Algebraic uniqueness of Kähler-Ricci flow limits on Fano varieties

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Kähler-Ricci flow

- Hamilton's Ricci flow: $\frac{dg}{dt} = -2Ric(g)$ powerful tool in geometric analysis: Perelman's proof of Geometriziation Conjecture.
- Ricci flow on Kähler manifolds

  (X, J, g) a Kähler manifold. J an integrable complex structure.
  Kähler form $\omega(\cdot, J\cdot) = g(\cdot, \cdot)$ is closed: $d\omega = 0$.

  Kähler class: $[\omega] \in H^2(X, \mathbb{R})$.

  Ricci flow preserves the Kähler condition: Kähler-Ricci flow

  $$\frac{d\omega}{dt} = -Ric(\omega).$$

  $Ric(\omega)$ is the (closed) Ricci curvature form:

  $$Ric(\omega) = -\sqrt{-1}\partial\bar{\partial} \log \omega^n,$$

  which represents the first Chern class $c_1(-K_X) = c_1(\wedge^n T_{hol} X) \in H^2(X, \mathbb{R})$.

Normalized Kähler-Ricci flow

Evolution of Kähler class: $\frac{d}{dt}[\omega_t] = -c_1(X) \quad \Rightarrow \quad [\omega_t] = [\omega] - tc_1(X)$.

Set $T_{\max} = \sup\{t; \quad [\omega_t] = [\omega] - tc_1(X)$ is a Kähler class$\}$.

Theorem (Tian-Zhang, Tsuji)

The smooth Kähler-Ricci flow exists if and only if $t \in [0, T_{\max})$.

If X is Fano: $-K_X := \wedge^n T_{hol}X$ is ample, $[\omega] = c_1(X) = c_1(-K_X) > 0$.
$T_{\max} = \max\{t; c_1(X) - tc_1(X) > 0\} = 1$.

Volume collapsed at $T_{\max}$: $\int_X \omega_t^n = (1-t)^n \int_X \omega^n \longrightarrow 0$ as $t \to 1$.

(volume) normalized Kähler-Ricci flow: $\tilde{\omega}_t = \frac{\omega_t}{1-t} \in c_1(X)$, $\tilde{t} = -\log(1-t)$:

$$\frac{d\tilde{\omega}}{d\tilde{t}} = -Ric(\tilde{\omega}_t) + \tilde{\omega}_t \quad \xrightarrow{\text{change notation}} \quad \frac{d\omega_t}{dt} = -Ric(\omega_t) + \omega_t.$$

Corollary (H. Cao)

Smooth solution ω_t to the NKR exists for all $t \in [0, +\infty)$.

Fano manifolds and shrinking Kähler-Ricci soliton

Fano manifolds ($c_1(-K_X) = c_1(\wedge^n T_{hol}X) > 0$) are important building blocks of projective varieties (Mori fibre space in MMP).

Examples:

- projective space $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, toric Fano manifolds.
- $\mathbb{P}^2$, blow-up projective plane $\mathbb{P}^2 \sharp k\overline{\mathbb{P}^2}$, $1 \le k \le 8$.
- hypersurface $\{F(Z_0, \ldots, Z_{n+1}) = 0\} \subset \mathbb{P}^{n+1}$ with degree $\le n+1$.

(Shrinking) Kähler-Ricci soliton (KRS) on Fano manifolds:

$$Ric(\omega) = \omega + \mathscr{L}_\xi \omega$$

where ξ is a holomorphic vector field and $\mathscr{L}$ is the Lie derivative. If $\sigma_\xi(t)$ is the 1-psg generated by ξ, $\omega(t) = \sigma_\xi(t)^*\omega$ solution to the NKF.
If $\xi = 0$, KRS is called Kähler-Einstein (KE) metric.

Theorem (Tian-Zhu, Dervan-Székelyhidi)

If X admits a KRS, then any NKR $(X, \omega(t))$ converges to ω_{KR} as $t \to +\infty$.

