Recent progress on the YTD conjecture for cscK metrics

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Table of Contents

1. Introduction and results
2. Variational approach and analytic criterion
3. Test configurations and $\mathbb{G}$-uniform K-stability
4. Maximal geodesic rays and approximation by TC
5. K-stability for models and Fujita approximation

Uniformatization Theorem for Riemann Surfaces

Riemann surface: surface with a complex structure:

Topology	Metric	Curvature
$\mathbb{S}^2 = \mathbb{CP}^1$	spherical	1
$\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$	flat	0
$\Sigma_g = \mathbb{B}^1/\pi_1(\Sigma_g)$	hyperbolic	-1

Σ_g closed oriented surface of genus $g \geq 2$.
$\mathbb{B}^1 = \{z \in \mathbb{C}; |z| < 1\}$.

Generalization for higher dimensional complex projective manifolds?

Kähler manifolds and Kähler metrics

X: complex manifold, $\{(U_\alpha, z_1, \dots, z_n)\}$.
Kähler form: a smooth closed positive $(1,1)$-form:

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad (g_{i\bar{j}}) > 0.$$

$d\omega = 0 \Longrightarrow$ Kähler class $[\omega] \in H^2(X, \mathbb{R}) \cap H^{1,1}_{\bar{\partial}}(X, \mathbb{C})$.

$\partial\bar{\partial}$-Lemma: any Kähler form in $[\omega]$ can be written as

$$dd^c\varphi \quad := \quad \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u = \frac{\sqrt{-1}}{2\pi}\sum_{i,j}\left((\varphi_0)_{i\bar{j}} + u_{i\bar{j}}\right) dz^i \wedge d\bar{z}^j$$

where $\varphi = \varphi_0 + u$ is locally defined, while $u = \varphi - \varphi_0$ and $dd^c\varphi$ are globally defined.

Constant scalar curvature Kähler (cscK) metrics

Ricci curvature:

$$R_{i\bar{j}} := Ric(dd^c\varphi)_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det\left(\varphi_{k\bar{l}}\right).$$

Scalar curvature:

$$\begin{aligned} S(dd^c\varphi) &= g^{i\bar{j}} R_{i\bar{j}} = -g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det\left(\varphi_{k\bar{l}}\right) \\ &= -(g_{0i\bar{j}} + u_{i\bar{j}})^{-1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det(g_{0k\bar{l}} + u_{k\bar{l}}). \end{aligned}$$

cscK equation is a 4-th order highly nonlinear equation:

$$S(u) := S(dd^c\varphi) = \underline{S}.$$

$\underline{S}$ is a topological constant:

$$\underline{S} = \frac{n\langle c_1(-K_X) \wedge [\omega]^{n-1}, X\rangle}{\langle [\omega]^n, X\rangle}.$$

Kähler metric as curvature forms

If $[\omega] \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$, then $[\omega] = c_1(L)$ for an ample holomorphic line bundle L over X and $\omega = dd^c\varphi$ for a Hermitian metric $e^{-\varphi}$ on L.

Holomorphic line bundle: transition functions $f_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$.

$$L = \left(\bigsqcup_\alpha U_\alpha \times \mathbb{C}\right) / \{s_\alpha = f_{\alpha\beta} s_\beta\}.$$

Hermitian metrics: $e^{-\varphi} := \{e^{-\varphi_\alpha}\}$ Hermitian metric on L:

$$e^{-\varphi_\alpha} = |f_{\alpha\beta}|^2 e^{-\varphi_\beta}.$$

$\partial\bar\partial$-lemma: Fix any reference metric $e^{-\varphi_0}$, then $\exists u \in C^\infty(X)$ s.t.

$$e^{-\varphi} = e^{-\varphi_0} e^{-u}.$$

Chern curvature: globally defined closed $(1,1)$-form

$$dd^c\varphi = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi_\alpha.$$

(Uniform) Yau-Tian-Donaldson (YTD) conjecture

Conjecture (YTD conjecture)

(X, L) admits a cscK metric if and only if (X, L) is $\mathrm{Aut}(X, L)_0$-uniformly K-stable for test configurations.

