Generalized YTD conjecture on Fano varieties

Chi Li

Department of Mathematics, Purdue University

Workshop on Geometric Analysis, September 21, 2020

Table of Contents

Uniformatization Theorem for Riemann Surfaces

Riemann surface: surface with a complex structure:

Topology	Metric	Curvature
$\mathbb{S}^2 = \mathbb{CP}^1$	spherical	1
$\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$	flat	0
$\Sigma_{\mathfrak{g}} = \mathbb{B}^1/\pi_1(\Sigma_{\mathfrak{g}})$	hyperbolic	-1

Riemannian metric: $\mathrm{g} = E|dz|^2 = \frac{\partial^2\varphi}{\partial z \partial \bar{z}}|dz|^2 = \frac{1}{4}\Delta\varphi|dz|^2$.

$$\begin{aligned} & \text{Constant Gauss/Ricci curvature equation} \\ = \; & \text{1-dimensional complex Monge-Ampère equation} \end{aligned}$$

$$Ric(\omega) = \lambda\omega \Longleftrightarrow -\Delta \log \Delta\varphi = \lambda\Delta\varphi \Longleftrightarrow \Delta\varphi = e^{-\lambda\varphi}.$$

Kähler manifolds and Kähler metrics

X: complex manifold; J: $TX \to TX$ integrable complex structure;
g: Kähler metric, $\mathrm{g}(J\cdot, J\cdot) = \mathrm{g}(\cdot,\cdot)$ and $d\omega = 0$.

$$\omega = \mathrm{g}(\cdot, J\cdot) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} \omega_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad (\omega_{i\bar{j}}) > 0.$$

Kähler class $[\omega] \in H^2(X, \mathbb{R})$.

Fact ($\partial\bar{\partial}$-Lemma): Set $\mathrm{dd^c} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$. Any $\omega' \in [\omega]$ is of the form

$$\omega_u := \omega + \mathrm{dd^c} u := \omega + \frac{\sqrt{-1}}{2\pi} \sum_{i,j} \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} dz^i \wedge d\bar{z}^j.$$

Kähler metric as curvature forms

$L \to X$: a $\mathbb{C}$-line bundle with holomorphic transition $\{f_{\alpha\beta}\}$.
$e^{-\varphi} := \{e^{-\varphi_\alpha}\}$ Hermitian metric on L:

$$e^{-\varphi_\alpha} = |f_{\alpha\beta}|^2 e^{-\varphi_\beta}. \tag{1}$$

Definition: L is positive (=ample) if $\exists\ e^{-\varphi} = \{e^{-\varphi_\alpha}\}$ on L s.t.

$$\omega + \mathrm{dd^c} u = \mathrm{dd^c}\varphi := \mathrm{dd^c}\varphi_\alpha > 0. \tag{2}$$

Anticanonical line bundle: $-K_X = \wedge^n T_{\mathrm{hol}} X$, $K_X = \wedge^n T^*_{\mathrm{hol}} X$.

Fact: {smooth volume forms}={Hermitian metrics on $-K_X$ }

$$\begin{aligned} c_1(X) \ni Ric(\omega) &= -\mathrm{dd^c} \log \omega^n \\ &= -\frac{\sqrt{-1}}{2\pi} \sum_{i,j} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(\omega_{k\bar{l}}) dz^i \wedge d\bar{z}^j. \end{aligned}$$

Kähler-Einstein metric and Monge-Ampère equation

KE equation:

$$Ric(\omega_u) = \lambda\, \omega_u \iff (\omega + \mathrm{dd^c} u)^n = e^{h_\omega - \lambda u} \omega^n \tag{3}$$

$$\iff \det\left(\omega_{i\bar{j}} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) = e^{h_\omega - \lambda u} \det(\omega_{i\bar{j}}).$$

$\lambda = -1$	Solvable (Aubin, Yau)	$c_1(X) < 0$
$\lambda = 0$	Solvable (Yau)	$c_1(X) = 0$
$\lambda = 1$	$\exists$ obstructions	$c_1(X) > 0$

Fano manifolds

X Fano: $c_1(X) > 0 \iff \exists$ Kähler metric ω with $Ric(\omega) > 0$.

1. $\dim_{\mathbb{C}} = 1$: $\mathbb{P}^1 = S^2$.
2. $\dim_{\mathbb{C}} = 2$: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2 \sharp k \overline{\mathbb{P}^2}$, $1 \leq k \leq 8$ (del Pezzo).
3. $\dim_{\mathbb{C}} = 3$: 105 deformation families (Iskovskikh, Mori-Mukai)
4. Smooth hypersurface in $\mathbb{P}^n$ of degree $< n + 1$;
5. Toric Fano manifolds

Fact: there are finitely many deformation family in each dimension (Campana, Kollár-Miyaoka-Mori, Nadel '90).

