Some notes on Futaki invariant

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1 Analytic Definition of Futaki Invariant

Let X be an n dimensional normal variety. Assume it's Fano, i.e. its anticanonical line bundle K_X^{-1} is ample. If X is smooth, then for any Kähler form ω in $[c_1(x)]$, by $\partial\bar{\partial}$-lemma, we have a smooth function h_ω, such that

$$Ric(\omega) - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}h_\omega$$

We call $h_\omega := -\log\frac{\omega_h^n}{\eta_h}$ the Ricci potential of ω. Let v be a holomorphic vector field on X, i.e. v is of type (1,0) and $\bar{\partial}v = 0$. Then the Futaki invariant is defined to be

$$F_{c_1(X)}(v) = \int_X v(h_\omega)\omega^n \tag{1}$$

It's a holomorphic invariant, as a character on the Lie algebra of holomorphic vector field, and independent of the choice of the Kähler form in $c_1(X)$. See [Fu]. The necessary condition of existence of Kähler-Einstein metric on X is that the Futaki invariant vanishes.

In [DT], the Futaki invariant is generalized to the singular case. When X is possibly singular normal, first use $|kK_X^{-1}|$ to embed X into projective spaces, $\phi_k = \phi_{|kK_X^{-1}|} : X \hookrightarrow \mathbb{CP}^{N_k}$. h_{FS} is the Fubini-Study metric determined by an inner product on $H^0(X, kK_X^{-1})$. $h = (\phi_k^* h_{FS})^{1/k}$ is an Hermitian metric on K_X^{-1}. Note that on the smooth part of X, Hermitian metrics on K_X^{-1} one-to-one corresponds to volume forms. If $\{z_i\}$ is a local holomorphic coordinate, denote $dz_1 \wedge \cdots \wedge dz_n$ by dz, and $d\bar{z}_1 \wedge \cdots d\bar{z}_n$ by $d\bar{z}$, the correspondence is given by

$$h \mapsto \sqrt{-1}^n \frac{dz_1 \wedge \cdots dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n}{|dz_1 \wedge \cdots \wedge dz_n|^2_{h^{-1}}} = \sqrt{-1}^n \frac{dz \wedge d\bar{z}}{|dz|^2_{h^{-1}}} =: \eta_h$$

$|dz|^{-2}_{h^{-1}} = |\partial_{z_1} \wedge \cdots \wedge \partial_{z_n}|^2_h$ is the induced Hermitian metric on K_X by the metric dual. On the smooth part of X,

$$\omega_h := \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial\log h = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{\eta_h}{\sqrt{-1}^n dz \wedge d\bar{z}} =: -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\eta_h$$

is a Kähler form, its Ricci curvature is: $Ric(\omega_h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det\omega_h^n$.

$$Ric(\omega_h) - \omega_h = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{\omega_h^n}{\eta_h}$$

So the Ricci potential is $h_{\omega_h} = -\log\frac{\omega_h^n}{\eta_h}$.

$$\begin{aligned} -\int_{X_{sm}} v(\log\frac{\omega_h^n}{\eta_h})\omega_h^n &= -\int_{X_{sm}} v(\frac{\omega_h^n}{\eta_h})\eta_h = -\int_{X_{sm}} (L_v\omega_h^n - \frac{L_v\eta_h}{\eta_h}\omega_h^n) \\ &= \int_{X_{sm}} div_{\eta_h}(v)\omega_h^n = \frac{1}{n+1}\int_{X_{sm}} (div_{\eta_h}(v) + \omega_h)^{n+1} \end{aligned}$$

In [DT], it's proved this is still a well defined holomorphic invariant. Note that in local holomorphic coordinate, $L_v d\bar{z}_i = 0$, so

$$\frac{L_v(\eta_h)}{\eta_h} = \frac{L_v(dz)}{dz} + v(\log|dz|^{-2}_{h^{-1}})$$

Note that the first term on the right is holomorphic , so

$$\bar{\partial} div_{\eta_h}(v) = -i_v \bar{\partial}\partial \log |dz|^{-2}_{h^{-1}} = -\frac{2\pi}{\sqrt{-1}}\, i_v\omega_h \tag{2}$$

