Recent progress on the YTD conjecture for cscK metrics

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Riemann surface: surface with a complex structure:

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<td>$\Sigma_g = \mathbb{B}^1/\pi_1(\Sigma_g)$</td>
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$\Sigma_g$ closed oriented surface of genus $g \geq 2$.

$\mathbb{B}^1 = \{z \in \mathbb{C}; |z| < 1\}$.

Generalization for higher dimensional complex projective manifolds?
$X$: complex manifold, $\{(U_\alpha, z_1, \ldots, z_n)\}$.

Kähler form: a smooth closed positive (1, 1)-form:

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad (g_{i\bar{j}}) > 0.$$  

$d\omega = 0 \implies$ Kähler class $[\omega] \in H^2(X, \mathbb{R}) \cap H^{1,1}_\partial(X, \mathbb{C})$.

$\partial\bar{\partial}$-Lemma: any Kähler form in $[\omega]$ can be written as

$$dd^c \varphi := \omega_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} \left( (\varphi_0)_{i\bar{j}} + u_{i\bar{j}} \right) dz^i \wedge d\bar{z}^j$$

where $\varphi = \varphi_0 + u$ is locally defined, while $u = \varphi - \varphi_0$ and $dd^c \varphi$ are globally defined.
Constant scalar curvature Kähler (cscK) metrics

Ricci curvature:

\[ R_{i\bar{j}} := \text{Ric}(dd^c \phi)_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det (\phi_{k\bar{l}}). \]

Scalar curvature:

\[ S(dd^c \phi) = g^{i\bar{j}} R_{i\bar{j}} = -g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det (\phi_{k\bar{l}}) \]

\[ = -(g^{0i\bar{j}} + u^{i\bar{j}})^{-1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det (g^{0k\bar{l}} + u_{k\bar{l}}). \]

cscK equation is a 4-th order highly nonlinear equation:

\[ S(u) := S(dd^c \phi) = S. \]

\( S \) is a topological constant:

\[ S = \frac{n \langle c_1(-K_X) \wedge [\omega]^{n-1}, X \rangle}{\langle [\omega]^n, X \rangle}. \]
Kähler metric as curvature forms

If $[\omega] \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$, then $[\omega] = c_1(L)$ for an ample holomorphic line bundle $L$ over $X$ and $\omega = \ddc \varphi$ for a Hermitian metric $e^{-\varphi}$ on $L$.

Holomorphic line bundle: transition functions $f_{\alpha \beta} \in \mathcal{O}(U_{\alpha} \cap U_{\beta})$.

$$L = \left( \bigsqcup_{\alpha} U_{\alpha} \times \mathbb{C} \right) / \{ s_\alpha = f_{\alpha \beta} s_\beta \}.$$

Hermitian metrics: $e^{-\varphi} := \{ e^{-\varphi_\alpha} \}$ Hermitian metric on $L$:

$$e^{-\varphi_\alpha} = |f_{\alpha \beta}|^2 e^{-\varphi_\beta}.$$

$\partial \bar{\partial}$-lemma: Fix any reference metric $e^{-\varphi_0}$, then $\exists u \in C^\infty(X)$ s.t.

$$e^{-\varphi} = e^{-\varphi_0} e^{-u}.$$

Chern curvature: globally defined closed $(1, 1)$-form

$$\ddc \varphi = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_\alpha.$$
Conjecture (YTD conjecture)

\((X, L)\) admits a cscK metric if and only if \((X, L)\) is \(\text{Aut}(X, L)_0\)-uniformly K-stable for test configurations.

The only if direction of this Conjecture is known to be true.

Example:
If \(L = -K_X\) ample, then \(X\) is Fano and cscK=Kähler-Einstein (Tian, Berman, Chen-Donaldson-Sun, Datar-Székelyhidi, Berman-Boucksom-Jonsson, Hisamoto, L. -Tian-Wang, L., Liu-Xu-Zhuang, ...)

Compare: Donaldson-Uhlenbeck-Yau’s theorem: a holomorphic vector bundle admits a Hermitian-Einstein metric if and only if the vector bundle is slope (poly)stable.
Main results

**Theorem (L. ’20)**

Let $G$ be a reductive subgroup of $\text{Aut}(X, L)_0$. If $(X, L)$ is $G$-uniformly K-stable for models (or for filtrations), then $(X, L)$ admits a cscK metric.

