Algebraicity of metric tangent cones via normalized volume and K-stability

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based on a series of works (joint with Yuchen Liu, Chenyang Xu and Xiaowei Wang)

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(M_i, g_i, J_i): a sequence of Kähler-Einstein Fano manifolds:

$$Ric(\omega_i) = \omega_i, \quad \omega_i = g_i(\cdot, J_i \cdot) \in 2\pi c_1(M, J_i) > 0.$$

Gromov's compactness: (M_i, g_i) sub-sequentially converges to a limit metric space (X, d_∞) in the Gromov-Hausdorff topology.

Question: How regular is the limit (X, d_∞)?
Answer: X is homeomorphic to a normal projective variety such that

1. X is a Fano: $-mK_X$ is an ample line bundle for some $m > 0 \in \mathbb{Z}$.
2. X has a weak Kähler-Einstein metric $\Longrightarrow$ X has Klt singularities.

Tian (proved dim 2 case and reduced it to a partial C^0-estimate conjecture)
Donaldson-Sun (proved the partial C^0-estimate conjecture)

Further question: What does the metric look like near the singularity of X?

Metric tangent cones

Metric tangent cone is the first order approximation of the metric structure:

$$C_x X := \lim_{r_k \to 0+}^{p-GH} \left(X, x, \frac{d_X}{r_k} \right)$$

is a metric cone (Cheeger-Colding). The limit a priori could depend on $\{r_k\}$.
General results:

1. (Cheeger-Colding-Tian) $(C_x X)^{\mathrm{sing}}$ has complex Hausdorff codimension at least 2. $(C_x X)^{\mathrm{reg}}$ is Ricci-flat Kähler cone.
2. (Donaldson-Sun) $C_x X$ is homeomorphic to an affine variety with an effective torus action (generated by the Reeb vector field) and is uniquely determined by the *metric* structure on the GH limit X.

Conjecture (Donaldson-Sun)

$C_x X$ depends only on the algebraic structure of the germ $x \in X$.

If true, the object $C_x X$ is a canonically new algebraic object associated to the Klt singularity. No metric structure needed!
The goal of this talk is to explain our work proving that this is indeed true.

Define a map $D_{\mathrm{metric}} : \mathcal{O}_{X,x} \to [0, \infty]$: for any $f \in \mathcal{O}_{X,x}$,

$$D_{\mathrm{metric}}(f) = \limsup_{r \to 0} \frac{\max_{z \in B(p,r)} \log |f(z)|}{\log r}.$$

Assume $D_{\mathrm{metric}}(\mathcal{O}_{X,x}) =: \Gamma = \{\lambda_i\}$. Let $\mathcal{F}_i = \{f \in \mathcal{O}_{X,x}; D(f) \geq \lambda_i\}$

$$R_{D_{\mathrm{metric}}} := \bigoplus_{\lambda_i \in \Gamma} \frac{\mathcal{F}_i}{\mathcal{F}_{i+1}}.$$

Theorem (Donaldson-Sun, '15)

1. D_{metric} *is a pseudovaluation,* $R_{D_{\mathrm{metric}}}$ *is finitely generated and* $W = \mathrm{Spec}(R_{D_{\mathrm{metric}}})$ *is a normal affine variety.*
2. *The metric tangent cone* $C_x X$ *is the central fibre of a torus equivariant degeneration of* W*, through affine varieties in* $\mathbb{C}^N$ *under the torus action.*

Rephrase Donaldson-Sun's Conjecture: D_{metric} is uniquely determined by the *algebraic* structure of the germ $x \in X$.