2 Calculation by Log Resolution

Assume $\tilde{X}$ is an equivariant log resolution of singularity of X such that

$$K_{\tilde{X}}^{-1} = \pi^* K_X^{-1} - \sum_i a_i E_i$$

E_i are exceptional divisors with normal crossings. v lifts to be a smooth holomorphic vector field $\tilde{v}$ on $\tilde{X}$, which is tangential to each exceptional divisor E_i. Let S_i be the defining section of $[E_i]$, so $E_i = \{S_i = 0\}$. Let h_i be an Hermitian metric on $[E_i]$ and $R_{h_i} = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log h_i$ be the corresponding curvature form. By $\partial\bar{\partial}$ lemma (or Hodge theory), there is an Hermitian metric $\tilde{h}$ on $K_{\tilde{X}}^{-1}$ such that its curvature form $R_{\tilde{h}} = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log \tilde{h} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log \eta_{\tilde{h}}$ satisfies

$$R_{\tilde{h}} = \pi^*\omega_h - \sum_i a_i R_{h_i}$$

So

$$\pi^*(Ric(\omega_h) - \omega_h) \quad = \quad -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log \frac{\pi^*\omega_h^n}{\eta_{\tilde{h}}} + \frac{\sqrt{-1}}{2\pi}\sum_i a_i \partial\bar{\partial}\log |S_i|^2_{h_i}$$

$$\pi^* h_{\omega_h} = -\log \frac{\pi^*\omega_h^n}{\eta_{\tilde{h}}} + \sum_i a_i \log |S_i|^2_{h_i} + C$$

$$\int_{X_{sm}} v(h_{\omega_h})\omega_h^n = \int_{X\setminus \cup_i E_i} \pi^*(v(h_{\omega_h}))\pi^*\omega_h^n = \int_{X\setminus\cup_i E_i} -\tilde{v}(\frac{\pi^*\omega_h^n}{\eta_{\tilde{h}}})\eta_{\tilde{h}} + \sum_i a_i \tilde{v}(\log |S_i|^2_{h_i})\pi^*\omega_h^n$$

$\tilde{v}(\frac{\pi^*\omega_h^n}{\eta_{\tilde{h}}})$ is a smooth function on $\tilde{X}$.

Lemma 1. *$\theta_i = \tilde{v}(\log |S_i|^2_{h_i})$ extends to a smooth function on $\tilde{X}$ such that*

$$\frac{\sqrt{-1}}{2\pi}\bar{\partial}\theta_i = -i_{\tilde{v}} R_{h_i}$$

Proof. It's clearly true away from exceptional divisors. Let $p \in E_i$, in a neighborhood U of p, choose a local frame e_i of $[E_i]$, $S_i = f_i e_i$, and $E_i = \{f_i = 0\}$. We assume E_i is smooth at p, so we can take f_i to be a coordinate function, say z_1. Since $\tilde{v}$ is tangent to E_i, $\tilde{v}$ is of the form

$$\tilde{v}(z) = z_1 b_1(z)\partial_{z_1} + \sum_{i>1} c_i(z)\partial_{z_i}$$

$b_1(z)$, $c_i(z)$ are holomorphic functions near p. Now

$$\theta_i = \tilde{v}(\log |z_1|^2) + \tilde{v}(\log |e_i|^2_{h_i})$$

the second term is smooth near p, and

$$\tilde{v}(\log |z_1|^2) = \frac{\tilde{v}(z_1)}{z_1} = b_1(z)$$

is holomorphic near p. Also

$$\bar{\partial}\theta_i = \bar{\partial}(v(\log |e_i|^2_{h_i})) = -i_v \bar{\partial}\partial \log |e_i|^2_{h_i} = -\frac{2\pi}{\sqrt{-1}} i_v R_{h_i} \tag{3}$$