Obstructions and uniqueness

1. $\exists$ KE $\Longrightarrow \mathrm{Aut}(X)$ is reductive: $\mathrm{Aut}(X)_0$ is the complexification of a compact Lie group (Matsushima)

2. Futaki invariant: $\forall$ holomorphic vector field ξ ,

$$\exists \text{ KE} \quad \Longrightarrow \quad \mathrm{Fut}(\xi) := \int_X \xi(h_\omega)\omega^n = 0. \tag{4}$$

3. Energy coerciveness (Tian, Tian-Zhu, Phong-Song-Sturm-Weinkove, Darvas-Rubinstein, Hisamoto)

4. K-stability (Tian, Donaldson)
   Ding stability (Berman, Boucksom-Jonsson)
   all equivalent (**L.**-Xu, Berman-Boucksom-Jonsson, Fujita).

Uniqueness: KE metrics are unique up to $\mathrm{Aut}(X)_0$ (Bando-Mabuchi, Berndtsson)

Theorem (Tian, Chen-Donaldson-Sun, Berman)

A Fano manifold X admits a KE metric if and only if X is K-stable ($\mathrm{Aut}(X)$ is discrete), or X K-polystable ($\mathrm{Aut}(X)$ is continuous).

We extend this theorem in two directions (with different proofs with those of the above):

1. Any (singular) $\mathbb{Q}$-Fano variety X (**L.**-Tian-Wang, **L.**)
2. Kähler-Ricci g-soliton (Han-**L.**)

Remark: In particular, we recover the above theorem with K-polystability replaced by appropriate ($\mathbb{G}$-)uniform K-stability. Conjecturally, uniform K-stability is equivalent to K-stability, which is reduced to an algebraic geometric problem.

$\mathbb{Q}$-Fano varieties: building blocks of algebraic varieties

Definition

$\mathbb{Q}$-Fano variety X is a normal projective variety satisfying:

1. *Fano: $\mathbb{Q}$-line bundle $-K_X$ is ample.*
2. *klt (Kawamata log terminal): $\forall s_\alpha^* \sim dz^1 \wedge \ldots dz^n \in \mathcal{O}_{K_X}(U_\alpha)$*

$$\int_{U^{\mathrm{reg}}} (\sqrt{-1}^{n^2} s_\alpha^* \wedge \bar{s}_\alpha^*) < +\infty. \tag{5}$$

Let $\mu : Y \to X$ be a resolution of singularities (Hironaka)

$$K_Y = \mu^* K_X + \sum_i (A_X(E_i) - 1) E_i. \tag{6}$$

The condition (5) $\iff \mathrm{mld} := \min_i A_X(E_i) > 0$.

Fact: (Birkar '16) ϵ-klt (i.e. $\mathrm{mld} \geq \epsilon > 0$) Fanos are bounded.

Fact: KE equation (only) makes sense on all $\mathbb{Q}$-Fano varieties.

Klt singularities: Examples

1. Smooth points, Orbifold points = (normal) Quotient singularities
2. $X = \{F(z_1, \ldots, z_{n+1}) = 0\} \subset \mathbb{C}^{n+1}$ with F homogenenous $\deg(F) < n+1$ s.t. X has an isolated singularity at 0.
3. Orbifold cones over log-Fano varieties. weighted homogeneous examples:

$$z_1^2 + z_2^2 + z_2^2 + z_3^{2k} = \mathcal{C}\left((\mathbb{P}^1 \times \mathbb{P}^1, (1 - \frac{1}{k})\Delta), H \right)$$

$$z_1^2 + z_2^2 + z_2^2 + z_3^{2k+1} = \mathcal{C}\left((\mathbb{P}^2, (1 - \frac{1}{2k+1})D), H \right)$$

4. ($\mathbb{Q}$-Gorenstein) deformation of Klt singularities are also Klt singularities.

KE equation on $\mathbb{Q}$-Fano varieties

Hermitian metric on the ($\mathbb{Q}$-)line bundle $-K_X$: $e^{-\varphi} = \{e^{-\varphi_\alpha}\}$ s.t.

$$e^{-\varphi_\alpha} = |s_\alpha|^2 e^{-\varphi}, \quad \{s_\alpha\} \text{ trivializing sections of } -K_X$$

Kähler-Einstein equation on Fano varieties:

$$\boxed{(\mathrm{dd^c}\varphi)^n = e^{-\varphi}} := |s_\alpha|^2 e^{-\varphi}\left(\sqrt{-1}^{n^2} s_\alpha^* \wedge \bar{s}_\alpha^*\right). \tag{7}$$

$\omega = \mathrm{dd^c}\psi \Rightarrow u = \varphi - \psi$ is globally defined. Then (7)$\Longleftrightarrow$ (3).

(weak) KE potential: generalized solutions in pluripotential sense.