Examples of metric tangent cones

3-dimensional A_k singularities:

$$X = \{z_1^2 + z_2^2 + z_3^2 + z_4^{k+1} = 0\} \subset \mathbb{C}^4.$$

X degenerates, via $(z_1, z_2, z_3, z_4) \to (t^2 z_1, t^2 z_2, t^2 z_3, t^\alpha z_4)$ for $\alpha > \frac{4}{k+1}$, to

$$X' := \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C} \cong \{z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{C}^4$$

Metric tangent cones:

k	$\frac{4}{k+1}$	W	$C_x X$	ξ_0 on $C_x X$
$0, 1, 2$	> 1	X	X	$(k+1, k+1, k+1, 2)$
3	$= 1$	X	X'	$(2, 2, 2, 1)$
$k \geq 4$	< 1	X'	X'	$(2, 2, 2, 1)$

(1)

The Ricci-flat Kähler cone metric on $C_x X$: $g = dr^2 + r^2 g_{M^{2n-1}}$.
The holomorphic vector field $\xi_0 = r\partial_r - iJ(r\partial_r)$ is called the Reeb vector field.
We say that $(Z, \xi_0) := (C_x X, \xi_0)$ is a Fano cone with the Reeb vector field ξ_0.

Overview II: Ricci-flat Kähler cone and K-stability

Definition (Collins-Székelyhidi, generalizing Fano case of Tian and Donaldson)

A Fano cone (Z, ξ_0) is K-semistable (resp. K-polystable) if for any T-equivariant degeneration $\mathcal{Z}$ to another Fano cone (Z_0, ξ_0), $\mathrm{Fut}(\mathcal{Z}) \geq 0$ (and $= 0$ iff $\mathcal{Z}$ is induced by a holomorphic vector field on Z).

Theorem (Collins-Székelyhidi, L.-Xu)

If a (Klt) Fano cone (Z, ξ_0) admits a Ricci-flat Kähler cone metric, then (Z, ξ_0) is K-polystable.

This says that $(Z, \xi_0) := (C_x X, \xi_0)$ is K-polystable.

Theorem (L.-Xu '17)

If a Fano cone W equivariantly degenerates to a K-polystable Fano cone, then W is K-semistable.

This means that W is K-semistable and we say that D_{metric} is a K-semistable valuation.

Donaldson-Sun's conjecture follows from the following main results, which are proved using only tools from algebraic geometry.

Theorem (L.-Xu '16-'17, (see below for notations))

For any Klt singularity, a K-semistable valuation is the unique minimizer of the normalized volume functional among all quasi-monomial valuations.

This implies D_{metric} and W are uniquely determined by $x \in X$.

Theorem (L.-Wang-Xu '18)

Any K-semistable Fano cone W degenerates to a K-polystable Fano cone Z. Moreover, such a Z is uniquely determined by W.

This implies $Z := C_x X$ is uniquely determined by W.

Let (X, x) be a normal singularity such that mK_X is locally generated over an open set U by a nowhere vanishing holomorphic section s.
(X, x) is Klt if:

$$\int_{U^{\mathrm{reg}}} \sqrt{-1}^{mn^2} (s \wedge \bar{s})^{1/m} < +\infty. \tag{2}$$

How to check this? Choose a log resolution $\mu : Y \to X$ and write:

$$\mu^*(s \wedge \bar{s})^{\frac{1}{m}} = h(z) \prod_i |z_i|^{2a_i} dz \wedge d\bar{z},$$

where $h(z)$ is a nowhere vanishing function. Then (2) is satisfied if and only if $a_i > -1$ for every i.

The Klt condition can be formulated algebraically: Write

$$K_Y = \mu^* K_X + \sum_i a_i E_i$$

X is Klt if and only if $A(\mathrm{ord}_{E_i}) := a_i + 1 > 0$ for all i. Examples include:

1. $\dim_{\mathbb{C}} X = 2$. Klt=isolated quotient singularity $\mathbb{C}^2/G$.
2. $\dim_{\mathbb{C}} X = 3$. partial classification ($\{$terminal$\}$ (classified) $\subset$ $\{$canonical$\}$ $\subset$ Klt)
3. Isolated quotient singularities and $\mathbb{Q}$-Gorenstein toric singularities are Klt.
4. Fano cone singularity (X, ξ_0): Klt singularity with an effective torus action and an attractive point (and a distinguished Reeb vector field).