Hamilton-Tian conjecture

Conjecture (Hamilton-Tian (HT) conjecture)

As $t \to +\infty$, any NKR on a Fano manifold $(X, \omega(t))$ converges to $(X_\infty, \omega_\infty)$ in the Gromov-Hausdorff topology where

- X_∞ *is a possibly singular normal Fano variety* $(\mathrm{codim}_{\mathbb{C}} X_\infty^{\mathrm{sing}} \geq 2)$.
- ω_∞ *is a KRS on* X_∞^{reg} *and the convergence is locally smooth on* X^{reg}.
- $X_\infty = \overline{X_\infty^{\mathrm{reg}}}$ *(metric completion).*

- Perelman:
  - W functional; Smooth convergence $\Rightarrow$ the limit is a shrinking KRS
  - Geometric estimates: non-collapsing, diameter bound, Ricci potential bound (written by Sesum-Tian)
- Q. Zhang: Uniform Sobolev, volume doubling property $\Rightarrow$ weak GH limit
- Tian-Z. Zhang: dim ≤ 3, HT conjecture true
- Chen-B. Wang, Bamler: HT conjecture is true for higher dimension.

Question of uniquenss

Chen-Sun-Wang showed that X_∞ can be constructed via an algebraic process: $X \rightsquigarrow Y \rightsquigarrow X_\infty$. The algebraic process a priori depend on the flow $\omega(t)$, or equivalently the initial metric $\omega(0)$. However they further conjectured:

Theorem (Han-L. , Chen-Sun-Wang's Conjecture)

$(X_\infty, \omega_\infty)$ *does not depend on the initial metric* $\omega(0)$. *In particular, the limit only depends on the algebraic structure of* X.

Remark:

- Given a normal $\mathbb{Q}$-Fano variety X_∞, the KR soliton on X_∞ is unique up to automorphisms (Tian-Zhu, Berndtsson, Berman-Boucksom-Eyssidieux-Guedj-Zeriahi, Berman-Witt-Nyström).
- There was an analogue conjecture in the local setting about uniqueness of algebraic metric tangent cones proposed by Donaldson-Sun and proved by **L.** -Wang-Xu. The proof of the above result is motivated by the local case.

Examples

- (weighted K-stable case) If X admits a KRS ω_∞ (for example if X is toric Fano), then $X = X_\infty$ (Tian-Zhu, Dervan-Székelyhidi).
- (K-semistable case) There exists a 3-dimensional Fano manifold which does not have any KRS metric, but degenerates to a Y with a KE metric. Then $X_\infty = Y$. (Tian-Mukai-Umemura, Sun-Y. Wang, Tian-Zhu)
- (weighted K-unstable case) There exists a smooth Fano compacification X of $SO(4, \mathbb{C})$ which does not admit KRS. X degenerates to a singular Fano variety Y that admits a KR soliton. $X_\infty = Y$ (Y. Li-Tian-Zhu, Y. Li-Z. Li) This example shows that the non-normalized Kähler-Ricci flow in general does not have type-I singularity since the blowup limit is indeed singular.

Steps of proofs for uniqueness

Achieved in several steps, also based on previous works of Chen-Sun-Wang and Dervan-Székelyhidi:

- Construction of a special filtration $\mathcal{F}_\infty$ using the flow. This induces a two-step degeneration: $X \rightsquigarrow Y = X_0^{\mathcal{F}_\infty} \rightsquigarrow X_\infty$ (Chen-Sun-Wang).

- $\mathcal{F}_\infty$ minimizes a functional $\mathbf{H}^{\mathrm{NA}}$ defined for all filtrations (Dervan-Székelyhidi, Hisamoto, Han-**L.**).

  The functional $\mathbf{H}^{\mathrm{NA}}$ arises from the work of Perelman and Tian-Zhang-Zhang-Zhu.