We have implications and conjecture they are all equivalent:

$\text{Aut}(X, L)_0$-uniformly K-stable for models $\implies$ cscK $\implies$ $\text{Aut}(X, L)_0$-uniformly K-stable for test configurations

Applications: proving the YTD conjecture for polarized spherical varieties (including polarized toric varieties):

**Corollary (observed by Yuji Odaka)**

A polarized spherical manifold $(X, L)$ admits a cscK metric if and only if $(X, L)$ is $G$-uniformly K-stable.

$G$ a complex reductive group, $B \subset G$ a Borel subgroup. $G/H$ spherical homogeneous if $B$ has an open orbit. $X$ is spherical: $X = \overline{G/H}$ is a $G$-equivariant compactification.
Toric case

Toric Kähler manifold: a $\mathbb{T} := (\mathbb{C}^*)^n$ action with an open dense orbit. Compact toric Kähler manifold $\iff$ moment convex polytope $\Delta$

$d\lambda$: Lebesgue measure; $d\sigma$: boundary (lattice normalized) measure.

For any piecewise linear concave rational function $f$ on $\Delta$, define:

$$M_{\text{NA}}^T(f) = -\int_{\partial \Delta} f d\sigma + \frac{\partial \Delta}{|\Delta|} \int_{\Delta} f d\lambda;$$

$$J_{\text{NA}}^T(f) = \inf \left\{ \max_{\Delta} f_\xi - \frac{1}{|\Delta|} \int_{\Delta} f_\xi d\lambda; \xi \in N_\mathbb{R} = \text{Lie}(\mathbb{S}^1)^n \right\}$$

where $f_\xi = f + \langle \cdot, \xi \rangle$. $J_{\text{NA}}^T$ is some normalized $L^1$-norm.

$\mathbb{T}$-uniform K-stable: there exists $\gamma > 0$ such that:

$$M_{\text{NA}}^T(f) \geq \gamma \cdot J_{\text{NA}}^T(f).$$

Corollary (Hisamoto, based on Chen-Cheng, Donaldson, Zhou-Zhu, ...)

A toric $X$ admits a cscK metric iff it is $\mathbb{T}$-uniformly K-stable.
A necessary condition: vanishing of Futaki invariant

If \( v \in \text{aut}(X, L) \) is a holomorphic \((1, 0)\) vector field with Hamiltonian function \( \theta_v: \iota_v \omega = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \theta_v \).

Define \textit{Futaki invariant} (independent of \( \omega \in [\omega] \)):

\[
\text{Fut}(v) = \int_X \theta_v (S(\omega) - S) \omega^n \\
= \int_X (-\text{Ric}(\omega) + \Delta \theta_v) \wedge (\omega + \theta_v)^n + \frac{S}{n+1}(\omega + \theta_v)^{n+1}.
\]

\( \mathbb{G} \)-uniformly K-stable \( \Rightarrow \) K-semistable \( \Rightarrow \) Futaki invariant \( \equiv 0 \).

\textbf{Toric case:} \quad \text{Fut} = \left|\frac{\partial \Delta}{|\Delta|}\right| \text{cent}(\Delta, d\lambda) - \text{cent}(\partial \Delta, d\sigma).

\textbf{Toric Fano case:} \quad \text{Reflective polytope} \leftrightarrow \text{Toric Fano manifold}

\text{Wang-Zhu’s existence result: KE} \leftrightarrow \text{Fut} \equiv 0.
Space of smooth Kähler metrics:

$$\mathcal{H} = \{ \varphi = \varphi_0 + u; u \in C^\infty(X), \omega_0 + dd^c u = dd^c \varphi > 0 \}.$$ 

Monge-Ampère energy (Aubin-Yau):

$$E(\varphi) = \int_0^1 \int_X \varphi(dd^c \varphi)^n dt = \frac{1}{n+1} \sum_k \int_X u \omega_k^k \wedge \omega^{n-k}.$$ 

Finite energy metrics as Completion of $\mathcal{H}$ (Cegrell, Guedj-Zeriahi)

$$\mathcal{E}^1 = \{ \varphi \in PSH(X, [\omega]); E(\varphi) := \inf \{ E(\tilde{\varphi}); \tilde{\varphi} \geq \varphi, \tilde{\varphi} \in \mathcal{H} \} > -\infty \}.$$ 

Strong topology on $\mathcal{E}^1$: $\varphi_m \to \varphi$ strongly if $\varphi_m \to \varphi$ in $L^1(\omega^n)$ and $E(\varphi_m) \to E(\varphi)$. 