The only if direction of this Conjecture is known to be true.

Example:
If $L = -K_X$ ample, then X is Fano and cscK=Kähler-Einstein (Tian, Berman, Chen-Donaldson-Sun, Datar-Székelyhidi, Berman-Boucksom-Jonsson, Hisamoto, **L.** -Tian-Wang, **L.** , Liu-Xu-Zhuang, ...)

Compare: Donaldson-Uhlenbeck-Yau's theorem: a holomorphic vector bundle admits a Hermitian-Einstein metric if and only if the vector bundle is slope (poly)stable.

Main results

Theorem (L. '20)

Let $\mathbb{G}$ be a reductive subgroup of $\mathrm{Aut}(X, L)_0$. If (X, L) is $\mathbb{G}$-uniformly K-stable for models (or for filtrations), then (X, L) admits a cscK metric.

We have implications and conjecture they are all equivalent:
$\mathrm{Aut}(X, L)_0$-uniformly K-stable for models $\Longrightarrow$ cscK
$\Longrightarrow$ $\mathrm{Aut}(X, L)_0$-uniformly K-stable for test configurations

Applications: proving the YTD conjecture for polarized spherical varieties (including polarized toric varieties):

Corollary (observed by Yuji Odaka)

A polarized spherical manifold (X, L) admits a cscK metric if and only if (X, L) is $\mathbb{G}$-uniformly K-stable.

G a complex reductive group, $B \subset G$ a Borel subgroup.
G/H spherical homogeneous if B has an open orbit.
X is spherical: $X = \overline{G/H}$ is a G-equivariant compactification.

Toric case

Toric Kähler manifold: a $\mathbb{T} := (\mathbb{C}^*)^n$ action with an open dense orbit.

Compact toric Kähler manifold $\longleftrightarrow$ moment convex polytope Δ

$d\lambda$: Lebesgue measure; $d\sigma$: boundary (lattice normalized) measure.

For any piecewise linear concave rational function f on Δ, define:

$$\mathbf{M}^{\mathrm{NA}}(f) = -\int_{\partial\Delta} f d\sigma + \frac{|\partial\Delta|_\sigma}{|\Delta|} \int_\Delta f d\lambda;$$

$$\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(f) = \inf\left\{\max_\Delta f_\xi - \frac{1}{|\Delta|}\int_\Delta f_\xi d\lambda; \xi \in N_{\mathbb{R}} = \mathrm{Lie}((S^1)^n)\right\}$$

where $f_\xi = f + \langle \cdot, \xi\rangle$. $\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}$ is some normalized L^1-norm.

$\mathbb{T}$-uniform K-stable: there exists $\gamma > 0$ such that:

$$\mathbf{M}^{\mathrm{NA}}(f) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(f).$$

Corollary (Hisamoto, based on Chen-Cheng, Donaldson, Zhou-Zhu, ...)

A toric X admits a cscK metric iff it is $\mathbb{T}$-uniformly K-stable.

A necessary condition: vanishing of Futaki invariant

If $v \in \mathrm{aut}(X, L)$ is a holomorphic $(1,0)$ vector field with Hamiltonian function θ_v: $\iota_v \omega = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \theta_v$.
Define Futaki invariant (independent of $\omega \in [\omega]$):

$$\begin{aligned} \mathrm{Fut}(v) &= \int_X \theta_v (S(\omega) - \underline{S}) \omega^n \\ &= \int_X (-Ric(\omega) + \Delta \theta_v) \wedge (\omega + \theta_v)^n + \frac{\underline{S}}{n+1} (\omega + \theta_v)^{n+1}. \end{aligned} \tag{1}$$

$\mathbb{G}$-uniformly K-stable $\Rightarrow$ K-semistable $\Rightarrow$ Futaki invariant $\equiv 0$.