Fact: Obstructions/uniqueness continue to hold for $\mathbb{Q}$-Fano case.

Face: Solutions are smooth on X^{reg}.

Fact: Aubin and Yau's theorems hold on projective varieties with Klt singularities (Eyssidieux-Guedj-Zeriahi based on Kołodziej).

Yau-Tian-Donaldson conjecture on Fano varieties

Theorem (L.-Tian-Wang, L. '19)

A $\mathbb{Q}$-Fano variety X has a KE potential if (and only if) X is $\mathrm{Aut}(X)_0$-uniformly K/Ding-stable.

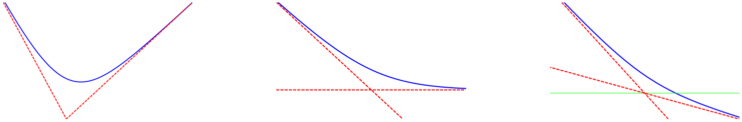
1. X Smooth (Chen-Donaldson-Sun, Tian, Datar-Székelyhidi).
2. $\mathbb{Q}$-Gorenstein smoothable (Spotti-Sun-Yao, **L.**-Wang-Xu);
3. *Good* (e.g. crepant) resolution of singularities (**L.**-Tian-Wang).

Proofs in above special cases depend on compactness/regularity theory in metric geometry and do NOT generalize to the general singular case.

4. X smooth & $\mathrm{Aut}(X)$ discrete: Berman-Boucksom-Jonsson (BBJ) in 2015 proposed an approach using pluripotential theory/non-Archimedean analysis. Our work greatly extends their work by removing the two assumptions.

Variational point of view

Consider energy functionals on a pluripotential version of Sobolev space, denoted by $\mathcal{E}^1(X, -K_X)$ (Cegrell, Guedj-Zeriahi)



(a) Proper (b) Bounded (c) Unbounded

There is a distance-like energy:

$$\mathbf{J}(\varphi) = \mathbf{\Lambda}(\varphi) - \mathbf{E}(\varphi) \quad \sim \quad \sup(\varphi - \psi) - \mathbf{E}(\varphi) > 0. \tag{8}$$

E is the primitive of complex Monge-Ampère operator:

$$\mathbf{E}(\varphi) = \frac{1}{V}\int_0^1 dt \int_X \dot{\varphi}(\mathrm{dd^c}\varphi)^n, \quad \frac{d}{dt}\mathbf{E}(\varphi) = \frac{1}{V}\int_X \dot{\varphi}(\mathrm{dd^c}\varphi)^n. \tag{9}$$

Analytic criterion for KE potentials

Energy functional with KE as critical points:

$$\mathbf{L}(\varphi) = -\log\left(\int_X e^{-\varphi}\right), \quad \mathbf{D} = -\mathbf{E} + \mathbf{L}. \tag{10}$$

The Euler-Lagrangian equation is just the KE equation:

$$\delta\mathbf{D}(\delta\varphi) = \frac{1}{V}\int_X (\delta\varphi)\left(-(\mathrm{dd^c}\varphi)^n + C\cdot e^{-\varphi}\right). \tag{11}$$

Analytic criterion (generalizing Tian-Zhu, Phong-Song-Sturm-Weinkove):

Theorem (Darvas-Rubinstein, Darvas, Di-Nezza-Guedj, Hisamoto, based on the compactness by BBEGZ and uniqueness by Berndtsson)

A $\mathbb{Q}$-Fano variety X admits a KE potential if and only if

1. $\mathrm{Aut}(X)_0$ *is reductive* (with center $\mathbb{T} \cong (\mathbb{C}^*)^r$)
2. *Moser-Trudinger type inequality: there exist $\gamma > 0$ and $C > 0$ s.t.* $\forall \varphi \in \mathcal{E}^1(X, -K_X)^{\mathbb{K}}$,

$$\mathbf{D}(\varphi) \geq \gamma \cdot \inf_{\sigma\in\mathbb{T}} \mathbf{J}(\sigma^*\varphi) - C. \quad (\mathrm{Aut}(X)_0\text{-coercive})$$

Contact with algebraic geometry: Test configurations (Tian, Donaldson)

A test configuration (TC) $(\mathcal{X}, \mathcal{L}, \eta)$ of $(X, -K_X)$ consists of:

1. $\pi : \mathcal{X} \to \mathbb{C}$: a $\mathbb{C}^*$-equivariant family of projective varieties;
2. $\mathcal{L} \to \mathcal{X}$: a $\mathbb{C}^*$-equiv. semiample holomorphic $\mathbb{Q}$-line bundle;
3. $\eta : (\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^* \cong (X, -K_X) \times \mathbb{C}^*$.