Assume S is a Fano manifold: $-K_S$ is ample. Assume $K_S^{-1} = rL$ with $r \in \mathbb{Q}_{>0}$ for a holomorphic line bundle L.
Contraction of zero section S, or extraction of S from the affine cone:

$$S \subset Y \xrightarrow{\mu} C(S, L) := \mathrm{Spec}_{\mathbb{C}} \left(\bigoplus_{k=0}^{+\infty} H^0(S, kL) \right).$$

$(C(S, L), \xi_0)$ is a Fano cone singularity where ξ_0 is the holomorphic vector field corresponding to the $\mathbb{Z}$-grading.

Examples:

- $S = \mathbb{CP}^{n-1}$, $r = \frac{1}{n}$, $L = H := \mathcal{O}_{\mathbb{CP}^{n-1}}(1)$, $X = \mathbb{C}^n$, $\xi_0 = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$.
- $S = \{F(Z_1, \ldots, Z_{n+1}) = 0\} \subset \mathbb{P}^n$ with $d < n+1$, $r = \frac{1}{n+1-d}$ and $L = H|_M$, $X = \{F(z_1, \ldots, z_{n+1}) = 0\} \subset \mathbb{C}^{n+1}$.

More generally, $S = (S, \Delta)$ can be a Fano orbifold and have Klt singularities.

Example: $X = \{z_1^2 + z_2^2 + z_3^2 + z_4^{k+1} = 0\}$ are Fano cones (with $\xi_0 = (k+1, k+1, k+1, 2)$) over the Fano orbifold:

$$(S, \Delta) = \begin{cases} (\mathbb{P}^2, \frac{k}{k+1} C) & k \text{ even}, \\ (\mathbb{P}^1 \times \mathbb{P}^1, \frac{k-1}{k+1} \Delta(\mathbb{P}^1)) & k \text{ odd} \end{cases}$$

(X, ξ_0) admits a Ricci-flat Kähler cone metric if and only if $0 \leq k \leq 3$ (Martelli-Sparks-Yau, L.-Sun, see (1))

Example: $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C} \cong \{z_1^2 + z_2^2 + z_3^2 = 0\} = C((S', \Delta'), L')$ with $(S', \Delta') = (\mathbb{P}(1,1,2), \frac{1}{2} D) = \mathbb{P}^2/\mathbb{Z}_2$. Reeb vector field $\xi_0 = (2, 2, 2, 1)$.

A consequence of deep results from Minimal Model Program (MMP):
Any Klt singularity can degenerate to a Fano orbifold cone (associated to a plt blow-up).
So the Fano cones can be considered as prototypes of Klt singularities.

General Fano cones

Example: Let $\sigma \subset N_{\mathbb{R}}$ be a rational polyhedral cone. $X := X_\sigma$ is the associated toric variety. For any $\xi_0 \in \mathrm{int}(\sigma)$, (X, ξ_0) is a Fano cone singularity (assuming $\mathbb{Q}$-Gorenstein).

General Fano cone singularity $x \in X := \mathrm{Spec}_{\mathbb{C}}(A)$:

- X : a normal Klt singularity with an effective torus $T := (\mathbb{C}^*)^d$ action.
- there is a unique closed point $x \in X$ that is in the orbit closure of any T-orbit.
- a distinguished Reeb vector $\xi_0 \in \mathfrak{t}_{\mathbb{R}}^+$.

(Co-)characters: $M = \mathrm{Hom}(T, \mathbb{C}^*)$. $N := \mathrm{Hom}(\mathbb{C}^*, T)$.

Weight decomposition: $A = \bigoplus_{\lambda \in \Gamma} A_\lambda, \quad \Gamma \subset M$

Reeb cone: $\sigma := \mathfrak{t}_{\mathbb{R}}^+ = \{\xi \in N_{\mathbb{R}}; \langle \lambda, \xi \rangle > 0 \text{ for any } \lambda \in \Gamma \backslash \{0\}\}$

Moment cone: $\sigma^\vee = \mathrm{Span}_{\mathbb{R}}(\Gamma) \subset M$.

In general, there is a combinatorial description using the theory of T-varieties via divisorial polytopes (Altmann-Hausen, Ilten-Süss, ...).

Assume $(X, x) = (\mathrm{Spec}_{\mathbb{C}}(R), \mathfrak{m})$ where R is a local integral domain which is a finitely generated $\mathbb{C}$-algebra.