Toric case: $\mathrm{Fut} = \frac{|\partial \Delta|}{|\Delta|} \mathrm{cent}(\Delta, d\lambda) - \mathrm{cent}(\partial \Delta, d\sigma)$.

Toric Fano case: Reflective polytope $\leftrightarrow$ Toric Fano manifold
Wang-Zhu's existence result: KE $\leftrightarrow$ $\mathrm{Fut} \equiv 0$.

Space of Kähler metrics

Space of smooth Kähler metrics:

$$\mathcal{H} = \{\varphi = \varphi_0 + u; u \in C^\infty(X), \omega_0 + dd^c u = dd^c\varphi > 0\}.$$

Monge-Ampère energy (Aubin-Yau):

$$\mathbf{E}(\varphi) = \int_0^1 \int_X \dot{\varphi}(dd^c\varphi)^n dt = \frac{1}{n+1}\sum_k \int_X u\omega_u^k \wedge \omega^{n-k}.$$

Finite energy metrics as Completion of $\mathcal{H}$ (Cegrell, Guedj-Zeriahi)

$$\mathcal{E}^1 = \{\varphi \in \mathrm{PSH}(X, [\omega]); \\ \mathbf{E}(\varphi) := \inf\{\mathbf{E}(\tilde{\varphi}); \tilde{\varphi} \geq \varphi, \tilde{\varphi} \in \mathcal{H}\} > -\infty\}.$$

Strong topology on $\mathcal{E}^1$: $\varphi_m \to \varphi$ strongly if $\varphi_m \to \varphi$ in $L^1(\omega^n)$ and $\mathbf{E}(\varphi_m) \to \mathbf{E}(\varphi)$.

Geodesic rays

Definition

Given $\varphi_1, \varphi_2 \in \mathcal{E}^1$, a geodesic segment joining φ_1, φ_2 is:

$$\Phi = \sup\{\tilde{\Phi} \in \mathrm{PSH}(X \times [s_1, s_2] \times S^1, p_1^* L); \tilde{\Phi}(\cdot, s_i) \leq \varphi_i, i = 1, 2\}.$$

A geodesic ray emanating from φ_0 is a map $\Phi : \mathbb{R}_{\geq 0} \to \mathcal{E}^1$ s.t. $\forall s_1, s_2 \in \mathbb{R}_{\geq 0}$, $\Phi|_{[s_1, s_2]}$ is the geodesic segment joining $\varphi(s_1)$ and $\varphi(s_2)$, and $\Phi(\cdot, 0) = \varphi_0$.

Geodesics originates from Mabuchi's L^2-metric on $\mathcal{H}$ and satisfies the Homogeneous Complex Monge-Ampère (HCMA) equation in pluripotential sense (Semmes, Donaldson):

$$(\sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} = 0.$$

Fact: $\mathbf{E}$ linear along geodesics: $\sqrt{-1}\partial\bar{\partial}\mathbf{E} = \int_X (\sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} = 0$.

Mabuchi functional (K-energy): Chen-Tian's formula:

$$\begin{aligned} \mathbf{M}(\varphi) &= -\int_0^1 dt \int_X \dot{\varphi} \cdot (S(\varphi(t)) - \underline{S})(dd^c\varphi(t))^n \\ &= \mathbf{H}(\varphi) + \mathbf{E}^{K_X}(\varphi) + \underline{S}\,\mathbf{E}(\varphi). \end{aligned}$$

Decomposition: Entropy, twisted energy, Monge-Ampère energy

$$\mathbf{H}(\varphi) = \int_X \log \frac{(dd^c\varphi)^n}{\Omega} (dd^c\varphi)^n.$$

The Euler-Lagrange equation of $\mathbf{M}$ is the cscK equation:

$$\begin{aligned} \frac{d}{dt}\mathbf{M}(\varphi) &= \int_X \dot{\varphi}(-\Delta(\log(dd^c\varphi)^n) + \underline{S})(dd^c\varphi)^n \\ &= -\int_X \dot{\varphi}(S(\varphi) - \underline{S})(dd^c\varphi)^n. \end{aligned}$$