Any test configuration is generated by a one-parameter subgroup of $GL(N_m)$ (with $m \gg 1$) under the Kodaira embedding

$$X \longrightarrow \mathbb{P}(H^0(X, -mK_X)^*) \cong \mathbb{P}^{N_m - 1}.$$

A test configuration is called special if the central fibre $\mathcal{X}_0$ is a $\mathbb{Q}$-Fano variety.

Slopes along subgeodesics

Under the isomorphism η, any smooth psh metric on $\mathcal{L} \to \mathcal{X}$ induces a family of smooth psh metrics $\Phi = \{\varphi(t)\}$ on $(X, -K_X)$.

Theorem (Ding-Tian, Paul-Tian, Phong-Sturm-Ross, Berman, Boucksom-Hisamoto-Jonsson)

For any $\mathbf{F} \in \{\mathbf{E}, \mathbf{J}, \mathbf{L}, \mathbf{D}\}$,

$$\mathbf{F}'^{\infty}(\Phi) := \lim_{t \to +\infty} \frac{\mathbf{F}(\varphi(t))}{-\log |t|^2} \tag{12}$$

exists, and is equal to an algebraic invariant $\mathbf{F}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$.

Coercivity (2) $\implies$ $\mathbf{D}'^{\infty}(\Phi) \geq \gamma \cdot \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}'^{\infty}(\sigma_\xi(t)^* \Phi)$.

Test configurations=smooth non-Archimedean metrics

Boucksom-Jonsson: Test configuration defines a non-Archimedean metric, represented by a function on the space of valuations:

$$\phi(v) = \phi_{(\mathcal{X},\mathcal{L})}(v) = -G(v)(\Phi), \quad v \in X_{\mathbb{Q}}^{\mathrm{div}}. \tag{13}$$

$G(v)(\Phi)$: the generic Lelong number of Φ with respect to $G(v)$.
Non-Archimedean functionals:

$$\begin{aligned}
\mathbf{E}^{\mathrm{NA}}(\phi) &= \frac{1}{V}\frac{\bar{\mathcal{L}}^{\cdot(n+1)}}{n+1} = \int_{\mathcal{X}_0} \theta_\eta(\varphi)(\mathrm{dd^c}\varphi)^n, \\
\mathbf{\Lambda}^{\mathrm{NA}}(\phi) &= \frac{1}{V}\bar{\mathcal{L}} \cdot p_1^*(-K_X)^{\cdot n} = \sup_{\mathcal{X}_0}(\theta_\eta(\varphi)), \\
\mathbf{J}^{\mathrm{NA}}(\phi) &= \mathbf{\Lambda}^{\mathrm{NA}}(\phi) - \mathbf{E}^{\mathrm{NA}}(\phi), \\
\mathbf{L}^{\mathrm{NA}}(\phi) &= \inf_{v \in X_{\mathbb{Q}}^{\mathrm{div}}} (A_X(v) + \phi(v)).
\end{aligned}$$

$X_{\mathbb{Q}}^{\mathrm{div}}$: space of divisorial valuations; $G(v)$: Gauss extension.

K-stability and Ding-stability

Definition-Theorem (Berman, Hisamoto, Boucksom-Hisamoto-Jonsson)

X KE implies that it is $\mathrm{Aut}(X)_0$-uniformly Ding-stable: $\exists \gamma > 0$ (slope) such that for all $\mathrm{Aut}(X)_0$-equivariant test configurations

$$\mathbf{D}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}, \eta + \xi). \tag{14}$$

When $\mathrm{Aut}(X)_0$ is discrete: (14) can be written as

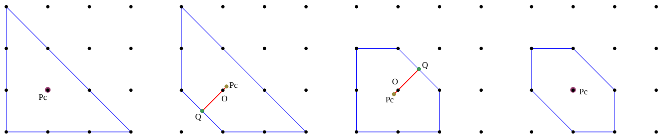
$$\mathbf{L}^{\mathrm{NA}}(\phi) \geq (1-\gamma)\mathbf{E}^{\mathrm{NA}}(\phi)(+\gamma \mathbf{\Lambda}^{\mathrm{NA}}(\phi)).$$

Based on Minimal Model Program (MMP) devised in [**L.**-Xu '12]:

1. Equivalent to K-stability of Tian and Donaldson (BBJ, Fujita).
2. valuative criterions (Fujita, **L.**, Boucksom-Jonsson).
3. algebraically checkable for (singular) Fano surfaces, and Fano varieties with large symmetries (e.g. all toric Fano varieties)

Examples: toric Fano manifolds

Toric manifolds $\leftrightarrow$ lattice polytopes. Fano $\leftrightarrow$ reflexive polytope.



(d) $\mathbb{P}^2$ (e) $\mathbb{P}^2\sharp\overline{\mathbb{P}^2}$ (f) $\mathbb{P}^2\sharp 2\overline{\mathbb{P}^2}$ (g) $\mathbb{P}^2\sharp 3\overline{\mathbb{P}^2}$

Set $\beta(X) = \sup\{t; \exists\, \omega \in 2\pi c_1(X) \text{ s.t. } Ric(\omega) > t\omega\} \in (0,1]$.