- Use algebraic method to prove that the minimizing special filtration of $\mathbf{H}^{\mathrm{NA}}$ is unique and hence the uniqueness of $\mathcal{F}_\infty$ and $\mathcal{X}_0^{\mathcal{F}_\infty}$ (Han-**L.** , Blum-Liu-Xu-Zhuang).

- Prove that the uniqueness of degeneration of $\mathcal{X}_0^{\mathcal{F}_\infty}$ to X_∞ (Han-**L.** , based on **L.** -Wang-Xu).

Torus action and filtrations

Assume that ξ generates a torus $\mathbb{T} \cong (\mathbb{C}^*)^r$ action on $(X, -K_X)$. $\mathbb{T}$ also acts on $-K_X = \wedge^n TX$ by pushforward. Then we have a weight decomposition on $R_m = H^0(X, -mK_X)$ for any $m \in \mathbb{Z}_{\geq 0}$:

$$R_m = \bigoplus_{\alpha \in \mathbb{Z}^r} R_{m,\alpha}, \quad R_{m,\alpha} = \{s \in R_m; t \circ s = t^\alpha s = \prod_i t_i^{\alpha_i} s\}$$

which induces a filtration $\mathcal{F} = \mathcal{F}_{\mathrm{wt}_\xi}$: for any $\lambda \in \mathbb{R}$:

$$\mathcal{F}^\lambda R_m = \mathcal{F}^\lambda_{\mathrm{wt}_\xi} R_m = \bigoplus \{R_{m,\alpha}; \langle \alpha, \xi \rangle \geq \lambda\}.$$

Definition (cf. Boucksom-H. Chen)

A filtration of $R(X) = \bigoplus_{m=0}^{+\infty} H^0(X, -mK_X)$: data $\{\mathcal{F}^\lambda R_m; \lambda \in \mathbb{R}, m \in \mathbb{N}\}$ satisfying:

1. $\mathcal{F}^{\lambda'} R_m \subseteq \mathcal{F}^\lambda R_m$ *if* $\lambda' \geq \lambda$.
2. $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} = \mathcal{F}^{\lambda+\lambda'} R_{m+m'}$.
3. $\cap_{\lambda' > \lambda} \mathcal{F}^{\lambda'} R_m = \mathcal{F}^\lambda R_m$.
4. $\exists$ e_+ *and* e_- *s.t.* $\mathcal{F}^{me_+} R_m = 0$ *and* $\mathcal{F}^{me_-} R_m = R_m$.

Intuition: $\omega(t) = \sqrt{-1}\partial\bar{\partial}\varphi_t \sim \sigma_\xi(t)^*\omega_\infty$. ξ generates a $(\mathbb{C}^*)^r$-action on the Fano variety $(X_\infty, -K_{X_\infty})$.

$$v_{\{\omega(t)\}}(s) := \lim_{t\to+\infty} \frac{\log|s|^2_{\varphi_t}}{t} \sim \lim_{t\to+\infty} \frac{\log|(\sigma_\xi)_* s|^2_{\varphi_0}}{t}.$$

Chen-Sun-Wang's Filtration $\mathcal{F}^\lambda R_m = \{s \in R_m; v_{\{\omega(t)\}}(s) \geq \lambda\}$.
Extended Rees algebra:

$$\mathcal{RF} = \bigoplus_{m,\lambda} t^{-\lambda}\mathcal{F}^\lambda R_m.$$

The central fibre $\mathcal{X}_0^{\mathcal{F}} = \mathrm{Proj}(\bigoplus_{m,\lambda} \mathcal{F}^\lambda R_m/\mathcal{F}^{>\lambda}R_m)$ admits a holomorphic vector field ξ_∞ that generates a $(\mathbb{C}^*)^r$-action.
We say that $\mathcal{F}$ is a special filtration if $\mathcal{RF}$ is finitely generated and the central fibre $\mathcal{X}_0^{\mathcal{F}}$ is normal Fano variety

Theorem (Chen-Sun-Wang)

$\mathcal{F}_\infty$ is a special filtration. Moreover, there is an equivariant degeneration of $(\mathcal{X}_0^{\mathcal{F}_\infty}, \xi_\infty)$ to (X_∞, ξ_∞).