CscK metrics are minimizers of Mabuchi functional

Theorem (Berman-Berndtsson, Berman-Darvas-Lu)

$\mathbf{M}$ *is convex along geodesics in* $\mathcal{E}^1$. *It is linear if and only if the geodesic is generated by holomorphic vector fields.*

Consequences of convexity:

Theorem (Berman-Berndtsson, Berman-Darvas-Lu)

CscK metrics obtain the minimum of $\mathbf{M}$ *over* $\mathcal{E}^1$. *Moreover (smooth) cscK metrics are unique up to* $Aut(X, [\omega])_0$.

Previous results: Chen-Tian, Donaldson and Mabuchi.

Variational criterion

$\mathbb{G} = \mathbb{K}^{\mathbb{C}} \subset \mathrm{Aut}(X, [\omega])_0$: a reductive Lie group, $\mathbb{T}$ the center of $\mathbb{G}$.

Definition (Tian, refined by Darvas-Rubinstein and Hisamoto)

M *is coercive (resp. $\mathbb{G}$-coercive) if there exists $\gamma > 0$ such that for any $\varphi \in \mathcal{H}$ (resp. $\varphi \in \mathcal{H}^{\mathbb{K}}$),*

$$\mathbf{M}(\varphi) \geq \gamma \cdot \mathbf{J}(\varphi) \quad (\textit{ resp.} \quad \geq \gamma \cdot \inf_{\sigma \in \mathbb{T}} \mathbf{J}(\sigma^* \varphi) \ )$$

where $\mathbf{J}(\varphi)$ is a distance-like (or norm-like) functional.

We have hard analytic results:

Theorem (Chen-Cheng; Darvas-Rubinstein, Berman-Darvas-Lu)

Tian's properness conjecture is true: there exists a cscK metric in $(X, [\omega])$ if and only if **M** *is Aut$(X, [\omega])_0$-coercive.*

Hisamoto, **L.** : $\mathrm{Aut}(X, [\omega])_0$ can be replaced by any connected reductive $\mathbb{G}$ that contains a maximal torus of $\mathrm{Aut}(X, [\omega])_0$.

Criterion via destabilizing geodesic rays

For a geodesic ray Φ and a functional $\mathbf{F}$ defined over $\mathcal{E}^1$, set:

$$\mathbf{F}'^{\infty}(\Phi) = \lim_{s\to+\infty} \frac{\mathbf{F}(\varphi(s))}{s}.$$

Fact: The limits exist for all $\mathbf{F} \in \{\mathbf{E}, \mathbf{E}^{K_X}, \mathbf{H}, \mathbf{M}, \mathbf{J}, \mathbf{J}_{\mathbb{T}}\}$.

Based on convexity of $\mathbf{M}$ and compactness result of Berman-Boucksom-Eyssidieux-Guedj-Zeriahi: destabilizing sequence produces destabilizing a geodesic ray:

Theorem (Darvas-He, Darvas-Rubinstein, Berman-Boucksom-Jonsson; $\mathbb{G}$-equivariant case: Hisamoto, L.)

$\mathbf{M}$ *is not* $\mathbb{G}$*-coercive (iff* $(X, [\omega])$ *has no cscK) iff there exists a geodesic ray* Φ*, such that*

$$\mathbf{M}'^{\infty}(\Phi) \le 0, \quad \mathbf{J}_{\mathbb{T}}'^{\infty}(\Phi) = 1.$$

Difficulty to YTD: (algebraic) regularity of "optimal" destabilizing rays.

