# Canonical Kähler Metrics and Stability of Algebraic Varieties

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# Kähler manifolds and Kähler metrics

$X$: $n$-dim compact complex manifold. $\{z_i\}$ holomorphic coordinates.

Kähler form (metric): $\omega = \sqrt{-1} \sum_{i,j=1}^{n} \omega_{i\bar{j}} dz^i \wedge d\bar{z}^j$, $(\omega_{i\bar{j}}) > 0$

satisfies $d\omega = 0 \quad \longrightarrow \quad$ Kähler class $[\omega] \in H^2(X, \mathbb{R})$.

Fact ($dd^c$-Lemma): Fix a reference Kähler form $\omega_0$, any other Kähler form in $[\omega]$ can be written as

$$\omega = \omega_\varphi := \omega_0 + dd^c\varphi, \qquad \omega_{i\bar{j}} = (\omega_0)_{i\bar{j}} + \varphi_{i\bar{j}}$$

where $\varphi : X \to \mathbb{R}$ is called a Kähler potential and

$$dd^c\varphi = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} \varphi_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

$L \to X$ a holomorphic line bundle

Kodaira: $L$ is ample if for $m \gg 1$, there exists an embedding:

$\iota : X \hookrightarrow \mathbb{P}^N = \mathbb{P}(H^0(X, L^{\otimes m})^*)$, $z \mapsto [s_0(z), \ldots, s_N(z)]$.

Restriction of Fubini-Study metric: on the locus of $s_j \neq 0$,

$$\omega = \frac{1}{m} dd^c \log \left( \frac{\sum_{i=0}^N |s_i|^2}{|s_j|^2} \right) > 0$$

represents the first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$.

# Holomorphic sectional curvature

Curvature tensor: $R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 \omega_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + \omega^{r\bar{q}} \frac{\partial \omega_{k\bar{q}}}{\partial z_i} \frac{\partial \omega_{r\bar{l}}}{\partial \bar{z}_j}$.

Uniformization for constant holomorphic sectional curvature $\Longleftrightarrow$

$$R_{i\bar{j}k\bar{l}} = \mu(\omega_{i\bar{j}}\omega_{k\bar{l}} + \omega_{i\bar{l}}\omega_{k\bar{j}})$$

| $X$ | $\omega$ | $\mu$ |
|---|---|---|
| $\mathbb{B}^n/\pi_1(X)$ | $-dd^c \log(1-\lvert z\rvert^2)$ | $-1$ |
| $\mathbb{C}^n/\pi_1(X)$ | $dd^c \lvert z\rvert^2$ | $0$ |
| $\mathbb{P}^n$ | $dd^c \log(1+\lvert z\rvert^2)$ | $1$ |

$\mathbb{B}^n = \{z \in \mathbb{C}^n; |z| < 1\}, \quad \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$.

Ricci: $Ric(\omega) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} R_{i\bar{j}} dz_i \wedge d\bar{z}_j \in c_1(X) := c_1(\det TX)$:

$$R_{i\bar{j}} := \sum_{i,j} \omega^{k\bar{l}} R_{i\bar{l}k\bar{j}} \quad = \quad -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det(\omega_{k\bar{l}}).$$

Scalar curvature:

$$S(\omega) \quad = \quad \sum_{i,j} \omega^{i\bar{j}} R_{i\bar{j}}.$$

The average scalar curvature $\underline{S}$ is a topological constant:

$$\underline{S} = \frac{1}{\mathrm{V}} \int_X n \cdot c_1(X) \wedge [\omega]^{n-1}, \quad \mathrm{V} = \int_X [\omega]^n.$$

# Constant scalar curvature Kähler (cscK) metrics

CscK equation is a 4-th order nonlinear PDE:

$$\underline{S} = S(\omega_\varphi) = -((\omega_0)_{i\bar{j}} + \varphi_{i\bar{j}})^{-1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det((\omega_0)_{k\bar{l}} + \varphi_{k\bar{l}}).$$

Obstructions to the existence of cscK in $[\omega]$:

• (Matsushima-Lichnerowicz) cscK $\Longrightarrow \mathrm{Aut}(X, [\omega])$ is reductive (the complexification of the compact isometry group).