Definition

A real valuation on X with center x is a function $v : R \to \mathbb{R} \cup \{+\infty\}$ satisfying:

1. $v(f+g) \geq \min\{v(f), v(g)\}, \quad \forall f, g \in R$;
2. $v(f \cdot g) = v(f) + v(g), \quad \forall f, g \in R$;
3. $v(0) = +\infty$, *and* $v(a) = 1$ *for any* $a \in \mathbb{C}^*$;
4. $v(f) > 0$ *for any* $f \in \mathfrak{m}$.

One should think of v as a measure of vanishing order of f around $x \in X$. Denote by $\mathrm{Val}_{X,x}$ the space of all real valuations centered at $x \in X$. If $v \in \mathrm{Val}_{X,x}$, then $\lambda v \in \mathrm{Val}_{X,x}$ for any $\lambda > 0$.

1. Divisorial valuations. Let $\mu : Y \to X$ be a birational morphism and E is a Weil divisor on Y. Define: for any $f \in \mathcal{O}_x$

$$\mathrm{ord}_E(f) = \mathrm{ord}(\mu^* f).$$

2. Monomial valuations on $\mathbb{C}^n$. Fix $\xi \in \mathbb{R}^n_+$, for any $f \in \mathbb{C}[z_1, \dots, z_n]$, define:

$$v_\xi(f) = \min \left\{ \sum_{\mathbf{m}} m_i \xi_i ; f = \sum_{\mathbf{m}} a_{\mathbf{m}} z^{\mathbf{m}}, a_{\mathbf{m}} \neq 0 \right\}.$$

3. Quasi-monomial valuations: monomial valuations on Y on some birational morphism $\mu : Y \to X$. Quasi-monomial valuations include all divisorial valuations and the following
Quasi-monomial valuation from torus actions: Assume $X = \mathrm{Spec}_{\mathbb{C}}(A)$ is a Fano cone singularity with $A = \bigoplus_{\lambda \in \Gamma} A_\lambda$. For any $\xi \in \mathfrak{t}^+_{\mathbb{R}}$,

$$v_\xi(f) = \min \left\{ \langle \xi, \lambda \rangle ; f = \sum_{\lambda} f_\lambda, f_\lambda \neq 0. \right\}$$

v_ξ is divisorial if and only if $\xi \in \mathfrak{t}^+_{\mathbb{Q}}$.

General construction: For any $v \in \mathrm{Val}_{X,x}$, $\Gamma = v(R)$ is an ordered semigroup. Γ-graded sequence of valuative ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_\lambda; \lambda \in \Gamma\}$:

$$\mathfrak{a}_\lambda(v) = \{f \in R; v(f) \geq \lambda\}, \quad \mathfrak{a}_{>\lambda}(v) = \{f \in A; v(f) > \lambda\}.$$

Associated graded ring of v:

$$\mathrm{gr}_v R = \bigoplus_{\lambda \in \Gamma} \mathfrak{a}_\lambda(v)/\mathfrak{a}_{>\lambda}(v)$$

Suppose $\mathrm{gr}_v R$ is finite generated then $W := \mathrm{Spec}_{\mathbb{C}}(\mathrm{gr}_v R)$ is an affine variety with an effective torus action.

Recall: For metric tangent cones, Donaldson-Sun's work implies:
There is a valuation v determined by the metric structure of X such that W is well defined and degenerates to the metric tangent cone $C_x X$.
Questions 1: How to characterize such v?
Question 2: How to characterize $C_x X$ in terms of v?

Motivated by result of Martelli-Sparks-Yau from Sasaki-Einstein geometry:

Definition (L. '15, the normalized volume)

$$\begin{aligned}\widehat{\mathrm{vol}} := \widehat{\mathrm{vol}}_{X,x} : \mathrm{Val}_{X,x} &\longrightarrow \mathbb{R}_{>0} \cup \{+\infty\} \\ v &\mapsto A_X(v)^n \cdot \mathrm{vol}(v).\end{aligned}$$

- $A_X(v)$: log disrepancy of v satisfying: $A_X(v) = A_Y(v) + \mathrm{ord}_v(K_{Y/X})$

  X Klt $\Longleftrightarrow A_X(v) > 0$ for any $v \in \mathrm{Val}_X$.