Contact with algebraic geometry: Test configurations (Tian, Donaldson)

A test configuration (TC) $(\mathcal{X}, \mathcal{L})$ is a $\mathbb{C}^*$-equivariant degeneration of (X, L):

1. $\pi : \mathcal{X} \to \mathbb{C}$: a $\mathbb{C}^*$-equivariant family of projective varieties;
2. $\mathcal{L} \to \mathcal{X}$: a $\mathbb{C}^*$-equiv. semiample holomorphic $\mathbb{Q}$-line bundle;
3. $\eta : (\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^* \cong (X, L) \times \mathbb{C}^*$.

Trivial test configuration: $(X_{\mathbb{C}}, L_{\mathbb{C}}) := (X, L) \times \mathbb{C}$.

$(\mathcal{X}, \mathcal{L})$ is dominating if there is a $\mathbb{C}^*$-equivariant birational morphism $\rho : \mathcal{X} \to X \times \mathbb{C}$.

Under the isomorphism η, psh metrics on $\mathcal{L}|_{\pi^{-1}(\mathbb{C}^*)}$ are considered as *subgeodesic* rays on (X, L).

Geodesic rays from test configurations

For any TC $(\mathcal{X}, \mathcal{L})$, there are many smooth subgeodesic ray which extend to be a smooth psh metrics on $\mathcal{L}$.

Theorem (Phong-Sturm)

$\forall$ test configuration $(\mathcal{X}, \mathcal{L})$, $\exists$ a unique geodesic ray $\Phi = \Phi_{(\mathcal{X},\mathcal{L})}$ emanating from φ_0 s.t. Φ extends to a bounded psh metric on $\mathcal{L}$.

Φ is obtained by solving the HCMA on a resolution $\tilde{\mathcal{X}} \to \mathcal{X}$:

$$(\mu^*(dd^c\tilde{\Phi}) + U)^{n+1} = 0; \quad U|_{X\times S^1} = 0,$$

where $\tilde{\Phi}$ is any smooth positively curved Hermitian metric on $\mathcal{L}$.
In general the solution $\Phi := \tilde{\Phi} + U$ is at most $C^{1,1}$ (Phong-Sturm, Chu-Tosatti-Weinkove).

Non-Archimedean functionals of TCs

For any TC $(\mathcal{X}, \mathcal{L})$, set:

$$\begin{aligned}
\mathbf{E}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) &= \frac{\bar{\mathcal{L}}^{\cdot n+1}}{n+1}; \\
\mathbf{J}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) &= \bar{\mathcal{L}} \cdot L_{\mathbb{P}^1}^{\cdot n} - \frac{\bar{\mathcal{L}}^{\cdot n+1}}{n+1} \geq 0; \\
\mathbf{H}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) &= K^{\log}_{\bar{\mathcal{X}}/X\times\mathbb{P}^1} \cdot \bar{\mathcal{L}}^{\cdot n} \geq 0; \\
(\mathbf{E}^{K_X})^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) &= K_X \cdot \bar{\mathcal{L}}^{\cdot n}; \\
\mathbf{M}^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) &= K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^{\cdot n} + \frac{\underline{S}}{n+1}\bar{\mathcal{L}}^{\cdot n+1} \\
&= \mathbf{H}^{\mathrm{NA}} + (\mathbf{E}^{K_X})^{\mathrm{NA}} + \underline{S}\,\mathbf{E}^{\mathrm{NA}}.
\end{aligned}$$

$\mathbf{M}^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ generalizes the Futaki invariant in (1).

Similar non-Archimedean functional $(\mathbf{E}^{K_X})^{\mathrm{NA}}$.

Slopes at infinitity = non-Archimedean functional

Theorem

Let Φ be the geodesic ray associated to a TC $(\mathcal{X}, \mathcal{L})$. Then

1. *[Tian, Zhang, Phong-Sturm-Ross, Boucksom-Hisamoto-Jonsson]* *For* $\mathbf{F} \in \{\mathbf{E}, \mathbf{J}, \mathbf{E}^{K_X}\}$, $\mathbf{F}'^{\infty}(\Phi) = \mathbf{F}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$.
2. *[Hisamoto] For* $\mathbb{G}$*-equivariant TC,* $\mathbf{J}'^{\infty}_{\mathbb{T}}(\Phi) = \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L})$ *where where*
$$\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L}) = \inf_{\xi \in N_{\mathbb{R}}} \mathbf{J}^{\mathrm{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi).$$
3. *[**L.** , based on Tian, Boucksom-Hisamoto-Jonsson, Xia]* $\mathbf{H}'^{\infty}(\Phi) = \mathbf{H}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$.