□

So the Futaki invariant can be written as

$$
\begin{aligned}
F_{c_1(X)}(v) &= \int_{\tilde{X}} (\frac{L_{\tilde{v}}\eta_{\tilde{h}}}{\eta_{\tilde{h}}} + \sum_i a_i\theta_i)(R_{\tilde{h}} + \sum_i a_i R_{h_i})^n \\
&= \frac{1}{n+1}\int_{\tilde{X}} (div_{\tilde{\eta}}(\tilde{v}) + \sum_i a_i\theta_i + R_{\tilde{h}} + \sum_i a_i R_{h_i})^{n+1}
\end{aligned}
$$

Now by (2) and (3), $(div_{\tilde{\eta}}(\tilde{v}) + \sum_i a_i\theta_i + R_{\tilde{h}} + \sum_i a_i R_{h_i})$ is an equivariantly closed form, so we can apply localization formula to this integral. See [BGV], [Ti2] for localization formula.

Remark 1. *Note that at any zero point p of $\tilde{v}$, the divergence $div_{\tilde{\eta}}(\tilde{v})$ is well defined independent of volume forms. Also by the proof of previous lemma, if $p \in E_i$, $\theta_i(p) = b_1(p)$ is the weight on the normal bundle of E_i at p, otherwise $\theta_i(p) = 0$. In any case, if $q = \pi(p) \in X$, then $div(\tilde{v})(p) + \sum_i a_i\theta_i(p)$ is the weight on $K_X^{-1}|_q$.*

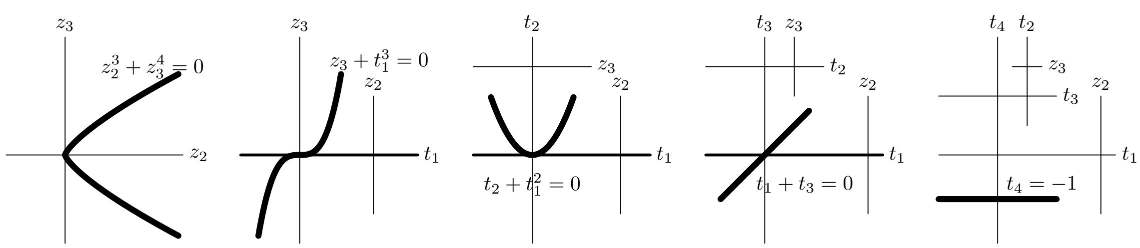
3 An example of calculation

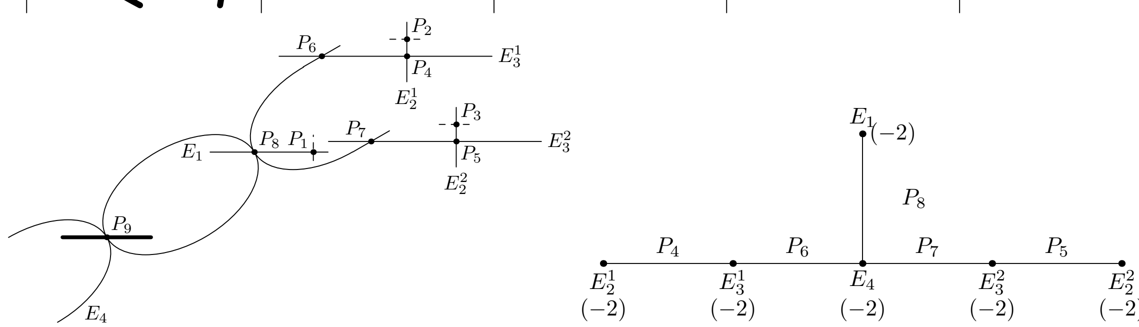
We calculate an example from [DT] using log resolution.

X is the hypersurface given by $F = Z_0Z_1^2 + Z_1Z_3^2 + Z_2^3$. v is given by $\lambda(t) = diag(1, e^{6t}, e^{4t}, e^{3t})$. The zero points of v are $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,0,1]$.

$[1,0,0,0]$ is an A-D-E singular point of type E_6. Locally, it's $\mathbb{C}^2/\Gamma$, Γ is the lifting to $SU(2)$ of the symmetric group of Tetrahedron in $SO(3)$. $|\Gamma| = 24$. After a (nonlinear) change of coordinate, we change it to the standard form $z_1^2 + z_2^3 + z_3^4$. The vector field is given by $v = 6z_1\partial_{z_1} + 4z_2\partial_{z_2} + 3z_3\partial_{z_3}$. By viewing the surface as a two-fold covering of $\mathbb{C}^2$, branched along a singular curve, we can equivariantly resolve the singularity by blowup and normalization (at the origin of each step). See [BPV].