Part II: Perelman's $\mathcal{W}$-functional

g a Riemannian metric, $f \in C^\infty(X, \mathbb{R})$. Perelman introduced:

$$\mathcal{W}(g, f) \quad = \quad \int_X (\Delta f + S + f)\, e^{-f} dV_g$$

$S = S(g) = g^{ij}(Ric)_{ij}$ is the scalar curvature. Set

$$\lambda(g) = \inf \left\{ \mathcal{W}(g, f); f \in C^\infty(X, \mathbb{R}), \int e^{-f} dV_g = 1 \right\}.$$

The first variation:

$$\delta\lambda(g) \cdot \delta g = -(2\pi)^{-n} \int_X \langle Ric(g) - g + \mathrm{Hess}(f), \delta g \rangle e^{-f} dV_g.$$

This implies that if $(X, g(t)) \to (X_\infty, g_\infty)$ smoothly (in Cheeger-Gromov topology), then g_∞ should be a shrinking KR soliton.

$\mathcal{W}$-functional and TZZZ's invariant

On a Fano manifold with volume $\mathrm{V} = \langle c_1(X)^n, [X] \rangle$, we have:

$$Ric(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h, \quad S(\omega) - n = \Delta h.$$

Any holomorphic vector field ξ has a (canonically normalized) Hamiltonian function $\theta = \theta_\xi$:

$$\iota_\xi \omega = \sqrt{-1}\bar{\partial}\theta_\xi, \quad \theta := \theta_\xi = \frac{\mathscr{L}_\xi(e^h\omega^n)}{e^h\omega^n} \Rightarrow \Delta\theta + \theta^i h_i + \theta = 0.$$

Set $f = \theta + \mathbf{H}_X(\xi)$ with $\mathbf{H}_X(\xi) := \log\left(\int_X e^{-\theta_\xi}\omega^n\right)$ s.t. $\int_X e^{-f}\omega^n = 1$.

$$\mathcal{W}(g, f) = n\mathrm{V} + \int_X (\Delta\theta + \theta^i h_i + \theta + \mathbf{H}_X(\xi))e^{-f}\omega^n + n\mathrm{V} = n\mathrm{V} + \mathbf{H}_X(\xi).$$

Theorem (Tian-Zhang-Zhang-Zhu)

For any holomorphic vector field $\xi \in \mathfrak{aut}(X)_0^{\mathrm{red}}$,

$$\sup\{\lambda(\omega); \omega \text{ is } \mathrm{Im}(\xi)\text{−invariant}\} \leq n\mathrm{V} + \mathbf{H}_X(\xi).$$

H-invariant for special filtrations

For any filtration $\mathcal{F}$, define a generalized Duistermaat-Heckman measure:

$$\mathrm{DH}(\mathcal{F}) = \lim_{m\to+\infty} \frac{n!}{m^n} \sum_i \delta_{\frac{\lambda_i^{(m)}}{m}}$$

where $\{\lambda_i^{(m)}\}$ are values where dimension of $\{\mathcal{F}^\lambda R_m; \lambda \in \mathbb{R}\}$ jumps. Set

$$\begin{aligned}
\tilde{\mathbf{S}}(\mathcal{F}) &= -\log\left(\frac{1}{\mathrm{V}}\int_{\mathbb{R}} e^{-\lambda}\mathrm{DH}(\mathcal{F})(\lambda)\right) \\
\mathbf{L}^{\mathrm{NA}}(\mathcal{F}) &= \lim_{m\to+\infty} \mathrm{lct}\left(X\times\mathbb{C}, \frac{1}{m}\sum_i \mathcal{I}_{\mathcal{F}^i R_m} t^{-i}; (t)\right) \\
\mathbf{H}^{\mathrm{NA}}(\mathcal{F}) &= \mathbf{L}^{\mathrm{NA}}(\mathcal{F}) - \tilde{\mathbf{S}}(\mathcal{F}).
\end{aligned}$$