$\mathbb{G}$-uniform K-stability

Definition (Hisamoto, generalizing Tian, Donaldson, Székelyhidi, Dervan, Boucksom-Hisamoto-Jonsson)

(X, L) is $\mathbb{G}$-uniformly K-stable if there exists $\gamma > 0$ such that for any $\mathbb{G}$-equivariant test configuration $(\mathcal{X}, \mathcal{L})$,

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L}). \tag{2}$$

Proposition (Hisamoto for $\mathrm{Aut}(X, L)_0$, L. for more general $\mathbb{G}$)

Assume that (X, L) admits a cscK metric. If $\mathbb{G}$ contains a maximal torus of $\mathrm{Aut}(X, L)_0$, then (X, L) is $\mathbb{G}$-uniformly K-stable.

Combining the above discussions, the Yau-Tian-Donaldson conjecture now is reduced to two questions:

Q1: Can we approximate any destabilizing geodesic ray Φ by geodesic rays from test configurations?

Q2: Can we approximate the slopes $\mathbf{F}'^{\infty}$ along any destabilizing geodesic ray by $\mathbf{F}^{\mathrm{NA}} = \mathbf{F}'^{\infty}$ of TC's for $\mathbf{F} \in \{\mathbf{E}, \mathbf{E}^{K_X}, \mathbf{J}, \mathbf{H}\}$?

More precisely, the YTD conjecture is reduced to the following

Conjecture: there exist TC's $(\mathcal{X}_m, \mathcal{L}_m)$ such that

$$\mathbf{J}_{\mathbb{T}}^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) \to \mathbf{J}_{\mathbb{T}}'^{\infty}(\Phi), \quad \limsup_{m\to+\infty} \mathbf{M}^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) \leq \mathbf{M}'^{\infty}(\Phi). \tag{3}$$

TC's associated to geodesic rays

For any geodesic ray Φ, Berman-Boucksom-Jonsson constructed a sequence of test configurations $(\mathcal{X}_m, \mathcal{L}_m)_m$ such that $\Phi_m \geq \Phi$ for any $m \gg 1$.

Their construction is based on Demailly's approximation:

1. Consider the multiplier ideal sheaf (MIS) over $X \times \mathbb{C}$:

$$\mathcal{J}(m\Phi)(\mathcal{U}) = \left\{ f \in \mathcal{O}(\mathcal{U}); \int_{\mathcal{U}} |f|^2 e^{-m\Phi} < +\infty \right\}.$$

2. $\mu_m : \mathcal{X}_m = Bl_{\mathcal{J}(m\Phi)} X_{\mathbb{C}} \to X_{\mathbb{C}}$, $\mathcal{L}_m = \mu_m^* L_{\mathbb{C}} - \frac{1}{m+m_0} E_m$. Using the Nadel vanishing and global generation property of MIS, $(\mathcal{X}_m, \mathcal{L}_m)$ is a test configuration of (X, L)

Maximal geodesic rays

Taking $\lim_{m\to+\infty}\Phi_{(\mathcal{X}_m,\mathcal{L}_m)}$:

Theorem (Berman-Boucksom-Jonsson)

For any geodesic ray Φ, there exists a unique "maximal" geodesic ray $\hat{\Phi}$ satisfying:

1. $\hat{\Phi} \geq \Phi$.
2. $\mathcal{J}(\lambda\hat{\Phi}) = \mathcal{J}(\lambda\Phi)$ *for any* $\lambda > 0$.