• (Calabi-Futaki invariant) a functional for holomorphic vector fields that vanishes if there exists a cscK metric in $[\omega]$.

# (Uniform) Yau-Tian-Donaldson (YTD) conjecture

The main goal of this talk is to discuss the following conjecture:

**Conjecture ((Uniform) YTD conjecture)**

*Let $L$ be an ample line bundle over $X$. $(X, L)$ admits a cscK metric if and only if $(X, L)$ is uniformly K-stable.*

A comparison:

| holomorphic vector bundles | projective manifolds |
|---|---|
| Hitchin-Kobayashi | Yau-Tian-Donaldson |
| Hermitian-Einstein metrics | cscK metrics |
| slope stability | (strengthened) K-stability |
| coherent subsheaves | test configurations |
| Donaldson-Uhlenbeck-Yau | open in general |

# A recent progress: a model version of YTD

**Theorem (L. '20)**

*If $(X, L)$ is uniformly K-stable over models, then it admits a cscK metric, i.e. there exists cscK metric in $c_1(L)$.*

These known implications are conjectured to be equivalent:

uniformly K-stable over models $\Longrightarrow$ cscK $\Longrightarrow$
uniformly K-stable (over test configurations)

**Corollary (YTD for spherical varieties, observed by Odaka)**

*A polarized spherical manifold $(X, L)$ admits a cscK metric if and only if $(X, L)$ is uniformly K-stable.*

Spherical manifolds: compactification of certain homogeneous spaces of reductive Lie groups (including all toric manifolds).

# Kähler-Einstein case: $c_1(X) = \lambda c_1(L)$

CscK equation reduces to a complex Monge-Ampère equation

$$Ric(\omega) = \lambda\omega \Longleftrightarrow (\omega_0 + dd^c\varphi)^n = e^{-\lambda\varphi}\Omega_0.$$

1. $\lambda = -1$ existence (Aubin, Yau)
2. $\lambda = 0$ existence (Yau)
3. $\lambda = 1$ there are obstructions.

- In this Fano case, the YTD conjecture has been confirmed and generalized to even singular Fano varieties:

  Tian, Berman, Chen-Donaldson-Sun, Datar-Székelyhidi; Berman-Boucksom-Jonsson, **L.**-Tian-Wang, **L.**, Liu-Xu-Zhuang; K.Zhang; ...

- Weighted Kähler-Ricci soliton case: Tian-Zhu, Berman-Witt-Nyström, Datar-Székelyhidi, Han-**L.**, Lahdili.

# Space of Kähler metrics

Space of smooth Kähler potentials:

$$\mathcal{H} = \{\varphi \in C^\infty(X, \mathbb{R}); \omega_\varphi = \omega_0 + dd^c\varphi > 0\}.$$

Space of Kähler forms $\overline{\mathcal{H}} = \{\omega_\varphi; \varphi \in \mathcal{H}\}$. Volume: $\mathrm{V} = \int_X [\omega]^n$.

Monge-Ampère energy: $\mathbf{E} : \mathcal{H} \to \mathbb{R}$ satisfies

$$\delta\mathbf{E} \cdot \delta\varphi = \frac{1}{\mathrm{V}} \int_X \delta\varphi \cdot \omega_\varphi^n.$$

**J**-Norm of Kähler forms relative to $\omega_0$:

$$\begin{aligned} \mathbf{J}(\omega_\varphi) &= \frac{1}{\mathrm{V}} \int_X \varphi\omega_0^n - \mathbf{E}(\varphi) \\ &= \sum_{k=1}^{n} \frac{k}{n+1} \frac{1}{\mathrm{V}} \int_X \frac{\sqrt{-1}}{2\pi} \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega_0^{k-1} \wedge \omega_\varphi^{n-k} \geq 0. \end{aligned}$$