  Example/Key Observation: For valuations induced by torus actions:

  $$A_X(v_\xi) = \frac{\mathcal{L}_\xi \Omega}{\Omega}$$

  where Ω is a $(\mathbb{C}^*)^d$-equivariant nowhere vanishing holomorphic n-form.
- $\mathrm{vol}(v) = \lim_{m\to+\infty} \frac{\dim_{\mathbb{C}}(A/\mathfrak{a}_m(v))}{m^n/n!}$ (Ein-Lazarsfeld-Smith).

Basic properties of normalized volume functional:

1. $\widehat{\mathrm{vol}}(\lambda v) = \widehat{\mathrm{vol}}(v)$ for any $\lambda > 0$.
2. $\widehat{\mathrm{vol}}(v) \geq C \frac{A_X(v)}{v(\mathfrak{m})} \geq C \cdot \mathrm{lct}(\mathfrak{m}) > 0$ (L. '15).

Conjecture (Proposed by L., Li-Xu)

Given any Klt singularity $x \in X = \mathrm{Spec}(R)$, *there is a unique minimizer* v *up to rescaling. Furthermore,* v *is quasi-monomial, with a finitely generated associated graded ring such that* $(Z := \mathrm{Spec}(\mathrm{gr}_v(R)), \xi_v)$ *is a K-semistable Fano cone singularity.*

- Existence of minimizer: H. Blum used de-Fernex-Ein-Mustaţă's technique of generic limits (for attacking ACC conjecture) to prove the existence.
- Uniqueness:
  - Divisorival minimizers are unique (L.-Xu '16)
  - On semistable Fano cone, quasi-monomial minimizers are unique (L.-Xu).
- Regularity of minimizer:
  - True for valuations from Gromov-Hausdorff limits, wide open in general
  - The quasi-monomial part is implied by a conjecture of Jonsson-Mustaţă (which is related to the openness conjecture).

Theorem (L., L.-Liu, L.-Xu, '15-'17)

A Fano cone (Z, ξ_0) is K-semistable if and only if v_{ξ_0} is a minimizer of $\widehat{\mathrm{vol}}$.

This is a generalization of the minimization result by Martelli-Sparks-Yau who considered valuations from torus actions.

Idea of Proof:

- Reduce to the torus invariant valuations;
- Derivative of normalized volume is the Futaki invariant;
- The normalized volume is convex along "equivariant rays".

Example: $\widehat{\mathrm{vol}}(0, \mathbb{C}^n/G) = \frac{n^n}{|G|}$, $\quad \widehat{\mathrm{vol}}(x, (X, d_\infty)) = n^n \cdot \lim_{r\to 0} \frac{\mathrm{vol}(B(x,r))}{\mathrm{vol}(B(0,\mathbb{C}^n))}$

Related development: valuative criterion of K-(semi)stability (L., Fujita) and uniform K-stability (by Fujita, Blum-Jonsson)

Theorem (Y. Liu, L.-Xu)

$$\widehat{\mathrm{vol}}(x,X) = \inf_{\mathfrak{a}} \mathrm{lct}(\mathfrak{a})^n \mathrm{mult}(\mathfrak{a}) = \inf_{Y/X} \mathrm{vol}_x(-(K_Y+E)) = \inf_{S\ plt} \widehat{\mathrm{vol}}(\mathrm{ord}_S)$$

$E = \mu^{-1}(x)_{\mathrm{red}}$ and vol_x is the local volume studied by Fulger:

$$\mathrm{vol}_x(-(K_Y+E)) = \lim_{m\to+\infty} \frac{\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/\mu_*(\mathcal{O}_Y(-m(K_Y+E)))}{m^n/n!}.$$

Important consequence: Minimizers v computes $\mathrm{lct}(\mathfrak{a}_\bullet(v))$.