$\beta(X) = 1 \Longleftrightarrow \mathrm{KE} \overset{Wang-Zhu}{\Longleftrightarrow} P_c = O \Longleftrightarrow \mathrm{Aut}(X)_0$ – uniformly K-stable.

Theorem (L. '09)

If $P_c \neq O$, then $\beta(X_\triangle) = \left|\overline{OQ}\right| / \left|\overline{P_c Q}\right|$, where $Q = \overrightarrow{P_c O} \cap \partial\triangle$.

Example: $\beta(\mathbb{P}^2\sharp\overline{\mathbb{P}^2}) = 6/7$(Székelyhidi), $\beta(\mathbb{P}^2\sharp 2\overline{\mathbb{P}^2}) = 21/25$.

Valuative criterion

Any divisor E on Y ($\xrightarrow{\mu} X$) defines a valuation $v := \mathrm{ord}_E$.

$$\begin{aligned} \mathrm{vol}(\mathcal{F}_v^{(x)}) &= \lim_{m\to+\infty} \frac{h^0(X, -m\mu^* K_X - \lceil mx \rceil E)}{m^n/n!} \\ S(v) &= \frac{1}{V}\int_0^{+\infty} \mathrm{vol}(\mathcal{F}_v^{(x)})dx = \frac{1}{V}\int_0^{+\infty} x(-d\mathrm{vol}(\mathcal{F}^{(x)})). \end{aligned}$$

$S(v)$ is some average vanishing order of holomorphic sections.

Theorem (Fujita, L.)

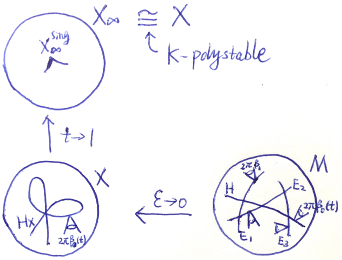
1. $(X, -K_X)$ *is uniformly Ding/K-stable if and only if* $\exists \delta > 1$ *such that for any* $v \in X_{\mathbb{Q}}^{\mathrm{div}}$, $A_X(v) \geq \delta S(v)$.
2. $(X, -K_X)$ *is* $\mathrm{Aut}(X)_0$*-uniformly stable if* $\mathrm{Aut}(X)_0$ *is reductive,* $\mathrm{Fut} \equiv 0$ *and* $\exists \delta > 1$ *such that for any* $v \in X_{\mathbb{Q}}^{\mathrm{div}}$, *there exists* $\xi \in N_{\mathbb{R}}$ *s.t.* $A_X(v_\xi) \geq \delta S(v_\xi)$.

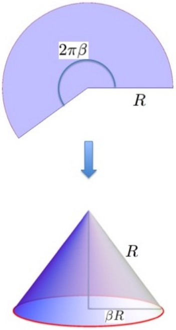
Proof of a special singular case (**L.**-Tian-Wang '17)

1. Take a resolution of singularities $\mu : Y \to X$:

$$-K_Y = \left(\mu^*(-K_X) - \epsilon \sum_i \theta_i E_i\right) + \overbrace{\sum_i (1 - A(E_i) + \epsilon\theta_i) E_i}^{B_\epsilon}.$$

2. Prove that $(Y, B_\epsilon = \sum_i (1 - \beta_{i,\epsilon}) E_i)$ is uniformly K-stable when $0 < \epsilon \ll 1$, by using the valuative criterion.
3. If the cone angle $0 < 2\pi A(E_i) \leq 2\pi$, then (Y, B_ϵ) is a log Fano pair and we can construct KE metrics ω_ϵ on Y with edge cone singularities along E_i.
4. Prove that as $\epsilon \to 0$, ω_ϵ converges to a KE metric in potential=metric=algebraic sense.
   Techniques include: pluripotential theory, Cheeger-Colding-Tian theory for edge cone Kähler-Einstein metrics, partial C^0-estimates.





Serious difficulties when cone angle is bigger than 2π:

$$\beta_{i,\epsilon} = A(E_i) - \epsilon\theta_i > 1 \Longleftrightarrow B_\epsilon = B_\epsilon^+ - B_\epsilon^- \text{non-effective.}$$

Fortunately, a different strategy initiated by Berman-Boucksom-Jonsson.

Key observation: the valuative/non-Archimedean side works for in-effective pairs.

BBJ's proof in case X smooth and $\mathrm{Aut}(X)$ discrete

Proof by contradiction: Assume **D** (and **M**) not coercive.