# Energy functionals

Mabuchi functional and Chen-Tian's decomposition:

$$\begin{aligned} \delta \mathbf{M}(\omega_\varphi) \cdot \delta\varphi &= -\frac{1}{\mathrm{V}} \int_X \delta\varphi \cdot (S(\omega_\varphi) - \underline{S})\omega_\varphi^n \\ \mathbf{M}(\omega_\varphi) &= \mathbf{H}(\omega_\varphi) + \mathbf{E}^{K_X}(\varphi) + \underline{S}\, \mathbf{E}(\varphi). \end{aligned}$$

Entropy and twisted MA-energy:

$$\begin{aligned} \mathbf{H}(\omega_\varphi) &= \frac{1}{\mathrm{V}} \int_X \log \frac{\omega_\varphi^n}{\omega_0^n} \omega_\varphi^n \\ \delta \mathbf{E}^{K_X} \cdot \delta\varphi &= -\frac{1}{\mathrm{V}} \int_X \delta\varphi \cdot n \cdot Ric(\omega_0) \wedge \omega_\varphi^{n-1}. \end{aligned}$$

$\omega_0$-plurisubharmonic ($\omega_0$-psh) potentials:
$\mathcal{P}(\omega_0) = \{\varphi \in L^1(X); \varphi$ is u.s.c. and $\omega_0 + dd^c\varphi \geq 0\}$.

Finite energy potentials:

$$\mathcal{E}^1 = \{\varphi \in \mathcal{P}(\omega_0); \inf\{\mathbf{E}(\tilde{\varphi}); \tilde{\varphi} \geq \varphi, \tilde{\varphi} \in \mathcal{H}\} > -\infty\}.$$

(Cegrell, Guedj-Zeriahi, Berman-Boucksom-Eyssidieux-Guedj-Zeriahi)

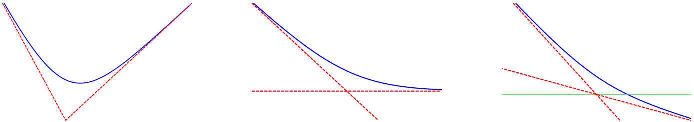
Geodesics in $\mathcal{E}^1$ (Mabuchi, Semmes, Donaldson, Chen, Darvas, ...)
The geodesic connecting $\varphi_0, \varphi_1 \in \mathcal{E}^1$ via an envelope:

$$\Phi = \sup\left\{\tilde{\Phi} \in \mathcal{P}(X \times [0,1] \times S^1, p_1^*\omega_0); \tilde{\Phi}(\cdot, i) \leq \varphi_i, i = 1, 2\right\}^*.$$

If $\Phi = \{\Phi_s(\cdot)\}_{s\in[0,1]}$ is smooth, then it satisfies the homogeneous complex Monge-Ampère equation $(p_1^*\omega_0 + dd^c\Phi)^{n+1} = 0$.

# Variational point of view

Facts: CscK metrics are minimizers of **M**-functional.



(a) Coercive (b) Bounded from below (c) Unbounded from below

Important results obtained by using pluripotential theory:

- All previous functionals can be defined on $\mathcal{E}^1$.
- **M** is convex along geodesics (Berman-Berndtsson, Chen-Tian).

# Variational criterion

maximal torus $\mathbb{T} \cong (\mathbb{C}^*)^r = ((S^1)^r)^{\mathbb{C}} \subset \mathrm{Aut}(X, [\omega])_0$ (reductive).

We now have an important existence criterion:

**Theorem (Chen-Cheng; Berman-Darvas-Lu ($\mathbb{T}$-version: Hisamoto, L.))**

*There exists a cscK metric in $(X, [\omega])$ if and only if* **M** *is ($\mathbb{T}$)-coercive, meaning that: there exist constants $\gamma, C > 0$ such that for any $\varphi \in \mathcal{H}^{(S^1)^r}$*

$$\mathbf{M}(\omega_\varphi) \geq \gamma \cdot \inf_{\sigma \in \mathbb{T}} \mathbf{J}(\sigma^* \omega_\varphi) - C.$$

In the Kähler-Einstein Fano case, the inequality is equivalent to a Moser-Trudinger type inequality and such type of results are due to Tian, Tian-Zhu, Phong-Song-Sturm-Weinkove, Darvas-Rubinstein.