1. $z_1^2 + z_2^3 + z_3^4 = 0$. $z_1 \mapsto e^{6t}z_1$, $z_2 \mapsto e^{4t}z_2$, $z_3 \mapsto e^{3t}z_3$.
2. $s_1^2 + z_3(z_3 + t_1^3) = 0$. $t_1 = \frac{z_2}{z_3} \mapsto e^t t_1$, $s_1 = \frac{z_1}{z_3} \mapsto e^{3t}s_1$.
3. $s_2^2 + t_2(t_2 + t_1^2) = 0$. $t_2 = \frac{z_3}{t_1} \mapsto e^{2t}t_2$, $s_2 = \frac{s_1}{t_1} \mapsto e^{2t}s_2$.
4. $s_3^2 + t_3(t_3 + t_1) = 0$. $t_3 = \frac{t_2}{t_1} \mapsto e^t t_3$, $s_3 = \frac{s_2}{t_1} \mapsto e^t s_3$.
5. $s_4^2 + t_4(t_4 + 1) = 0$. $t_4 = \frac{t_3}{t_1} \mapsto t_4$, $s_4 = \frac{s_3}{t_1} \mapsto s_4$.





The intersection diagram of Exceptional divisors is of type E_6. Assume

$$K_{\tilde{X}} = \pi^* K_X + \sum_i a_i E_i$$

Note that $\pi^* K_X \cdot E_i = 0$, then

$$K_{\tilde{X}} \cdot E_i = \sum_j a_j E_j \cdot E_i$$

By adjoint formula,

$$K_{\tilde{X}} \cdot E_i = K_{E_i} \cdot E_i - E_i^2 = 0$$

Because the intersection matrix $\{E_i \cdot E_j\}$ is negative definite, we have $a_i = 0$. So

$$K_{\tilde{X}} = \pi^* K_X$$

The zero points set of $\tilde{v}$ are: $\cup_{i=1}^5 \{P_i\} \cup E_4$.

1. equation near P_1 is: $u_1^2 + z_2(1 + t_1'^4 z_2) = 0$. $u_1 = \frac{z_1}{z_2} \mapsto e^{2t} u_1$, $t_1' = \frac{z_3}{z_2} \mapsto e^{-t} t_1'$.
2. equation near P_2, P_3 is: $u_2^2 + t_2'^3 z_3^2 + 1 = 0$. $t_2' = \frac{t_1}{z_3} \mapsto e^{-2t} t_2'$, $z_3 \mapsto e^{3t} z_3$.
3. equation near P_4, P_5 is: $u_3^2 + t_3'^2 t_2 + 1 = 0$. $t_3' = \frac{t_1}{t_2} \mapsto e^{-t} t_3'$, $t_2 \mapsto e^{2t} t_2$.
4. equation near E_4 (away from P_6, P_7) is: $s_4^2 + t_4(t_4 + 1) = 0$. (near P_6, P_7, the equation is $u_4^2 + t_4'^2 + 1 = 0$) $E_4 = \{t_1 = 0\}$. $t_1 \mapsto e^t t_1$.

So the contribution to the localization formula of Futaki invariant at point $[1, 0, 0, 0]$ is:

$$\frac{1}{-2} + 2\frac{1}{-6} + 2\frac{1}{-2} + \int_E \frac{1}{1 + c_1([E])} = \frac{1}{6}$$

the contributions from the other two fixed points are easily calculated, so the Futaki invariant is:

$$F_{c_1(X)}(v) = \frac{1}{3}\left(\frac{1}{6} + \frac{(-5)^3}{6} + \frac{(-2)^3}{-3}\right) = -6$$