Maximal geodesic rays are exactly those that can be approximated by algebraic ones (i.e. geodesic rays associated to TC's).

Theorem (Berman-Boucksom-Jonsson, L.)

$\forall$ maximal Φ, $\exists$ a sequence of TC's $(\mathcal{X}_m,\mathcal{L}_m)_m$ s.t.
$\mathbf{F}'^{\infty}(\Phi) = \lim_{m\to+\infty}\mathbf{F}'^{\infty}(\Phi_{(\mathcal{X}_m,\mathcal{L}_m)})$ *for* $\mathbf{F} \in \{\mathbf{E}, \mathbf{E}^{K_X}, \mathbf{J}, \mathbf{J}_{\mathbb{T}}\}$.

Conjecture: The same holds for **H**, or equivalently for **M**.

Destabilizing geodesic rays are maximal

Theorem (L. , '20)

A geodesic ray Φ satisfies $\mathbf{M}'^{\infty}(\Phi) < +\infty$ is necessarily maximal.

The proof uses two key ingredients: equisingularity of multiplier approximation (via a valuative description) and Jensen's inequality (motivated by Tian's α-type estimate): for arbitrary $\alpha > 0$,

$$\begin{aligned} C(\alpha) &> \log \int_{X \times \mathbb{D}} e^{\alpha(\hat{\Phi} - \Phi)} \Omega \sqrt{-1} dt \wedge d\bar{t} \\ &\geq \alpha \int_X (\hat{\varphi}(s) - \varphi(s))(dd^c \varphi(s))^n - \mathbf{H}_\Omega(\varphi(s)) - s \\ &\geq C\alpha \cdot (\mathbf{E}(\hat{\varphi}(s)) - \mathbf{E}(\varphi(s))) - \mathbf{H}(\varphi(s)) - s. \end{aligned}$$

$\overset{\text{take slope}}{\Longrightarrow}$ $\mathbf{E}'^{\infty}(\hat{\Phi}) \geq \mathbf{E}'^{\infty}(\Phi)$ $\overset{\mathbf{E}\text{ linear}}{\Longrightarrow}$ $\hat{\Phi} = \Phi$ (Dinew's comparison principle).

In the definition of a test configuration $(\mathcal{X}, \mathcal{L})$, if we don't require $\mathcal{L}$ to be semiample, then we say that $(\mathcal{X}, \mathcal{L})$ is a model of (X, L).

For any (big) model $(\mathcal{X}, \mathcal{L})$, one can still define:

$$\begin{aligned}
\mathbf{E}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{\langle \bar{\mathcal{L}}^{\cdot n+1} \rangle}{n+1} = \frac{\mathrm{vol}(\bar{\mathcal{L}})}{n+1}; \\
\mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \langle \bar{\mathcal{L}} \rangle \cdot L_{\mathbb{P}^1}^{\cdot n} - \frac{\langle \bar{\mathcal{L}}^{\cdot n+1} \rangle}{n+1}; \\
\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \langle \bar{\mathcal{L}}^{\cdot n} \rangle + \frac{\underline{S}}{n+1} \langle \bar{\mathcal{L}}^{\cdot n+1} \rangle.
\end{aligned}$$

Volume/restricted volume studied by Tsuji, Boucksom-Favre-Jonsson, Ein-Lazarsfeld-Mustață-Nakamaye-Popa:

$$K_{\bar{\mathcal{X}}} \cdot \langle \bar{\mathcal{L}}^{\cdot n} \rangle = \frac{d}{dt}\Big|_{t=0} \mathrm{vol}(\bar{\mathcal{L}} + t K_{\bar{\mathcal{X}}}).$$

Approximation slopes by NA invariants of models

Definition (L.)

(X, L) is $\mathbb{G}$-uniformly K-stable for models if $\exists\gamma > 0$ such that for any model $(\mathcal{X}, \mathcal{L})$, $\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L})$.