# Criterion via geodesic rays

Geodesic ray $\Phi = \{\Phi_s\}_{s\in\mathbb{R}_{\geq 0}} \subset \mathcal{E}^1$:

$\Phi|_{[s_1,s_2]}$ are geodesic segments for all $s_1, s_2 \in \mathbb{R}_{\geq 0}$.

For the functional $\mathbf{F} \in \{\mathbf{M}, \mathbf{J}_{\mathbb{T}}(\cdot) = \inf_{\sigma\in\mathbb{T}} \mathbf{J}(\sigma^*\cdot)\}$, set

$$\mathbf{F}'^{\infty}(\Phi) = \lim_{s\to+\infty} \frac{\mathbf{F}(\Phi_s)}{s}.$$

Fact: The limits exist (based on convexity properties).

**Theorem (Chen-Cheng; Darvas-Rubinstein, Berman-Boucksom-Jonsson ($\mathbb{T}$-version: Hisamoto, L.))**

$\mathbf{M}$ *is coercive if and only if* $\mathbf{M}'^{\infty}(\Phi) > 0$ *for any geodesic ray* $\Phi$ *satisfying* $\mathbf{J}_{\mathbb{T}}'^{\infty}(\Phi) = 1$ *(non-trivial).*

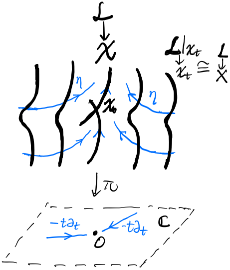
Existence of cscK metrics $\iff$ geodesic stability (Donaldson)

# Test configurations (Tian, Donaldson)

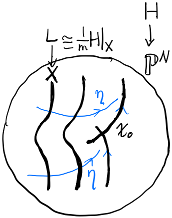
A test configuration (TC) $(\mathcal{X}, \mathcal{L}; \eta)$ for $(X, L)$ is the following data:

- $\pi : \mathcal{X} \to \mathbb{C}$: a normal family of varieties with $\mathcal{X}_t \cong X$ for $t \neq 0$;
- $\mathcal{L} \to \mathcal{X}$: a $\pi$-ample $\mathbb{Q}$-line bundle with $\mathcal{L}|_{\mathcal{X}_t} \cong L$ for $t \neq 0$;
- A $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ generated by a holomorphic vector field $\eta$ such

that $\pi$ is $\mathbb{C}^*$-equivariant and $\pi_* \eta = -t\partial_t$.




Intrinsic view




Extrinsic view

Trivial test configuration: $(X \times \mathbb{C}, p_1^* L; -t\partial_t)$.

Product test configuration: $(X \times \mathbb{C}, p_1^* L; \eta = v - t\partial_t)$.

# Non-Archimedean pluripotential theory I

$(X^{\mathrm{NA}}, L^{\mathrm{NA}})$ Berkovich analytification of $(X, L)$ with respect to the trivial norm on $\mathbb{C}$.

$\mathrm{Val}_X$ the space of real valuation is a dense subset $\subset X^{\mathrm{NA}}$.

- (S. Zhang, Boucksom-Favre-Jonsson) Correspondence:

$$\text{test configurations} \\ \updownarrow \\ \text{smooth non-Archimedean psh metrics on } L^{\mathrm{NA}}.$$

$(\mathcal{X}, \mathcal{L}) \longrightarrow$ non-Archimedean potential function on $X^{\mathrm{NA}}$:

$$\phi_{(\mathcal{X},\mathcal{L})}(v) = G(v)(\mathcal{L} - p_1^* L), \quad \forall v \in \mathrm{Val}_X.$$

$G(v) \in \mathrm{Val}_{X\times\mathbb{C}}$ : Gauss extension $G(v)(t) = 1$ and $G(v)|_{\mathbb{C}(X)} = v$.