Remark 2. *The contribution of the singular point can also be calculated using the localization formula for orbifolds given in [DT]. Note that the local uniformization is given by:*

$$\begin{array}{rcl} \pi_1 : \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2/\Gamma \subset \mathbb{C}^3 \\ (z_1, z_2) & \mapsto & [1, (z_1^4 + 2\sqrt{-3} z_1^2 z_2^2 + z_2^4)^3, 2(-3)^{\frac{3}{4}} z_1 z_2 (z_1^4 - z_2^4), -(z_1^8 + 14 z_1^4 z_2^4 + z_2^8)] \end{array}$$

So $\pi_1^* v = \frac{1}{2}(z_1 \partial_{z_1} + z_2 \partial_{z_2})$, *and*

$$\frac{1}{|\Gamma|} \frac{(div(\pi_1^* v))^{n+1}}{det(\nabla(\pi_1^* v)|_{T_z X})} = \frac{1}{24} \frac{1^3}{1/4} = \frac{1}{6}$$

4 Algebraic Definition

We can transform the expression of Futaki invariant (1) into another form:

$$F_{c_1(X)}(v) = -\int_X (S(\omega) - \omega) \theta_v \omega^n \tag{4}$$

where $S(\omega)$ is the scalar curvature of ω, and θ_v is the potential function of the vector field v satisfying

$$i_v \omega = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \theta_v$$

In this way, the Futaki invariant generalizes to any Kähler class. The vanishing of Futaki invariant is necessary for the existence of constant scalar Kähler metric in the fixed Kähler class.

Assume there is a $\mathbb{C}^*$ action on (X, L), there are induced actions on $H^0(X, L^k)$. Let w_k be the $k - th$ (Hilbert) weight of these actions. For k sufficiently large,

$$w_k = a_0 \frac{k^{n+1}}{n!} + a_1 \frac{k^n}{2n!} + O(k^{n-1}) \tag{5}$$

$$d_k = dimH^0(X, L^k) = b_1 \frac{k^n}{n!} + b_2 \frac{k^{n-1}}{2n!} + O(k^{n-2})$$

At least in the smooth case, one can show that (See [Do])

$$a_0 = \int_X \theta_v \omega^n, \ \ a_1 = \int_X S(\omega) \theta_v \omega^n \tag{6}$$

$$b_1 = \int_X \omega^n, \ \ b_2 = \int_X S(\omega) \omega^n$$

By this, Donaldson [Do] gives an algebro-geometric definition of Futaki invariant:

$$F_{c_1(L)}(v) = -\frac{a_1 b_1 - a_0 b_2}{b_1} \tag{7}$$

Remark 3. *Assume we can embed X into $\mathbb{P}(H^0(X, L)^*)$ using the complete linear system $|L|$ such that the $\mathbb{C}^*$ action is induced by a one parameter subgroup in $SL(d_1, \mathbb{C})$. Then we see that, at least in the smooth case, if we normalize θ_v, the (normalized) leading coefficient $((n+1)a_0)$ in the expansion (5) is the Chow weight of this $\mathbb{C}^*$ action.*

5 Futaki invariant of Complete Intersections

We will use the algebraic definition to calculate. Assume $X \in \mathbb{CP}^N$ is a complete intersection given by:

$$X = \cap_{\alpha=1}^r \{F_\alpha = 0\}$$

Assume $\deg F_\alpha = d_\alpha$, so

$$\deg X = \prod_\alpha d_\alpha$$

Let $R = \mathbb{C}[Z_0, \cdots, Z_N]$. X has homogeneous coordinate ring

$$R(X) = \mathbb{C}[Z_0, \cdots, Z_N]/(I(X)) = R/I(X)$$

$I(X)$ is the homogeneous ideal generated by homogeneous polynomial $\{F_\alpha\}$. It is well known that $R(X)$ has a minimal free resolution by Koszul complex:

$$0 \to R(-\sum_{\alpha=0}^r d_\alpha) \otimes (\mathbb{C} \cdot \prod_\alpha F_\alpha) \to \cdots \to \bigoplus_{\alpha<\beta}^r R(-d_\alpha - d_\beta) \cdot (\mathbb{C} \cdot (F_\alpha F_\beta)) \to \bigoplus_{\alpha=0}^r R(-d_\alpha) \otimes (\mathbb{C} \cdot F_\alpha) \to R \to R(X) \to 0$$