Variational approach and analytic criterion
**Definition**

Given $\varphi_1, \varphi_2 \in \mathcal{E}^1$, a geodesic segment joining $\varphi_1, \varphi_2$ is:

\[
\Phi = \sup\{\tilde{\Phi} \in \text{PSH}(X \times [s_1, s_2] \times S^1, p_1^* L); \tilde{\Phi}(\cdot, s_i) \leq \varphi_i, i = 1, 2\}.
\]

A geodesic ray emanating from $\varphi_0$ is a map $\Phi : \mathbb{R}_{\geq 0} \rightarrow \mathcal{E}^1$ s.t.

$\forall s_1, s_2 \in \mathbb{R}_{\geq 0}$, $\Phi|_{[s_1, s_2]}$ is the geodesic segment joining $\varphi(s_1)$ and $\varphi(s_2)$, and $\Phi(\cdot, 0) = \varphi_0$.

Geodesics originates from Mabuchi’s $L^2$-metric on $\mathcal{H}$ and satisfies the Homogeneous Complex Monge-Ampère (HCMA) equation in pluripotential sense (Semmes, Donaldson):

\[
(\sqrt{-1} \partial \bar{\partial} \Phi)^{n+1} = 0.
\]

**Fact:** $E$ linear along geodesics: $\sqrt{-1} \partial \bar{\partial} E = \int_X (\sqrt{-1} \partial \bar{\partial} \Phi)^{n+1} = 0$. 

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**Variational approach and analytic criterion**
Mabuchi functional (K-energy): Chen-Tian’s formula:

\[ M(\varphi) = - \int_0^1 dt \int_X \dot{\varphi} \cdot (S(\varphi(t)) - S)(dd^c \varphi(t))^n \]

\[ = H(\varphi) + E^{K_X}(\varphi) + S E(\varphi). \]

Decomposition: Entropy, twisted energy, Monge-Ampère energy

\[ H(\varphi) = \int_X \log \frac{(dd^c \varphi)^n}{\Omega} (dd^c \varphi)^n. \]

The Euler-Lagrange equation of \( M \) is the cscK equation:

\[ \frac{d}{dt} M(\varphi) = \int_X \dot{\varphi}(-\Delta (\log(dd^c \varphi)^n) + S)(dd^c \varphi)^n \]

\[ = - \int_X \dot{\varphi}(S(\varphi) - S)(dd^c \varphi)^n. \]
CscK metrics are minimizers of Mabuchi functional

Theorem (Berman-Berndtsson, Berman-Darvas-Lu)

\( M \) is convex along geodesics in \( \mathcal{E}^1 \). It is linear if and only if the geodesic is generated by holomorphic vector fields.

Consequences of convexity:

Theorem (Berman-Berndtsson, Berman-Darvas-Lu)

CscK metrics obtain the minimum of \( M \) over \( \mathcal{E}^1 \). Moreover (smooth) cscK metrics are unique up to \( \text{Aut}(X, [\omega])_0 \).

Previous results: Chen-Tian, Donaldson and Mabuchi.
Variational criterion

\( \mathbb{G} = \mathbb{K}^C \subset \text{Aut}(X, [\omega])_0 \): a reductive Lie group, \( \mathbb{T} \) the center of \( \mathbb{G} \).

**Definition (Tian, refined by Darvas-Rubinstein and Hisamoto)**

\( \mathbb{M} \) is coercive (resp. \( \mathbb{G} \)-coercive) if there exists \( \gamma > 0 \) such that for any \( \varphi \in \mathcal{H} \) (resp. \( \varphi \in \mathcal{H}^K \)),

\[
\mathbb{M}(\varphi) \geq \gamma \cdot J(\varphi) \quad (\text{resp. } \geq \gamma \cdot \inf_{\sigma \in \mathbb{T}} J(\sigma^* \varphi))
\]

where \( J(\varphi) \) is a distance-like (or norm-like) functional.