Example: A new interpretation of de-Fernex-Ein-Mustaţă's inequality:

$\mathbb{CP}^{n-1}$ is K-semistable

$\iff$ $\mathrm{lct}(\mathfrak{a})^n\mathrm{mult}(\mathfrak{a}) \geq n^n$ for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$

$\iff$ Arithmetic Mean - Geometric Mean inequality.

Assume (Z, ξ_0) is a Fano cone singularity with Reeb cone σ and moment cone $\sigma^\vee$. For any T-invariant quasi-monomial valuation v.

- Connect v_{ξ_0} with v by a path $\{v_t\}_{t\in(0,1)}$ of T-invariant quasi-monomial valuations.
- Use the tools of Newton-Okounkov to express $\mathrm{vol}(v_t)$ as volumes of varying convex bodies.
- Reduce to the following convex geometric problem.

Let $\tilde{\sigma} \subset \mathbb{R}^n$ be a strictly convex cone. Fix $u_0 \in \mathrm{int}(\tilde{\sigma}^\vee)$. Consider the map:

$$\{\xi \in \tilde{\sigma}; \langle u_0, \xi\rangle = 1\} = H^+_{u_0} \ni \xi \quad \mapsto \quad \Delta_\xi = \{y \in \tilde{\sigma}^\vee; \langle y, \xi\rangle \leq 1\}$$

Lemma (Gigena, 1978)

The function $\xi \mapsto \mathrm{vol}(\Delta_\xi)$ is proper and strictly convex on $H^+_{u_0}$ and hence has a unique minimizer ξ_0.

Toric Example: non-divisorial minimizers on the affine cone over $\mathbb{P}^2 \sharp \overline{\mathbb{P}^2}$ (Martelli-Sparks-Yau, Futaki-Ono-Wang, H. Blum)

Theorem (L.-Xu)

A divisorial valuation ord_S *is a minimizer if and only if*

1. *There is a plt blow up* $\mu : Y \to X$ *with* S *being the exceptional divisor, and*
2. *The log Fano pair* $(S, \mathrm{Diff}_S(0))$ *is K-semistable.*

Moreover, such a divisorial minimizer is unique if it exists.

Necessity of item 1 is also independently proved by H.Blum. The proof is based the the fact that ord_S computes $\mathrm{lct}(\mathfrak{a}_\bullet(\mathrm{ord}_S))$ and the following key result from MMP (used again and again in the following argument).

Theorem (Birkar-Casini-Hacon-McKernan)

Let $\mathscr{X}$ *be a normal projective variety,* $\mathscr{A} \subset \mathcal{O}_{\mathscr{X}}$ *an ideal sheaf and* $c > 0$. *Assume* ord_E *is a divisorial valuation which has center on* $\mathscr{X}$ *and satisfies:*

$$\mathrm{lct}(\mathscr{X}, c \cdot \mathscr{A}) < 1 \quad \text{and} \quad A_{\mathscr{X}}(\mathscr{E}) - c \cdot \mathrm{ord}_{\mathscr{E}}(\mathscr{A}) < 1.$$

Then $\mathscr{E}$ *can be extracted as a prime divisor on a birational model over* $\mathscr{X}$

Uniqueness of minimizers II: divisorial minimizers

Idea of Proof of Uniquenss: Fix a divisorial (plt) minimizer $S \subset Y \to X$.

1. Construct the degeneration $\mathcal{X}$ of X to $C((S,\Delta), -S|_S) \bigcup Y$ by the deformation to the normal cone (or using associated graded ring).
2. For any divisorial (plt) minimizer $S' \subset Y' \to X$, equivariantly degenerate ideals $\mathfrak{a}_\bullet(\mathrm{ord}_{S'})$.
3. Degenerate the model $Y' \to X$, equivalently extract divisor $S' \times \mathbb{C}$ over $X \times \mathbb{C}$. To do this, use minimizing property to find an ideal $\mathfrak{A}$ on $\mathcal{X}$ satisfying Theorem 9.
4. Use uniqueness in the torus invariant case on the central fibre to conclude $S' \cong S$ over the cone.
5. Contract the blown-up cone to conclude $S \cong S'$ over X. Algebraically, $\mathrm{ord}_{S'}(f) = \mathrm{ord}_{S'=S}(\mathrm{in}(f)) = \mathrm{ord}_S(f)$.