1. construct a destabilizing geodesic ray Φ in $\mathcal{E}^1(-K_X)$ such that
$$0 \geq \mathbf{D}'^{\infty}(\Phi) = -\mathbf{E}'^{\infty}(\Phi) + \mathbf{L}'^{\infty}(\Phi), \quad \mathbf{E}'^{\infty}(\Phi) = -1.$$
2. $\phi_m := (\mathrm{Bl}_{\mathcal{J}(m\Phi)}(X \times \mathbb{C}), \mathcal{L}_m = \pi_m^* p_1^*(-K_X) - \frac{1}{m+m_0} E_m)$. Just need to show that the TC Φ_m, $m \gg 1$ is destabilizing.
3. Comparison of slopes:
$$\Phi_m \geq \Phi - C \Longrightarrow \mathbf{E}^{\mathrm{NA}}(\phi_m) \geq \mathbf{E}'^{\infty}(\Phi) \quad \text{(FAILS when } X \text{ is singular!)}$$
$$\lim_{m\to+\infty} \mathbf{L}^{\mathrm{NA}}(\phi_m) = \mathbf{L}^{\mathrm{NA}}(\phi) = \mathbf{L}'^{\infty}(\Phi).$$
4. Contradiction to uniform stability:
$$\begin{aligned} & -1 = \mathbf{E}'^{\infty}(\Phi) \geq \mathbf{L}'^{\infty}(\Phi) = \mathbf{L}^{\mathrm{NA}}(\phi) \leftarrow \mathbf{L}^{\mathrm{NA}}(\phi_m) \\ \geq_{\mathrm{Stability}} \quad & (1-\gamma)\mathbf{E}^{\mathrm{NA}}(\phi_m) \geq (1-\gamma)\mathbf{E}'^{\infty}(\Phi) = \gamma - 1. \end{aligned}$$

Perturbed variational approach (**L.**-Tian-Wang, **L.**'19)

Proof by contradiction. Assume **D** (and **M**) not $\mathbb{G}$-coercive.

1. Construct a geodesic ray Φ in $\mathcal{E}^1(-K_X)$ as before.
2. Prove uniform stability of (Y, B_ϵ) for $0 < \epsilon \ll 1$.
3. Perturbed destabilizing geodesic sub-ray $\Phi_\epsilon = \mu^*\Phi + \epsilon\varphi_M$. Blow-up $\mathcal{J}(m\Phi_\epsilon)$ to get test configurations $\phi_{\epsilon,m} := (\mathcal{Y}_{\epsilon,m}, \mathcal{B}_{\epsilon,m}, \mathcal{L}_{\epsilon,m})$ of (Y, B_ϵ).
4. Comparison of slopes

$$\mathbf{E}^{\mathrm{NA}}(\phi_{\epsilon,m}) \geq \mathbf{E}'^{\infty}(\Phi_\epsilon) \quad \text{(true since } Y \text{ is smooth)}$$
$$\lim_{\epsilon\to 0} \mathbf{E}'^{\infty}(\Phi_\epsilon) = \mathbf{E}'^{\infty}(\Phi) \quad \text{(key new convergence)}$$
$$\lim_{\epsilon\to 0} \mathbf{L}^{\mathrm{NA}}(\phi_\epsilon) = \mathbf{L}^{\mathrm{NA}}(\phi) \quad \text{(key new convergence)}$$

5. Chain of contradiction to uniform stability of (Y, B_ϵ):

$$\begin{aligned} & -1 = \mathbf{E}'^{\infty}(\Phi) \geq \mathbf{L}'^{\infty}(\Phi) \leftarrow \mathbf{L}^{\mathrm{NA}}(\phi_\epsilon) \leftarrow \mathbf{L}^{\mathrm{NA}}(\phi_{\epsilon,m}) \\ \geq_{\mathrm{Stab.}} \quad & (1-\gamma_\epsilon)\mathbf{E}^{\mathrm{NA}}(\phi_{\epsilon,m}) \geq (1-\gamma_\epsilon)\mathbf{E}'^{\infty}(\Phi_\epsilon) \\ & \to (1-\gamma)\mathbf{E}'^{\infty}(\Phi) = \gamma - 1. \end{aligned}$$

New ingredients for continuous $\mathrm{Aut}(X)$ (**L.**'19)

1. Valuative criterion for $\mathrm{Aut}(X)_0$-uniform stability: $\exists\delta > 1$, s.t.

$$\inf_{v\in X_{\mathbb{Q}}^{\mathrm{div}}} \sup_{\xi\in N_{\mathbb{R}}} \left(A_X(v_\xi) - \delta S(v_\xi)\right) \geq 0. \tag{15}$$

2. Non-Archimedean metrics $\longleftrightarrow$ functions on $X_{\mathbb{Q}}^{\mathrm{div}}$.

$$\phi_\xi(v) = \phi(v_\xi) + \theta(\xi), \quad \theta(\xi) = A_X(v_\xi) - A_X(v). \tag{16}$$

3. Reduce the infimum (resp. supremum) to "bounded" subsets of $X_{\mathbb{Q}}^{\mathrm{div}}$ (resp. $N_{\mathbb{R}}$) (depending on *Strong Openness Conjecture*)
4. Delicate interplay between convexity and coerciveness of Archimedean and non-Archimedean energy.