We have hard analytic results:

**Theorem (Chen-Cheng; Darvas-Rubinstein, Berman-Darvas-Lu)**

*Tian’s properness conjecture is true: there exists a cscK metric in \( (X, [\omega]) \) if and only if \( \mathbb{M} \) is \( \text{Aut}(X, [\omega])_0 \)-coercive.*

Hisamoto, L.: \( \text{Aut}(X, [\omega])_0 \) can be replaced by any connected reductive \( \mathbb{G} \) that contains a maximal torus of \( \text{Aut}(X, [\omega])_0 \).
For a geodesic ray $\Phi$ and a functional $F$ defined over $\mathcal{E}^1$, set:

$$F'_{\infty}(\Phi) = \lim_{s \to +\infty} \frac{F(\varphi(s))}{s}.$$ 

**Fact:** The limits exist for all $F \in \{E, E^{K_X}, H, M, J, J_T\}$.

Based on convexity of $M$ and compactness result of Berman-Boucksom-Eyssidieux-Guedj-Zeriahi: destabilizing sequence produces destabilizing a geodesic ray:

**Theorem** (Darvas-He, Darvas-Rubinstein, Berman-Boucksom-Jonsson; $G$-equivariant case: Hisamoto, L.)

$M$ is not $G$-coercive (iff $(X, [\omega])$ has no $cscK$) iff there exists a geodesic ray $\Phi$, such that

$$M'_{\infty}(\Phi) \leq 0, \quad J'_T(\Phi) = 1.$$

Difficulty to YTD: (algebraic) regularity of “optimal” destabilizing rays.
A test configuration (TC) \((\mathcal{X}, \mathcal{L})\) is a \(\mathbb{C}^*\)-equivariant degeneration of \((X, L)\):

1. \(\pi : \mathcal{X} \to \mathbb{C}\): a \(\mathbb{C}^*\)-equivariant family of projective varieties;
2. \(\mathcal{L} \to \mathcal{X}\): a \(\mathbb{C}^*\)-equivariant semiample holomorphic \(\mathbb{Q}\)-line bundle;
3. \(\eta : (\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^* \cong (X, L) \times \mathbb{C}^*\).

**Trivial test configuration:** \((X_\mathbb{C}, L_\mathbb{C}) := (X, L) \times \mathbb{C}\).

\((\mathcal{X}, \mathcal{L})\) is **dominating** if there is a \(\mathbb{C}^*\)-equivariant birational morphism \(\rho : \mathcal{X} \to X \times \mathbb{C}\).

Under the isomorphism \(\eta\), psh metrics on \(\mathcal{L}|_{\pi^{-1}(\mathbb{C}^*)}\) are considered as **subgeodesic** rays on \((X, L)\).
For any TC \((\mathcal{X}, \mathcal{L})\), there are many smooth subgeodesic rays which extend to be a smooth psh metric on \(\mathcal{L}\).

**Theorem (Phong-Sturm)**

\[
\forall \text{ test configuration } (\mathcal{X}, \mathcal{L}), \exists \text{ a unique geodesic ray } \Phi = \Phi_{(\mathcal{X}, \mathcal{L})} \text{ emanating from } \varphi_0 \text{ s.t. } \Phi \text{ extends to a bounded psh metric on } \mathcal{L}.
\]

\(\Phi\) is obtained by solving the HCMA on a resolution \(\tilde{\mathcal{X}} \rightarrow \mathcal{X}\):

\[
(\mu^*(dd^c\tilde{\Phi}) + U)^{n+1} = 0; \quad U|_{\mathcal{X} \times S^1} = 0,
\]

where \(\tilde{\Phi}\) is any smooth positively curved Hermitian metric on \(\mathcal{L}\).

In general the solution \(\Phi := \tilde{\Phi} + U\) is at most \(C^{1,1}\) (Phong-Sturm, Chu-Tosatti-Weinkove).
Non-Archimedean functionals of TCs

For any TC \((\mathcal{X}, \mathcal{L})\), set:

\[
\begin{align*}
\mathbf{E}_{\mathcal{A}}^\mathcal{X}(\mathcal{X}, \mathcal{L}) & = \frac{\bar{\mathcal{L}} \cdot n+1}{n+1}; \\
\mathbf{J}_{\mathcal{A}}^\mathcal{X}(\mathcal{X}, \mathcal{L}) & = \bar{\mathcal{L}} \cdot \mathcal{L}_{\mathbb{P}^1} - \frac{\bar{\mathcal{L}} \cdot n+1}{n+1} \geq 0; \\
\mathbf{H}_{\mathcal{A}}^\mathcal{X}(\mathcal{X}, \mathcal{L}) & = \mathcal{K}^\log_{\mathcal{X}/\mathcal{X} \times \mathbb{P}^1} \cdot \bar{\mathcal{L}} \cdot n \geq 0; \\
(\mathbf{E}_{K_X}^\mathcal{X})_{\mathcal{A}}^\mathcal{X}(\mathcal{X}, \mathcal{L}) & = \mathcal{K}_X \cdot \bar{\mathcal{L}} \cdot n; \\
\mathbf{M}_{\mathcal{A}}^\mathcal{X}(\mathcal{X}, \mathcal{L}) & = \mathcal{K}^\log_{\mathcal{X}/\mathbb{P}^1} \cdot \bar{\mathcal{L}} \cdot n + \frac{S}{n+1} \bar{\mathcal{L}} \cdot n+1 \\
& = \mathbf{H}_{\mathcal{A}}^\mathcal{X} + (\mathbf{E}_{K_X}^\mathcal{X})_{\mathcal{A}}^\mathcal{X} + S \mathbf{E}_{\mathcal{A}}^\mathcal{X}.
\end{align*}
\]

\(\mathbf{M}_{\mathcal{A}}^\mathcal{X}(\mathcal{X}, \mathcal{L})\) generalizes the Futaki invariant in (1).

Similar non-Archimedean functional \((\mathbf{E}_{K_X}^\mathcal{X})_{\mathcal{A}}^\mathcal{X}\).
Slopes at infinity $\equiv$ non-Archimedean functional

Theorem

Let $\Phi$ be the geodesic ray associated to a TC $(\mathcal{X}, \mathcal{L})$. Then

1. \textit{[Tian, Zhang, Phong-Sturm-Ross, Boucksom-Hisamoto-Jonsson]}
   For $F \in \{E, J, E^{K_X}\}$, $F'_{\infty}(\Phi) = F^{\text{NA}}(\mathcal{X}, \mathcal{L})$.

2. \textit{[Hisamoto]}
   For $G$-equivariant TC, $J'_{\infty}(\Phi) = J^{\text{NA}}_{T}(\mathcal{X}, \mathcal{L})$ where
   \[ J^{\text{NA}}_{T}(\mathcal{X}, \mathcal{L}) = \inf_{\xi \in \mathbb{N}_R} J^{\text{NA}}(\mathcal{X}_\xi, \mathcal{L}_\xi). \]

3. \textit{[L., based on Tian, Boucksom-Hisamoto-Jonsson, Xia]}
   $H'_{\infty}(\Phi) = H^{\text{NA}}(\mathcal{X}, \mathcal{L})$. 

Test configurations and $G$-uniform K-stability.
**Definition (Hisamoto, generalizing Tian, Donaldson, Székelyhidi, Dervan, Boucksom-Hisamoto-Jonsson)**

\((X, L)\) is \(G\)-uniformly K-stable if there exists \(\gamma > 0\) such that for any \(G\)-equivariant test configuration \((\mathcal{X}, \mathcal{L})\),

\[
M^{NA}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot J_T^{NA}(\mathcal{X}, \mathcal{L}).
\]  

(2)

**Proposition (Hisamoto for \(\text{Aut}(X, L)_0\), \(L\). for more general \(G\))**

Assume that \((X, L)\) admits a cscK metric. If \(G\) contains a maximal torus of \(\text{Aut}(X, L)_0\), then \((X, L)\) is \(G\)-uniformly K-stable.
Combining the above discussions, the Yau-Tian-Donaldson conjecture now is reduced to two questions:

**Q1:** Can we approximate any destabilizing geodesic ray $\Phi$ by geodesic rays from test configurations?

**Q2:** Can we approximate the slopes $F'_{\infty}$ along any destabilizing geodesic ray by $F_{NA} = F'_{\infty}$ of TC's for $F \in \{E, E^{K_X}, J, H\}$?