# Algebraic (Non-Archimedean) functionals of TCs

Algebraic invariants defined using intersection products on the canonical compactification $\bar{\mathcal{L}} \to \bar{\mathcal{X}} = \mathcal{X} \cup_{X \times \mathbb{C}^*} (X \times (\mathbb{P}^1 \setminus \{0\}))$:

$$\begin{aligned}
\mathbf{E}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{\mathrm{V}} \frac{\bar{\mathcal{L}}^{\cdot n+1}}{n+1}; \\
\mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{\mathrm{V}} \bar{\mathcal{L}} \cdot L_{\mathbb{P}^1}^{\cdot n} - \frac{1}{\mathrm{V}} \frac{\bar{\mathcal{L}}^{\cdot n+1}}{n+1} \geq 0; \\
\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{\mathrm{V}} K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^{\cdot n} + \frac{\underline{S}}{n+1} \bar{\mathcal{L}}^{\cdot n+1}.
\end{aligned}$$

Example (Tian): $\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ recovers Calabi-Futaki invariant for product test configurations

# Uniform K-stability

We use a strengthened version of K-stability (Tian, Donaldson; Székelyhidi, Dervan, Boucksom-Hisamoto-Jonsson, Hisamoto).

**Definition**

*$(X, L)$ is ($\mathbb{T}$)-uniformly-K-stable (over test configurations) if there exists $\gamma > 0$ such that for any $\mathbb{T}$-equivariant TC $(\mathcal{X}, \mathcal{L})$,*

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L})$$

*where (with $\mathfrak{t}$ being the Lie algebra of $\mathbb{T}$)*

$$\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}, \mathcal{L}) = \inf_{\xi \in \mathfrak{t}_{\mathbb{Q}}} \mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \eta + \xi).$$

**Proposition (Berman-Darvas-Lu ($\mathbb{T}$-version: Hisamoto, L.))**

*Assume that $(X, L)$ admits a cscK metric. Then $(X, L)$ is uniformly K-stable.*

# K-stability of Fano varieties

$X$ a $\mathbb{Q}$-Fano variety if $-K_X$ is an ample $\mathbb{Q}$-line bundle and $X$ has mild (klt) singularities.

K-polystable (resp. stable): $\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq 0$ (resp. $> 0$) and $= 0$ only if $(\mathcal{X}, \mathcal{L})$ is a product (resp. trivial) test configuration.

**Theorem (Liu-Xu-Zhuang)**

*For any (possibly singular) $\mathbb{Q}$-Fano variety $X$, K-polystability is equivalent to the uniform K-stability.*

**Theorem (L.-Tian-Wang, L.)**

*For any $\mathbb{Q}$-Fano variety $X$, the existence of KE metric is equivalent to uniform K-stability.*

$\Longrightarrow$ strong version of YTD conjecture: KE $\Longleftrightarrow$ K-polystability

# Fano case: special test configurations

Definition(Tian): $(\mathcal{X}, \mathcal{L})$ is special if $\mathcal{X}_0$ is a $\mathbb{Q}$-Fano variety.

**Theorem (L.-Xu '14, Tian's conjecture)**

*To test the K-(poly)stability of $X$, it is enough to test on special test configurations.*

- Minimal Model Program: $\mathcal{X} \dashrightarrow \mathcal{X}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{X}^s$.
- Change of polarization: $\mathcal{L}_\lambda = \frac{\mathcal{L}+\lambda K_{\mathcal{X}}}{(1-\lambda)}, \quad \dot{\mathcal{L}} = \frac{\mathcal{L}+K_{\mathcal{X}}}{(1-\lambda)^2}$
- Monotonicity formula:

$$\frac{d}{dt}\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}_\lambda) = n(1-\lambda)^{-2}\mathcal{L}^{\cdot n-1} \cdot (K_{\mathcal{X}} + \mathcal{L})^2 \leq 0$$

by Zariski's lemma.