3-parameters approximation argument:

$$
\begin{aligned}
\mathbf{E}'^{\infty}(\Phi) &\geq \mathbf{L}'^{\infty}(\Phi) + O(k^{-1}) \\
&\leftarrow \mathbf{L}'^{\infty}(\Phi_\epsilon) + O(\epsilon, k^{-1}) \\
&\leftarrow \mathbf{L}^{\mathrm{NA}}(\phi_{\epsilon,m}) + O(\epsilon, m^{-1}, k^{-1}) \\
&= A(v_k) + \phi_{\epsilon,m}(v_k) \\
&= A(v_{k,-\xi_k}) + \phi_{\epsilon,m,-\xi_k}(v_{k,\xi_k}) \\
&\geq \delta S_{L_\epsilon}(v_{k,-\xi_k}) + \phi_{\epsilon,m,-\xi_k}(v_{k,\xi_k}) \\
&\geq \delta \mathbf{E}^{\mathrm{NA}}(\delta^{-1}\phi_{\epsilon,m,-\xi}) \\
&\geq (1-\delta^{-1/n})\mathbf{J}^{\mathrm{NA}}(\phi_{\epsilon,m,-\xi_k}) + \mathbf{E}^{\mathrm{NA}}(\phi_{\epsilon,m,-\xi_k}) \\
&= (1-\delta^{-1/n})\mathbf{\Lambda}^{\mathrm{NA}}(\phi_{\epsilon,m,-\xi_k}) + \delta^{-1/n}\mathbf{E}^{\mathrm{NA}}(\phi_{\epsilon,m,-\xi_k}) \\
&\geq (1-\delta^{-1/n})\mathbf{\Lambda}'^{\infty}(\Phi_{\epsilon,-\xi_k}) + \delta^{-1/n}\mathbf{E}'^{\infty}(\Phi_{\epsilon,-\xi_k}) \\
&= (1-\delta^{-1/n})\mathbf{J}'^{\infty}(\Phi_{\epsilon,-\xi_k}) + \mathbf{E}'^{\infty}(\Phi_{\epsilon,-\xi_k}) \\
&\geq (1-\delta^{-1/n})\chi + \mathbf{E}'^{\infty}(\Phi).
\end{aligned}
$$

Data for g-solitons

- $\mathbb{T} \cong (\mathbb{C}^*)^r$ acts effectively on a Fano manifold $(X, -K_X)$.
- $\mathfrak{t} = \mathrm{Lie}(\mathbb{T}) = \mathrm{Span}_{\mathbb{R}}\{\xi_1, \ldots, \xi_r\} \otimes \mathbb{C}$.
- $e^{-\varphi}$: smooth Hermitian metric on $-K_X$, with Kähler curvature form: $\mathrm{dd^c}\varphi > 0$.

Hamiltonian function:

$$\theta_{k,\varphi} = \frac{\mathscr{L}_{\xi_k} e^{-\varphi}}{e^{-\varphi}}, \quad \iota_{\xi_k}\mathrm{dd^c}\varphi = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\theta_{k,\varphi}.$$

Moment map and moment polytope:

$$\mathbf{m}_\varphi = (\theta_{1,\varphi}, \ldots, \theta_{r,\varphi}) : X \to P = \mathbf{m}_\varphi(X) \subset \mathbb{R}^r.$$

- $g : P \to \mathbb{R}_{>0}$: a smooth positive function on the moment polytope P
- $V_g := \int_X g(\mathbf{m}_\varphi)(\mathrm{dd^c}\varphi)^n = \int_P g(y)(\mathbf{m}_\varphi)_*(\mathrm{dd^c}\varphi)^n$.

g-Monge-Ampére equation:

$$\mathrm{MA}_g(\varphi) := g(\mathbf{m}_\varphi)(\mathrm{dd^c}\varphi)^n = \Omega. \tag{17}$$

Berman-Witt-Nyström: (17) as a complex version of optimal transport equation, which is always uniquely solvable (Calabi-Yau type results)

Kähler-Ricci g-soliton:

$$g(\mathbf{m}_\varphi)(\mathrm{dd^c}\varphi)^n = e^{-\varphi}. \tag{18}$$

1. $g = 1$: Kähler-Einstein.
2. $g = e^{\sum_k c_k\theta_k}$: Kähler-Ricci soliton (limits of Kähler-Ricci flow)

$$Ric(\mathrm{dd^c}\varphi) = \mathrm{dd^c}\varphi + \mathscr{L}_{\sum_k c_k\xi_k}\mathrm{dd^c}\varphi. \tag{19}$$