More precisely, the YTD conjecture is reduced to the following

**Conjecture:** there exist TC’s $(\mathcal{X}_m, \mathcal{L}_m)$ such that

$$J_{T_{\mathcal{X}}(\mathcal{X}_m, \mathcal{L}_m)} \to J_{T_{\infty}}(\Phi), \quad \limsup_{m \to +\infty} M_{NA}(\mathcal{X}_m, \mathcal{L}_m) \leq M'_{\infty}(\Phi). \quad (3)$$
For any geodesic ray $\Phi$, Berman-Boucksom-Jonsson constructed a sequence of test configurations $(X_m, L_m)_m$ such that $\Phi_m \geq \Phi$ for any $m \gg 1$. Their construction is based on Demailly’s approximation:

1. Consider the multiplier ideal sheaf (MIS) over $X \times \mathbb{C}$:

   \[ I(m\Phi)(U) = \left\{ f \in O(U); \int_U |f|^2 e^{-m\Phi} < +\infty \right\}. \]

2. $\mu_m : X_m = Bl_{I(m\Phi)} X_\mathbb{C} \to X_\mathbb{C}$, $L_m = \mu_m^* L_\mathbb{C} - \frac{1}{m+m_0} E_m$. Using the Nadel vanishing and global generation property of MIS, $(X_m, L_m)$ is a test configuration of $(X, L)$.
Taking $\lim_{m \to +\infty} \Phi(x_m, \mathcal{L}_m)$:

**Theorem (Berman-Boucksom-Jonsson)**

*For any geodesic ray $\Phi$, there exists a unique “maximal” geodesic ray $\hat{\Phi}$ satisfying:*

1. $\hat{\Phi} \geq \Phi$.  
2. $\mathcal{J}(\lambda \hat{\Phi}) = \mathcal{J}(\lambda \Phi)$ for any $\lambda > 0$.

Maximal geodesic rays are exactly those that can be approximated by algebraic ones (i.e. geodesic rays associated to TC’s).

**Theorem (Berman-Boucksom-Jonsson, L. )**

\[ \forall \text{ maximal } \Phi, \exists \text{ a sequence of TC's } (x_m, \mathcal{L}_m)_m \text{ s.t. } \]
\[ F^{\infty}(\Phi) = \lim_{m \to +\infty} F^{\infty}(\Phi(x_m, \mathcal{L}_m)) \text{ for } F \in \{E, E^{K_x}, J, J_T\}. \]

**Conjecture:** The same holds for $H$, or equivalently for $M$. 

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**Maximal geodesic rays and approximation by TC**
Destabilizing geodesic rays are maximal

Theorem (L., '20)

A geodesic ray $\Phi$ satisfies $M'_{\infty}(\Phi) < +\infty$ is necessarily maximal.

The proof uses two key ingredients: equisingularity of multiplier approximation (via a valuative description) and Jensen’s inequality (motivated by Tian’s $\alpha$-type estimate): for arbitrary $\alpha > 0$,

$$
C(\alpha) > \log \int_{X \times \mathbb{D}} e^{\alpha(\hat{\Phi} - \Phi)} \sqrt{-1} dt \wedge d\bar{t} \\
\geq \alpha \int_X (\hat{\varphi}(s) - \varphi(s))(dd^c \varphi(s))^n - H_\Omega(\varphi(s)) - s \\
\geq C \alpha \cdot (E(\hat{\varphi}(s)) - E(\varphi(s))) - H(\varphi(s)) - s.
$$

take slope $\implies E'_{\infty}(\hat{\Phi}) \geq E'_{\infty}(\Phi) \implies E_{\text{linear}} \hat{\Phi} = \Phi$ (Dinew’s comparison principle).
K-stability for models

In the definition of a test configuration \((\mathcal{X}, \mathcal{L})\), if we don’t require \(\mathcal{L}\) to be semiample, then we say that \((\mathcal{X}, \mathcal{L})\) is a model of \((X, L)\).

For any (big) model \((\mathcal{X}, \mathcal{L})\), one can still define:

\[
E^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \frac{\langle \bar{\mathcal{L}} \cdot n+1 \rangle}{n+1} = \frac{\text{vol}(\bar{\mathcal{L}})}{n+1};
\]
\[
J^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \langle \bar{\mathcal{L}} \rangle \cdot L_{\mathbb{P}^1} \cdot \frac{\langle \bar{\mathcal{L}} \cdot n+1 \rangle}{n+1};
\]
\[
M^{\text{NA}}(\mathcal{X}, \mathcal{L}) = K_{\mathcal{X}/\mathbb{P}^1}^{\text{log}} \cdot \langle \bar{\mathcal{L}} \cdot n \rangle + \frac{S}{n+1} \langle \bar{\mathcal{L}} \cdot n+1 \rangle.
\]

Volume/restricted volume studied by Tsuji, Boucksom-Favre-Jonsson, Ein-Lazarsfeld-Mustaţă-Nakamaye-Popa:

\[
K_{\bar{\mathcal{X}}} \cdot \langle \bar{\mathcal{L}} \cdot n \rangle = \frac{d}{dt} \bigg|_{t=0} \text{vol}(\bar{\mathcal{L}} + tK_{\bar{\mathcal{X}}}).
\]
Definition (L.)