# Fano case: Valuative criterion

- (Boucksom-Hisamoto-Jonsson) each irreducible components of central fibre defines a valuation $v_i = \mathrm{ord}_{\mathcal{X}_{0,i}}\big|_{\mathbb{C}(X)} = q \cdot \mathrm{ord}_E$.

- (**L.**) For any special test configuration $\mathcal{X}$,

$$\mathbf{M}^{\mathrm{NA}} =_{const.} A_X(E) - \frac{1}{\mathrm{V}} \int_0^{+\infty} \mathrm{vol}(\mu^*(-K_X) - \lambda E) d\lambda =: \beta(E).$$

$\longrightarrow$ valuative criterion: $X$ is K-stable if and only if $\beta(E) > 0$ for any prime divisor $E$ over $X$ (**L.**, Fujita, Blum-Xu)

- Blum-Liu-Xu and Liu-Xu-Zhuang:

special test configurations $\rightarrow$ special valuations $\rightarrow$ log canonical places of complements (a concept invented by Shokurov)
$\longrightarrow$ application of deep boundedness results (Birkar, Hacon -McKernan -Xu) to study K-stability.

# K-stability of Fano varieties: flourishing strong/deep results

1. Effective ways to test K-stability, identification of K-polystable Fano threefolds (Abban-Zhuang, Cheltsov, Fujita, Araujo et al., ...).

2. Algebraic construction of moduli space of K-stable log-Fano varieties (Blum-Xu et al., **L.**-Wang-Xu, Liu-Xu-Zhuang, ...) Explicit examples and crossing phenomenon (Odaka-Spotti-Sun, Ascher-DeVleming-Liu, ...)

3. Weighted version of K-stability; Optimal degeneration of Fano varieties (Berman-Witt-Nyström, Tian-Zhang-Zhang-Zhu, Chen-Sun-Wang, Dervan-Székelyhidi, Hisamoto, Han-**L.**, Blum-Liu-Xu-Zhuang)

# Local stability theory for klt singularity $(X, o)$

Martelli-Sparks-Yau's work on Sasaki-Einstein metrics motivates a functional on $\mathrm{Val}_{X,x}$ the space of real valuations with centers $x$.

$$\begin{aligned} \widehat{\mathrm{vol}} : \mathrm{Val}_{X,x} &\longrightarrow \mathbb{R}_{>0} \cup \{+\infty\}, \qquad (\mathbf{L.}'15) \\ v &\mapsto A_X(v)^n \cdot \mathrm{vol}(v). \end{aligned}$$

$A_X(v)$: log disrepancy. For any prime divisor $E \subset \tilde{X} \to X$,

$$A_X(\mathrm{ord}_E) := \mathrm{ord}_E(K_{\tilde{X}/X}) + 1 > 0 \quad \text{(klt condition)}.$$

$$\mathrm{vol}(v) = \lim_{m \to +\infty} \frac{\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/\{f \in \mathcal{O}_{X,x}; v(f) \geq m\})}{m^n/n!}$$

Works of **L.**, Xu, Blum, Liu, Zhuang:

**Theorem**

*For any klt singularity, there exists a unique valuation $v_* \in \mathrm{Val}_{X,x}$ such that $\widehat{\mathrm{vol}}(v_*) = \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v) > 0$ and induces a degeneration of $(X, x)$ to a K-semistable affine Fano cone.*

# Analytic vs. algebraic invariants of $(\mathcal{X}, \mathcal{L})$

Because $\mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_0} \cong p_1^* L|_{X\times\mathbb{C}^*}$, any plurisubharmonic Hermitian metric on $\mathcal{L}$ gives a ray $\tilde{\Phi} = \left\{\tilde{\Phi}_s; s = -\log|t|^2\right\} \subset \mathcal{P}(\omega_0)$.