Theorem (L. '20)

For any maximal Φ, $\exists$ sequence of models $(\mathcal{X}_m, \mathcal{L}_m)$ such that

$$\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}_m, \mathcal{L}_m) \to \mathbf{J}'^{\infty}_{\mathbb{T}}(\Phi), \quad \limsup_{m\to+\infty} \mathbf{M}^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) \leq \mathbf{M}'^{\infty}(\Phi).$$

The proof uses Archimedean/non-Archimedean pluripotential theory

Corollary

If (X, L) is $\mathbb{G}$-uniform K-stable for models, then there does not exist destabilizing geodesic ray Φ (with $\mathbf{M}'^{\infty}(\Phi) \leq 0$ and $\mathbf{J}'^{\infty}_{\mathbb{T}}(\Phi) = 1$) and hence (X, L) admits a cscK metric.

Approximation of big models: Fujita approximation

Fujita approximation (almost Zariski decomposition) studied by Fujita, Demailly-Ein-Lazarsfeld, Boucksom-Favre-Jonsson, Ein-Lazarsfeld-Mustață-Nakamaye-Popa, ...:

$$\mu_m : \mathcal{X}_m \to \mathcal{X}, \quad \mathcal{L}_m = \mu^* \mathcal{L} - \frac{1}{m} E_m$$

μ_m is blow-up of base ideals or some asymptotic multiplier ideal sheaves with exceptional divisor E_m. $\mathcal{L}_m$ is semiample satisfying

$$\lim_{m \to +\infty} \frac{h^0(\bar{\mathcal{X}}, m\bar{\mathcal{L}})}{m^{\dim \mathcal{X}}/(\dim \mathcal{X})!)} =: \mathrm{vol}(\bar{\mathcal{X}}, \bar{\mathcal{L}}) = \lim_{m \to +\infty} \mathrm{vol}(\bar{\mathcal{X}}_m, \bar{\mathcal{L}}_m).$$

Theorem (Boucksom-Jonsson, L.)

For any model $(\mathcal{X}, \mathcal{L})$, there exists a sequence of TC's $(\mathcal{X}_m, \mathcal{L}_m)$ s.t.

$$\mathbf{F}^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) = \mathbf{F}'^{\infty}(\Phi_m) \to \mathbf{F}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \mathbf{F}'^{\infty}(\Phi)$$

for $\mathbf{F} \in \{\mathbf{E}, \mathbf{E}^{K_X}, \mathbf{J}, \mathbf{J}_{\mathbb{T}}\}$ where $\Phi_m = \Phi_{(\mathcal{X}_m, \mathcal{L}_m)}$.

A purely algebro-geometric conjecture

We propose the following conjecture:

Conjecture (strengthened Fujita approximation $\Rightarrow$ approximation of $\mathbf{H}^{\mathrm{NA}}$ (conj. of Boucksom-Jonsson) $\Rightarrow$ YTD)

$\forall$ big line bundle $\bar{\mathcal{L}} \to \bar{\mathcal{X}}$, $\exists$ birational morphisms $\mu_m : \mathcal{X}_m \to \mathcal{X}$ and decompositions $\mu_m^* \bar{\mathcal{L}} = \mathcal{L}_m + \frac{1}{m} E_m$ with $\mathcal{L}_m$ semiample and E_m effective, s.t.

1. $\mathrm{vol}(\bar{\mathcal{L}}_m) \to \mathrm{vol}(\bar{\mathcal{L}})$.
2. *The derivatives also converge:*

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{vol}(\bar{\mathcal{L}}_m + tK_{\bar{\mathcal{X}}_m}) \to \frac{d}{dt}\bigg|_{t=0} \mathrm{vol}(\bar{\mathcal{L}} + tK_{\bar{\mathcal{X}}}). \tag{4}$$

True if (X, L) is spherical (since $\bar{\mathcal{X}}$ is a Mori dream space), in particular true for toric manifolds.

Thanks for your attention!