\((X, L)\) is \(\mathbb{G}\)-uniformly K-stable for models if \(\exists \gamma > 0\) such that for any model \((X, L)\), \(M^{\text{NA}}(X, L) \geq \gamma \cdot J^{\text{NA}}(X, L)\).

Theorem (L. ’20)

For any maximal \(\Phi\), \(\exists\) sequence of models \((X_m, L_m)\) such that

\[
J^{\text{NA}}(X_m, L_m) \to J^{\prime \infty}(\Phi), \quad \limsup_{m \to +\infty} M^{\text{NA}}(X_m, L_m) \leq M^{\prime \infty}(\Phi).
\]

The proof uses Archimedean/non-Archimedean pluripotential theory

Corollary

If \((X, L)\) is \(\mathbb{G}\)-uniform K-stable for models, then there does not exist destabilizing geodesic ray \(\Phi\) (with \(M^{\prime \infty}(\Phi) \leq 0\) and \(J^{\prime \infty}(\Phi) = 1\)) and hence \((X, L)\) admits a cscK metric.
Fujita approximation (almost Zariski decomposition) studied by Fujita, Demailly-Ein-Lazarsfeld, Boucksom-Favre-Jonsson, Ein-Lazarsfeld-Mustatǎ-Nakamaye-Popa, ...:

\[ \mu_m : \mathcal{X}_m \to \mathcal{X}, \quad \mathcal{L}_m = \mu^* \mathcal{L} - \frac{1}{m} \mathcal{E}_m \]

\( \mu_m \) is blow-up of base ideals or some asymptotic multiplier ideal sheaves with exceptional divisor \( \mathcal{E}_m \). \( \mathcal{L}_m \) is semiample satisfying

\[
\lim_{m \to +\infty} \frac{h^0(\mathcal{X}, m\mathcal{L})}{m^{\dim \mathcal{X}}/(\dim \mathcal{X})!} =: \text{vol}(\mathcal{X}, \mathcal{L}) = \lim_{m \to +\infty} \text{vol}(\mathcal{X}_m, \mathcal{L}_m).
\]

**Theorem (Boucksom-Jonsson, L.)**

For any model \((\mathcal{X}, \mathcal{L})\), there exists a sequence of TC’s \((\mathcal{X}_m, \mathcal{L}_m)\) s.t.

\[
F^{NA}(\mathcal{X}_m, \mathcal{L}_m) = F'\infty(\Phi_m) \to F^{NA}(\mathcal{X}, \mathcal{L}) = F'\infty(\Phi)
\]

for \( F \in \{E, E^{KX}, J, J_T\} \) where \( \Phi_m = \Phi(\mathcal{X}_m, \mathcal{L}_m) \).
We propose the following conjecture:

Conjecture (strengthened Fujita approximation $\Rightarrow$ approximation of $H_{NA}^B$ (conj. of Boucksom-Jonsson) $\Rightarrow$ YTD)

\[
\forall \text{ big line bundle } \mathcal{L} \to \mathcal{X}, \exists \text{ birational morphisms } \mu_m : \mathcal{X}_m \to \mathcal{X} \text{ and decompositions } \mu_m^* \mathcal{L} = \mathcal{L}_m + \frac{1}{m} E_m \text{ with } \mathcal{L}_m \text{ semiample and } E_m \text{ effective, s.t.}
\]

1. \( \text{vol}(\mathcal{L}_m) \to \text{vol}(\mathcal{L}). \)
2. \( \text{The derivatives also converge:} \)

\[
\frac{d}{dt} \bigg|_{t=0} \text{vol}(\mathcal{L}_m + tK_{\mathcal{X}_m}) \to \frac{d}{dt} \bigg|_{t=0} \text{vol}(\mathcal{L} + tK_{\mathcal{X}}). \quad (4)
\]

True if \((X, L)\) is spherical (since \(\mathcal{X}\) is a Mori dream space), in particular true for toric manifolds.
Thanks for your attention!