**Theorem (Phong-Sturm)**

*There exists a unique geodesic ray $\Phi = \Phi_{(\mathcal{X},\mathcal{L})}$ that corresponds to a bounded psh Hermitian metric on $\mathcal{L}$.*

Analytic slopes = Non-Archimedean invariants:

**Theorem**

*For any $\mathbf{F} \in \{\mathbf{E}, \mathbf{J}, \mathbf{E}^{K_X}, \mathbf{J}_{\mathbb{T}}, \mathbf{H}\}$, we have the identity:*

$$\mathbf{F}'^{\infty}(\Phi) = \mathbf{F}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}). \tag{1}$$

Many works: Tian, Phong-Ross-Sturm, Boucksom-Hisamoto -Jonsson, Hisamoto, **L.**, Xia.

# Approximation approach to (uniform) YTD conjecture

How about a general geodesic ray $\Phi$?

**Q1:** Is $\Phi$ approximable by geodesic rays associated to TC's?

**Q2:** Is $\mathbf{F}'^{\infty}(\Phi)$ approximable by $\mathbf{F}^{\mathrm{NA}}$-invariants of TC's?

The YTD conjecture is reduced to

Conjecture: Given a destabilizing geodesic ray $\Phi$, there exist test configurations $(\mathcal{X}_m, \mathcal{L}_m)$ s.t. as $m \to +\infty$,

1. $\Phi_m = \Phi_{(\mathcal{X}_m, \mathcal{L}_m)} \to \Phi$.
2. $\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}_m, \mathcal{L}_m) \to \mathbf{J}'^{\infty}_{\mathbb{T}}(\Phi), \quad \mathbf{M}^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) \to \mathbf{M}'^{\infty}(\Phi)$.

# Destabilizing geodesic rays are algebraically approximable

Darvas, Berman-Boucksom-Jonsson: geodesic rays in $\mathcal{E}^1$ in general not algebraically approximable. Surprisingly, non of them is destabilizing:

**Theorem (L. '20)**

*If a geodesic ray $\Phi$ satisfies $\mathbf{M}'^{\infty}(\Phi) < +\infty$, then there exist TC's $(\mathcal{X}_m, \mathcal{L}_m)$ whose associated geodesic rays $\Phi_m$ converge to $\Phi$. Moreover, for $\mathbf{F} \in \{\mathbf{E}, \mathbf{E}^{K_X}, \mathbf{J}_{\mathbb{T}}\}$, $\mathbf{F}^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) \to \mathbf{F}'^{\infty}(\Phi)$.*

These test configurations are constructed by blowing up multiplier ideals (Demailly-Ein-Lazarsfeld, Berman-Boucksom-Jonsson).

Recall $\mathbf{M} = \mathbf{H} + \mathbf{E}^{K_X} + \underline{S}\mathbf{E}$. Remaining conjecture:

**Conjecture (Boucksom-Jonsson, regularization conjecture)**

$\mathbf{H}'^{\infty}(\Phi)$ *is algebraically approximable.*

# K-stability over models

In the definition of TC $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$, if we don't require $\mathcal{L}$ to be $\pi$-ample, then we call $(\mathcal{X}, \mathcal{L})$ a model of $(X, L)$.

WLOG, we can assume $\bar{\mathcal{L}}$ is big (=ample+effective), and define:

$$
\begin{aligned}
\mathbf{E}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{\mathrm{V}} \frac{\langle \bar{\mathcal{L}}^{\cdot n+1} \rangle}{n+1} = \frac{1}{\mathrm{V}} \frac{\mathrm{vol}(\bar{\mathcal{L}})}{n+1}; \\
\mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{\mathrm{V}} \langle \bar{\mathcal{L}} \rangle \cdot L_{\mathbb{P}^1}^{\cdot n} - \frac{1}{\mathrm{V}} \frac{\langle \bar{\mathcal{L}}^{\cdot n+1} \rangle}{n+1}; \\
\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{\mathrm{V}} K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \langle \bar{\mathcal{L}}^{\cdot n} \rangle + \frac{S}{n+1} \langle \bar{\mathcal{L}}^{\cdot n+1} \rangle.
\end{aligned}
$$

Invariants of big line bundles: positive intersection products, volume, restricted volume=derivative of volume functional
Works of T. Fujita, Tsuji, Boucksom-Favre-Jonsson, Ein-Lazarsfeld -Mustaţă-Nakamaye-Popa.