Let $\lambda(t) \in PSL(N+1, \mathbb{C})$ be a one-parameter subgroup generated by $A = diag(\lambda_0, \cdots, \lambda_N)$, and v be the corresponding holomorphic vector field. Assume that

$$\sum_{i=0}^N \lambda_i Z_i \frac{\partial}{\partial Z_i} F(Z) = \mu_\alpha F_\alpha$$

$(\mathbb{C}^*)^2$ acts on $S(X)$. Let $a_{k,l} = dimS(X)_{k,l}$ be the dimensions of weight spaces, then this action has character:

$$Ch(S(X)) = \sum_{(k,l) \in \mathbb{N} \times \mathbb{Z}} a_{k,l} t_1^k t_2^l = \frac{\prod_{\alpha=1}^r (1 - t_1^{d_\alpha} t_2^{\mu_\alpha})}{\prod_{i=0}^N (1 - t_1 t_2^{\lambda_i})} = f(t_1, t_2)$$

The $k - th$ Hilbert weight is (note it's a finite sum)

$$w_k = \sum_{l \in \mathbb{Z}} a_{k,l} \times l$$

and

$$\sum_{k\in\mathbb{N}} w_k t_1^k = \frac{\partial f}{\partial t_2}\Big|_{t_2=1} = -\frac{\sum_\alpha(\mu_\alpha t_1^{d_\alpha}\prod_{\beta\neq\alpha}(1-t_1^{d_\beta}))}{(1-t_1)^{N+1}} + (\sum_i \lambda_i)t_1\frac{\prod_{\alpha=1}^r(1-t_1^{d_\alpha})}{(1-t_1)^{N+2}}$$

$$= -\frac{\sum_\alpha(\mu_\alpha t_1^{d_\alpha}\prod_{\beta\neq\alpha}(1+\cdots+t_1^{d_\beta-1}))}{(1-t_1)^{N+2-r}} + \lambda t_1\frac{\prod_{\alpha=1}^r(1+\cdots+t_1^{d_\alpha-1})}{(1-t_1)^{N+2-r}} \tag{8}$$

Lemma 2. *Let*

$$f(t) = \frac{g(t)}{(1-t)^{n+1}} = \frac{\sum_{i=0}^r a_i t^i}{(1-t)^{n+1}} = \sum_{k=0}^{+\infty} b_k t^k$$

then

$$b_k = \frac{k^n}{n!}g(1) + \frac{k^{n-1}}{2(n-1)!}((n+1)g(1) - 2g'(1)) + O(k^{n-2})$$

Proof.

$$f(t) = (\sum_{i=0}^r a_i t^i)\cdot\sum_{j=0}^\infty \binom{n+j}{n}t^j$$

So when $k \gg 1$,

$$\begin{aligned} b_k &= \sum_{i=0}^r a_i\binom{n+k-i}{n} = \sum_{i=0}^r a_i\frac{(n+k-i)\cdots(k-i+1)}{n!} \\ &= \frac{k^n}{n!}\sum_{i=0}^r a_i + \frac{k^{n-1}}{2(n-1)!}\sum_{i=0}^r a_i(n+1-2i) + O(k^{n-2}) \\ &= \frac{k^n}{n!}g(1) + \frac{k^{n-1}}{2(n-1)!}((n+1)g(1) - 2g'(1)) + O(k^{n-2}) \end{aligned}$$

□

Let $g(t) = -\sum_\alpha(\mu_\alpha t_1^{d_\alpha}\prod_{\beta\neq\alpha}(1+\cdots+t_1^{d_\beta-1})) + \lambda t_1\prod_{\alpha=1}^r(1+\cdots+t_1^{d_\alpha-1})$, $n = N+1-r$, let $\tilde{\mu}_\alpha = \mu_\alpha - \frac{\lambda}{N+1}d_\alpha$, then $\tilde{\mu}$ is invariant when $\lambda(t)$ differs by a diagonal matrix. by the lemma, we can get