We apply similar strategy to prove the uniqueness result for K-semistable valuations v (i.e. v is quasimonomial, $\mathrm{gr}_v(R)$ is finitely generated and $\mathrm{Spec}(\mathrm{gr}_v R)$ is a K-semistable Fano cone). The essential and technical results we proved are contained in the following:

Proposition (L.-Xu '17)

For a quasi-monomial minimizer v, we can find divisors $S_1, \dots, S_r$, s.t.

1. *there is a model $Y \to X$ which precisely extracts $S_1, \dots, S_r$ over x,*
2. *v is a monomial valuation w.r.t. (Y, E).*
3. *(Y, E) is log canonical, and $-K_Y - E$ is nef.*

If moreover $\mathrm{gr}_v(R)$ is finitely generated, then $X' = \mathrm{Spec}(\mathrm{gr}_v R)$ has Klt singularities.

Theorem 5: Uniqueness of K-polystable degenerations

Assume (X, ξ_0) degenerates to two K-polystable Fano cones $X_0^{(i)}, i = 1, 2$.

$$\begin{array}{ccccc} X_0^{(2)} & \overset{\mathcal{X}^{(2)} \leftarrow \mathcal{Y}_k^{(2)} \leftarrow \mathcal{E}_k^{(2)}}{\twoheadleftarrow\!\!\sim\!\sim\!\sim\!\sim\!\sim\!\sim} & X \longleftarrow Y_k \leftarrow E_k \\ \Big\wr\, \mathcal{X}'^{(2)} & & \Big\wr\, \mathcal{X}^{(1)} \leftarrow \mathcal{Y}_k \leftarrow \mathcal{E}_k = E_k \times \mathbb{C}^1 \\ X_0' & \overset{\mathcal{X}'^{(1)}}{\twoheadleftarrow\!\!\sim\!\sim\!\sim\!\sim\!\sim\!\sim} & X_0^{(1)} \longleftarrow Y_{k,0} \leftarrow E_k \end{array} \qquad (3)$$

Key arguments:

- Approximate ξ_0 by a sequence of divisorial valuations ord_{E_k}.
- Show that $E_k \times \mathbb{C}$ can be extracted: $\mathcal{Y}_k^{(2)} \to \mathcal{X}^{(2)}$. $\mathrm{Fut}(\mathcal{X}^{(\mathrm{i})}) = 0$ is crucial:
    - $\widehat{\mathrm{vol}}(E_k) = \widehat{\mathrm{vol}}(v_{\xi_0}) + O(k^{-2})$.
    - $X_0^{(i)}$ is K-semistable and hence has the volume minimizing property.
    - The equivariant degeneration of the ideal sheaf $\mathfrak{a}(\mathrm{ord}_{E_k})$ on $\mathcal{X}^{(2)}$ produces $\mathfrak{A}$ satisfying the condition of Theorem 9.
- Degenerate the model $\mathcal{Y}_k^{(2)} \to \mathcal{X}^{(2)}$ to complete the square.
- Show that $\mathrm{Fut}(\mathcal{X}'^{(\mathrm{i})}) = 0$ and that $\mathcal{X}_0'^{(i)}$ are Fano cones.

Recent applications of the study of metric tangent cones/normalized volumes

1. Determine the metric tangent cones a priori without knowing the metric. This is useful:
    1. Prove the polynomial asymptotics of Kähler-Einstein metrics near special (stable) isolated conical points (Hein-Sun).
    2. New examples of slow convergence of singular Kähler-Einstein metrics to metric tangent cones (Han-L.).
2. New (torus-equivariant) criterions for the K-semistability/K-polystability of Fano varieties (L., L.-Liu, L.-Wang-Xu)
3. Bound the singularities of K-semistable Fano varieties (Liu) and application to the construction of moduli (Liu-Xu, Spotti-Sun)
4. 2-dimensional logarithmic normalized volume is equal to Langer's local orbifold Euler number (Borbon-Spotti, L.)

Thanks for your attention!