# Non-Archimedean pluripotential theory II

- A model $(\mathcal{X}, \mathcal{L})$ determines a sequence of test configurations $(\mathcal{X}_m, \mathcal{L}_m) = (\mathrm{Bl}_{\mathrm{Bs}|m\mathcal{L}|}\mathcal{X}, \mu_m^*\mathcal{L} - \frac{1}{m}E_m)$

- (Boucksom-Favre-Jonsson) any model defines a continuous non-Archimedean psh metric on $L^{\mathrm{NA}}$:

$$\phi_{(\mathcal{X},\mathcal{L})} = \lim_{m\to+\infty} \phi_{(\mathcal{X}_m,\mathcal{L}_m)}.$$

- (Boucksom-Favre-Jonsson, **L.**) $\phi_{(\mathcal{X},\mathcal{L})}$ satisfies the non -Archimedean Monge-Ampère equation:

$$\mathrm{MA}^{\mathrm{NA}}(\phi_{(\mathcal{X},\mathcal{L})}) = \sum_i \langle \mathcal{L}^{\cdot n} \rangle \cdot \mathcal{X}_{0,i} \;\; \delta_{v_i}.$$

(Generalization of Chambert-Loir's formula)

# Approximation by models

**Theorem (L. '20)**

*For any destabilizing geodesic ray $\Phi$, $\exists$ models $(\mathcal{X}_m, \mathcal{L}_m)$ s.t.*

$$\mathbf{J}^{\mathrm{NA}}_{\mathbb{T}}(\mathcal{X}_m, \mathcal{L}_m) \to \mathbf{J}'^{\infty}_{\mathbb{T}}(\Phi),$$
$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}_m, \mathcal{L}_m) \to \mathbf{M}^{\mathrm{NA}}(\Phi_{\mathrm{NA}}) \leq \mathbf{M}'^{\infty}(\Phi).$$

Bouckson-Favre-Jonsson and Berman-Boucksom-Jonsson:

| | | |
|---|---|---|
| test configurations | $\longleftrightarrow$ | smooth NA psh metrics |
| $\cap$ | | $\cap$ |
| models | $\longrightarrow$ | continuous NA psh |
| $\cap$ | | $\cap$ |
| approximable geodesic rays | $\longleftrightarrow$ | finite energy NA psh |

Geodesic stability $\Rightarrow$ K-stability over models $\Rightarrow$ K-stability over test configurations

# An algebro-geometric conjecture

Both the original uniform YTD conjecture and Boucksom-Jonsson's regularization conjecture are implied by:

**Conjecture (strengthened Fujita approximation theorem)**

*Given a big line bundle $\bar{\mathcal{L}} \to \bar{\mathcal{X}}$, $\exists$ birational morphisms $\mu_m : \bar{\mathcal{X}}_m \to \bar{\mathcal{X}}$ and decompositions $\mu_m^* \bar{\mathcal{L}} = \bar{\mathcal{L}}_m + E_m$ with $\bar{\mathcal{L}}_m$ ample and $E_m$ effective, s.t.*

1. $\bar{\mathcal{L}}_m^{\cdot n+1} \to \mathrm{vol}(\bar{\mathcal{L}})$ *(conclusion of Fujita's theorem);*
2. *The next Riemann-Roch coefficients also converge:*

$$\bar{\mathcal{L}}_m^{\cdot n} \cdot K_{\bar{\mathcal{X}}_m} \to \frac{1}{n+1} \frac{d}{dt}\bigg|_{t=0} \mathrm{vol}(\bar{\mathcal{L}} + tK_{\bar{\mathcal{X}}}). \qquad (2)$$

True if $\bar{\mathcal{L}}$ admits a birational Zariski decomposition, in particular if $(X, L)$ is spherical ($\bar{\mathcal{X}}$ is then a Mori dream space).

Thanks for your attention!