Example: If ξ generates a torus action, and $\mathcal{F} = \mathcal{F}_{\mathrm{wt}_\xi}$ be the associated filtration. Then $\mathrm{DH}(\mathcal{F}) = (\theta_\xi)_*\omega^n$ and $\mathbf{H}^{\mathrm{NA}}(\mathcal{F}) = \mathbf{H}_X(\xi)$.

Minimization of $\mathbf{H}^{\mathrm{NA}}$-invariant

Theorem (Dervan-Székelyhidi, Hisamoto; Han-L.)

CSW's filtration minimizes the $\mathbf{H}^{\mathrm{NA}}$ functional among all filtrations. Moreover, there is an identity:

$$\sup\{\lambda(\omega); \omega \in c_1(X)\} - n\mathrm{V} = \mathbf{H}^{\mathrm{NA}}_X(\mathcal{F}_{\{\omega(t)\}}) = \mathbf{H}^{\mathrm{NA}}_{X_\infty}(\xi_\infty). \quad (1)$$

Algebraic proof of the first statement:

- (X_∞, ξ_∞) KRS implies that it is weighted K-polystable (an easier direction of the Yau-Tian-Donaldson conjecture).
- Because $(Y := \mathcal{X}_0^{\mathcal{F}_\infty}, \xi_\infty)$ equivariantly degenerates to the weighted K-polystable (X_∞, ξ_∞), one can show that (Y, ξ_∞) is weighted K-semistable, which implies $\mathbf{H}_Y(\hat{\mathcal{F}}) \geq \mathbf{H}^{\mathrm{NA}}_Y(\xi_\infty) = \mathbf{H}^{\mathrm{NA}}_{X_\infty}(\xi_\infty)$.
- For any filtration $\mathcal{F}$ on $R(X)$, consider the Initial Term Degeneration $\hat{\mathcal{F}}$ on $R(Y)$. One can then use lower-semicontinuity of $\mathbf{H}^{\mathrm{NA}}$ functional to show:

$$\mathbf{H}^{\mathrm{NA}}_X(\mathcal{F}) \geq \mathbf{H}^{\mathrm{NA}}_Y(\hat{\mathcal{F}}) \geq \mathbf{H}^{\mathrm{NA}}_Y(\xi_\infty) = \mathbf{H}^{\mathrm{NA}}_{X_\infty}(\xi_\infty).$$

Part III: Uniqueness of minimizers

Theorem (Han-L.)

A minimizing special filtration must be unique.

1. Assume that the special minimizing filtration $\mathcal{F}$ has the central fibre $(Y, \xi) := (\mathcal{X}_0^{\mathcal{F}}, \xi_\infty)$. For any other filtration $\mathcal{F}'$, use the technique of Initial Term Degeneration to get a filtration $\hat{\mathcal{F}}'$ on $R(Y)$ that is equivariant under the ξ action on Y. Moreover $\mathbf{H}_X^{\mathrm{NA}}(\mathcal{F}') \geq \mathbf{H}_Y^{\mathrm{NA}}(\hat{\mathcal{F}}') \geq \mathbf{H}_Y(\xi)$. If $\mathcal{F}'$ is a minimizing filtration, then $\hat{\mathcal{F}}'$ is also a minimizer of $\mathbf{H}_Y^{\mathrm{NA}}$.
2. $\mathcal{F}_{\mathrm{wt}_\xi}$ is also a minimizing filtration of $\mathbf{H}_Y^{\mathrm{NA}}$. Consider the an interpolation:

$$\hat{\mathcal{F}}_s'^{\lambda} R_{m,\alpha} := \hat{\mathcal{F}}'^{\frac{\lambda}{s} - \frac{1-s}{s}\langle \alpha, \xi\rangle} R_{m,\alpha}, \quad \hat{\mathcal{F}}_0' = \mathcal{F}_{\mathrm{wt}_\xi}, \quad \hat{\mathcal{F}}_1' = \hat{\mathcal{F}}'.$$

