
Proof by contradiction. Assume $D$ (and $M$) not $\mathcal{G}$-coercive.

1. Construct a geodesic ray $\Phi$ in $\mathcal{E}^1(-K_X)$ as before.
2. Prove uniform stability of $(Y, B_\epsilon)$ for $0 < \epsilon \ll 1$.
3. Perturbed destabilizing geodesic sub-ray $\Phi_\epsilon = \mu^* \Phi + \epsilon \varphi_M$. Blow-up $\mathcal{J}(m\Phi_\epsilon)$ to get test configurations $\phi_{\epsilon,m} := (Y_{\epsilon,m}, B_{\epsilon,m}, L_{\epsilon,m})$ of $(Y, B_\epsilon)$.
4. Comparison of slopes

\[
\mathcal{E}^{\text{NA}}(\phi_{\epsilon,m}) \geq \mathcal{E}'(\Phi_\epsilon) \quad \text{(true since $Y$ is smooth)}
\]
\[
\lim_{\epsilon \to 0} \mathcal{E}'(\Phi_\epsilon) = \mathcal{E}'(\Phi) \quad \text{(key new convergence)}
\]
\[
\lim_{\epsilon \to 0} \mathcal{L}^{\text{NA}}(\phi_\epsilon) = \mathcal{L}^{\text{NA}}(\phi) \quad \text{(key new convergence)}
\]

5. Chain of contradiction to uniform stability of $(Y, B_\epsilon)$:

\[
-1 = \mathcal{E}'(\Phi) \geq \mathcal{L}'(\Phi) \leftarrow \mathcal{L}^{\text{NA}}(\phi_\epsilon) \leftarrow \mathcal{L}^{\text{NA}}(\phi_{\epsilon,m}) \geq \text{Stab.} 
\]
\[
(1 - \gamma_\epsilon)\mathcal{E}^{\text{NA}}(\phi_{\epsilon,m}) \geq (1 - \gamma_\epsilon)\mathcal{E}'(\Phi_\epsilon) \]
\[
\rightarrow (1 - \gamma)\mathcal{E}'(\Phi) = \gamma - 1.
\]
1. Valuative criterion for $\text{Aut}(X)_0$-uniform stability: $\exists \delta > 1$, s.t.

$$\inf_{v \in X_0^{\text{div}}} \sup_{\xi \in N_\mathbb{R}} (AX(v_\xi) - \delta S(v_\xi)) \geq 0. \quad (15)$$

2. Non-Archimedean metrics $\leftrightarrow$ functions on $X_Q^{\text{div}}$.

$$\phi_\xi(v) = \phi(v_\xi) + \theta(\xi), \quad \theta(\xi) = AX(v_\xi) - AX(v). \quad (16)$$

3. Reduce the infimum (resp. supremum) to “bounded” subsets of $X_Q^{\text{div}}$ (resp. $N_\mathbb{R}$) (depending on Strong Openness Conjecture)

4. Delicate interplay between convexity and coerciveness of Archimedean and non-Archimedean energy.
3-parameters approximation argument:

\[ E'^{\infty}(\Phi) \geq L'^{\infty}(\Phi) + O(k^{-1}) \]
\[ \leftarrow L'^{\infty}(\Phi_{\epsilon}) + O(\epsilon, k^{-1}) \]
\[ \leftarrow L^{\text{NA}}(\phi_{\epsilon,m}) + O(\epsilon, m^{-1}, k^{-1}) \]
\[ = A(v_{k}) + \phi_{\epsilon,m}(v_{k}) \]
\[ = A(v_{k},-\xi_{k}) + \phi_{\epsilon,m,-\xi_{k}}(v_{k},\xi_{k}) \]
\[ \geq \delta S_{L_{\epsilon}}(v_{k},-\xi_{k}) + \phi_{\epsilon,m,-\xi_{k}}(v_{k},\xi_{k}) \]
\[ \geq \delta E^{\text{NA}}(\delta^{-1}\phi_{\epsilon,m,-\xi}) \]
\[ \geq (1 - \delta^{-1/n})J^{\text{NA}}(\phi_{\epsilon,m,-\xi_{k}}) + E^{\text{NA}}(\phi_{\epsilon,m,-\xi_{k}}) \]
\[ = (1 - \delta^{-1/n})\Lambda^{\text{NA}}(\phi_{\epsilon,m,-\xi_{k}}) + \delta^{-1/n}E^{\text{NA}}(\phi_{\epsilon,m,-\xi_{k}}) \]
\[ \geq (1 - \delta^{-1/n})\Lambda'^{\infty}(\Phi_{\epsilon,-\xi_{k}}) + \delta^{-1/n}E'^{\infty}(\Phi_{\epsilon,-\xi_{k}}) \]
\[ = (1 - \delta^{-1/n})J'^{\infty}(\Phi_{\epsilon,-\xi_{k}}) + E'^{\infty}(\Phi_{\epsilon,-\xi_{k}}) \]
\[ \geq (1 - \delta^{-1/n})\chi + E'^{\infty}(\Phi). \]
Data for $g$-solitons

- $\mathbb{T} \cong (\mathbb{C}^*)^r$ acts effectively on a Fano manifold $(X, -K_X)$.
- $t = \text{Lie}(\mathbb{T}) = \text{Span}_\mathbb{R}\{\xi_1, \ldots, \xi_r\} \otimes \mathbb{C}$.
- $e^{-\varphi}$: smooth Hermitian metric on $-K_X$, with Kähler curvature form: $dd^c \varphi > 0$.