3. $g = \sum_k c_k\theta_k$: Mabuchi soliton (limits of inverse Monge-Ampère flow)

Archimedean functionals:

$$\begin{aligned}
\mathbf{E}_g(\varphi) &= \frac{1}{V_g}\int_0^1 dt \int_X \dot{\varphi} g(\mathbf{m}_\varphi)(\mathrm{dd^c}\varphi)^n \\
\mathbf{\Lambda}_g(\varphi) &= \frac{1}{V_g}\int_X (\varphi-\varphi_0)(\mathrm{MA}_g(\varphi_0)-\mathrm{MA}_g(\varphi)) \\
\mathbf{J}_g(\varphi) &= \mathbf{\Lambda}_g(\varphi)-\mathbf{E}_g(\varphi) \\
\mathbf{D}_g(\varphi) &= -\mathbf{L}(\varphi)+\mathbf{E}_g(\phi).
\end{aligned}$$

Non-Archimdean functionals:

$$\begin{aligned}
\mathbf{E}_g^{\mathrm{NA}}(\phi) &= \frac{1}{V_g}\int_{\mathcal{X}_0} \theta_\eta(\varphi) g(\mathbf{m}_\varphi)(\mathrm{dd^c}\varphi)^n \\
\mathbf{\Lambda}_g^{\mathrm{NA}}(\phi) &= \sup_{\mathcal{X}_0} \theta_\eta(\varphi) = \mathbf{\Lambda}^{\mathrm{NA}}(\phi) \\
\mathbf{J}_g^{\mathrm{NA}}(\phi) &= \mathbf{\Lambda}_g^{\mathrm{NA}}(\phi)-\mathbf{E}_g^{\mathrm{NA}}(\phi) \\
\mathbf{D}_g^{\mathrm{NA}}(\phi) &= -\mathbf{L}^{\mathrm{NA}}(\phi)+\mathbf{E}_g^{\mathrm{NA}}(\phi) \\
S_g(v) &= \frac{1}{V_g}\int_0^{+\infty} \mathrm{vol}_g(-K_X - xv)dx.
\end{aligned}$$

Set

$$\mathbb{G} = \mathrm{Aut}(X, \mathbb{T}) := \{\sigma \in \mathrm{Aut}(X); \sigma \cdot x = x \cdot \sigma \ \ \forall x \in \mathbb{T}\}. \tag{20}$$

Theorem (Han-L. '20)

The following are equivalent:

1. $(X, \mathbb{T})$ *admits a Kähler-Ricci g-soliton.*
2. $\mathbf{D}_g$ *is* $\mathbb{G}$*-coercive.*
3. $\mathrm{Aut}(X, \mathbb{T})$*-uniformly g-Ding/K-stable.*
4. $\mathrm{Aut}(X, \mathbb{T})$*-uniformly g-Ding/K-stable among* $\mathbb{G} \times \mathbb{T}$*-equivariant special test configurations.*

Theorem (Han-L. '20)

$(X, \mathbb{T})$ *is* $\mathrm{Aut}(X, \mathbb{T})$*-uniformly g-Ding/K-stable if and only if* $\exists \delta > 1$ *s.t.*

$$\inf_{v \in X_{\mathbb{Q}}^{\mathrm{div}}} \sup_{\xi \in N_{\mathbb{R}}} \left(A_X(v_\xi) - \delta \cdot S_g(v_\xi)\right) \geq 0. \tag{21}$$

Fibration construction

For any $\vec{k} = (k_1, \ldots, k_r)$, set:

$$\begin{aligned} \mathbb{S}^{[\vec{k}]} &= S^{2k_1+1} \times \cdots \times S^{2k_r+1}, \\ (X^{[\vec{k}]}, L^{[\vec{k}]}) &= (X, L) \times \mathbb{S}^{[\vec{k}]}/(S^1)^r, \\ (\mathcal{X}^{[\vec{k}]}, \mathcal{L}^{[\vec{k}]}) &= (\mathcal{X}, \mathcal{L}) \times \mathbb{S}^{[\vec{k}]}/(S^1)^r. \end{aligned}$$

Applications to monomial $g = \prod_{\alpha=1}^{r} \theta_\alpha^{k_\alpha}$ (and to polynomial g):

1. Define $\mathrm{MA}_g(\varphi)$ for $\varphi \in (\mathcal{E}^1)^{(S^1)^r}$;
2. Prove the slope formula $\mathbf{F}_g'^{\infty} = \mathbf{F}_g^{\mathrm{NA}}$;
3. Prove the monotonicity formula for $\mathbf{D}_g^{\mathrm{NA}}$ along MMP.

For general smooth g, we use the Stone-Weierstrass approximation theorem to reduce to the polynomial case.

Thanks for your attention!