   Prove that $\mathbf{L}^{\mathrm{NA}}(\hat{\mathcal{F}}_s') \equiv s\mathbf{L}^{\mathrm{NA}}(\hat{\mathcal{F}}')$ and $f(s) = \tilde{\mathbf{S}}(\hat{\mathcal{F}}_s')$ is convex in $s \in [0,1]$. Moreover, it is strictly convex unless $d_2(\mathcal{F}, \mathcal{F}') = d_2(\mathcal{F}_{\mathrm{wt}_\xi}, \hat{\mathcal{F}}') = 0$ (distance considered by Boucksom-Jonsson).
3. Key observation: for special filtrations, $d_2(\mathcal{F}, \mathcal{F}') = 0$ if and only if $\mathcal{F} = \mathcal{F}'$. This depends on a valuative point of view.

Minimization problem for real valuations

(real) valuation: $\mathrm{Val}(X) \ni v : \mathbb{C}(X) = \text{field of rational functions} \to \mathbb{R} \cup \{+\infty\}$

- $v(f+g) \geq \min\{v(f), v(g)\}$ for any $f, g \in \mathbb{C}(X)$.
- $v(fg) = v(f) + v(g)$.
- $v(1) = 0, v(0) = +\infty$.

Any valuation v over $\mathbb{C}(X)$ defines a filtration: $\mathcal{F}_v R_m = \{s \in R_m; v(s) \geq \lambda\}$. Conversely any special filtration comes from a filtration (up to a translation). So $\mathcal{F}_{\omega(t)} = \mathcal{F}_{v_\infty}$ for some real valuation v_∞.
Define a functional:

$$\tilde{\beta}(v) := A(v) + \log\left(\frac{1}{\mathrm{V}}\int_X e^{-\lambda}\mathrm{DH}(\mathcal{F}_v)\right) \tag{2}$$

where $A(v)$ is the log discrepancy of v (Jonsson-Mustață).

Theorem (Han-L. , based on L. -Xu)

$$\inf_{v \in \mathrm{Val}(X)} \tilde{\beta}(v) = \inf_{\mathcal{F}} \mathbf{H}^{\mathrm{NA}}(\mathcal{F}) = \inf_{\text{special } \mathcal{F}} \mathbf{H}^{\mathrm{NA}}(\mathcal{F}).$$

Algebraic Hamilton-Tian conjecture

Theorem (Blum-Liu-Xu-Zhuang, L. -Wang-Xu, Han-L.)

For any normal Fano variety X, the following statements are true:

1. *There exists a unique minimizer v_∞ of $\tilde{\beta}$ over $\mathrm{Val}(X)$.*
2. *v_∞ is quasi-monomial and induces a special filtration with central fibre $\mathcal{X}_0$.*
3. *$\mathcal{X}_0$ is weighted K-semistable and there is a unique degeneration to a modifed K-polystable X_∞.*
4. *X_∞ admits a singular KR soliton.*

Remark:

1. $\mathcal{X}_0$ and X_∞ should be considered as certain optimal degenerations of X, which are non-linear analogues of Harder-Narasimhan filtration and Jordan-Hölder filtration in the stability theory of holomorphic vector bundles.
2. The proof is combination of (techniques developed in a series of) works by Blum-Liu-Xu, Liu-Xu-Zhuang, Han-**L.** and **L.** -Wang-Xu.

Thanks for your attention!