Hamiltonian function:

$$\theta_{k, \varphi} = \frac{\mathcal{L}_{\xi_k} e^{-\varphi}}{e^{-\varphi}}, \quad \iota_{\xi_k} dd^c \varphi = \frac{\sqrt{-1}}{2\pi} \partial \theta_{k, \varphi}.$$

Moment map and moment polytope:

$$\mathbf{m}_{\varphi} = (\theta_1, \varphi, \ldots, \theta_r, \varphi) : X \rightarrow P = \mathbf{m}_{\varphi}(X) \subset \mathbb{R}^r.$$

- $g : P \rightarrow \mathbb{R}_{>0}$: a smooth positive function on the moment polytope $P$
- $V_g := \int_X g(\mathbf{m}_{\varphi})(dd^c \varphi)^n = \int_P g(y)(\mathbf{m}_{\varphi})_*(dd^c \varphi)^n.$
\textbf{g-Monge-Ampère equation:}

\[ \text{MA}_g(\varphi) := g(m\varphi)(dd^c\varphi)^n = \Omega. \] (17)

Berman-Witt-Nyström: (17) as a complex version of optimal transport equation, which is always uniquely solvable (Calabi-Yau type results)

\textbf{Kähler-Ricci g-soliton:}

\[ g(m\varphi)(dd^c\varphi)^n = e^{-\varphi}. \] (18)

1. $g = 1$: Kähler-Einstein.
2. $g = e\sum_k c_k \theta_k$: Kähler-Ricci soliton (limits of Kähler-Ricci flow)

\[ \text{Ric}(dd^c\varphi) = dd^c\varphi + \mathcal{L} \sum_k c_k \xi_k dd^c\varphi. \] (19)

3. $g = \sum_k c_k \theta_k$: Mabuchi soliton (limits of inverse Monge-Ampère flow)
Archimedean functionals:

\[
E_g(\varphi) = \frac{1}{V_g} \int_0^1 dt \int_X \phi g(m_\varphi)(dd^c \varphi)^n
\]

\[
\Lambda_g(\varphi) = \frac{1}{V_g} \int_X (\varphi - \varphi_0)(MA_g(\varphi_0) - MA_g(\varphi))
\]

\[
J_g(\varphi) = \Lambda_g(\varphi) - E_g(\varphi)
\]

\[
D_g(\varphi) = -L(\varphi) + E_g(\varphi).
\]

Non-Archimdean functionals:

\[
E^{NA}_g(\phi) = \frac{1}{V_g} \int \theta_\eta(\phi) g(m_\varphi)(dd^c \varphi)^n
\]

\[
\Lambda^{NA}_g(\phi) = \sup_{x_0} \theta_\eta(\varphi) = \Lambda^{NA}(\phi)
\]

\[
J^{NA}_g(\phi) = \Lambda^{NA}_g(\phi) - E^{NA}_g(\phi)
\]

\[
D^{NA}_g(\phi) = -L^{NA}(\phi) + E^{NA}_g(\phi)
\]

\[
S_g(v) = \frac{1}{V_g} \int_0^{+\infty} \text{vol}_{g}(-K_X - xv)dx.
\]
Set
\[ G = \text{Aut}(X, \mathbb{T}) := \{ \sigma \in \text{Aut}(X); \sigma \cdot x = x \cdot \sigma \quad \forall x \in \mathbb{T} \}. \] (20)

Theorem (Han-L. '20)

The following are equivalent:

1. \((X, \mathbb{T})\) admits a Kähler-Ricci \(g\)-soliton.
2. \(D_g\) is \(G\)-coercive.
3. \(\text{Aut}(X, \mathbb{T})\)-uniformly \(g\)-Ding/K-stable.
4. \(\text{Aut}(X, \mathbb{T})\)-uniformly \(g\)-Ding/K-stable among \(G \times \mathbb{T}\)-equivariant special test configurations.

Theorem (Han-L. '20)

\((X, \mathbb{T})\) is \(\text{Aut}(X, \mathbb{T})\)-uniformly \(g\)-Ding/K-stable if and only if \(\exists \delta > 1\) s.t.
\[ \inf_{v \in X^\text{div}_Q} \sup_{\xi \in \mathbb{N}_R} (A_X(v_\xi) - \delta \cdot S_g(v_\xi)) \geq 0. \] (21)
For any \( \vec{k} = (k_1, \ldots, k_r) \), set:

\[
\mathbb{S}[\vec{k}] = S^{2k_1+1} \times \cdots \times S^{2k_r+1},
\]

\[
(X[\vec{k}], L[\vec{k}]) = (X, L) \times \mathbb{S}[\vec{k}]/(S^1)^r,
\]

\[
(X'[\vec{k}], L'[\vec{k}]) = (X, L) \times \mathbb{S}[\vec{k}]/(S^1)^r.
\]

Applications to monomial \( g = \prod_{\alpha=1}^r \theta_{\alpha}^{k_\alpha} \) (and to polynomial \( g \)):

1. Define \( MA_g(\varphi) \) for \( \varphi \in (\mathcal{E}^1)^{(S^1)^r} \);
2. Prove the slope formula \( F'_\infty = F_{g}^{\text{NA}} \);
3. Prove the monotonicity formula for \( D_g^{\text{NA}} \) along MMP.

For general smooth \( g \), we use the Stone-Weierstrass approximation theorem to reduce to the polynomial case.
Thanks for your attention!