$$g(1) = -\sum_\alpha \mu_\alpha\prod_{\beta\neq\alpha}d_\beta + \lambda\prod_\alpha d_\alpha = -\prod_\alpha d_\alpha(\sum_\beta\frac{\mu_\beta}{d_\beta} - \lambda) = -\prod_\alpha d_\alpha\left(\sum_\beta\frac{\tilde{\mu}_\beta}{d_\beta} - \frac{\lambda}{N+1}(N+1-r)\right) \tag{9}$$

$$\begin{aligned} (N-r+2)g(1) - 2g'(1) &= -\prod_\alpha d_\alpha\left((N+1-\sum_\beta d_\beta)\sum_\gamma\frac{\mu_\gamma}{d_\gamma} - \sum_\beta\mu_\beta - \lambda(N-\sum_\beta d_\beta)\right) \\ &= -\prod_\alpha d_\alpha\left((N+1-\sum_\beta d_\beta)\sum_\gamma\frac{\tilde{\mu}_\gamma}{d_\gamma} - \sum_\beta\tilde{\mu}_\beta - \frac{\lambda}{N+1}(N-r)(N+1-\sum_\alpha d_\alpha)\right) \end{aligned}$$

$$\begin{aligned} w_k = & -\prod_\alpha d_\alpha\sum_\beta\frac{\tilde{\mu}_\beta}{d_\beta}\frac{k^{N+1-r}}{(N+1-r)!} - \prod_\alpha d_\alpha\left((N+1-\sum_\beta d_\beta)\sum_\gamma\frac{\tilde{\mu}_\gamma}{d_\gamma} - \sum_\beta\tilde{\mu}_\beta\right)\frac{k^{N-r}}{2(N-r)!} + O(k^{N-r-1}) \\ & + \frac{\lambda}{N+1}k\cdot dimH^0(X,\mathcal{O}(k)) \end{aligned} \tag{10}$$

By (7), we can get the Futaki invariant

$$F_{c_1(\mathcal{O}(1))}(v) = -\prod_\alpha d_\alpha\left(\sum_\beta\tilde{\mu}_\beta - \frac{N+1-\sum_\gamma d_\gamma}{N+1-r}\sum_\beta\frac{\tilde{\mu}_\beta}{d_\beta}\right)$$

Remark 4. *In hypersurface case, the above formula becomes*

$$F_{c_1(\mathcal{O}(1))}(v) = -\frac{(d-1)(N+1)}{N}(\mu - \frac{\lambda}{N+1}d)$$

Apply this to the example in section 3, where $d = 3$, $N = 3$, $\lambda = 6+3+4 = 13$, $\mu = 12$, $\mathcal{O}(1) = K_X^{-1}$, then we get the same result as before.

$$F_{c_1(X)}(v) = -\frac{2\cdot 4}{3}(12 - \frac{13}{4}\cdot 3) = -6$$

Remark 5. *We can calculate directly the leading coefficient of w_k in (10)using the Lelong-Poincáre equation. Also see [Lu].*

Lemma 3 (Poincáre-Lelong equation)**.** *Assume L is a holomorphic line bundle on X, s is a nonzero holomorphic section of L, D is the zero divisor of s, i.e. $\{s = 0\}$ counted with multiplicities. h is an Hermitian metric on L, $R_h = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log h$ is its curvature form. Then in the sense of distribution, we have the identity*

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|s|_h^2 = \int_D -R_h$$

i.e., for any smooth $(2n-2)$ form η on X, we have

$$\frac{\sqrt{-1}}{2\pi}\int_X (\log|s|_h^2)\partial\bar{\partial}\eta = \int_D \eta - \int_X R_h\wedge\eta$$

Let $X_0 = \mathbb{CP}^N$, $X_{a+1} = X_a \cap \{F_a = 0\}$, then $X_0 \supset X_1 \cdots \supset X_r = X$. $\theta_v = \frac{\sum_i \lambda_i|Z_i|^2}{\sum_i |Z_i|^2}$, then $i_v\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\theta_v$. On X_{a-1}, by the lemma, we have

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{|F_a|^2}{(\sum_i|Z_i|^2)^{d_a}}\bigg|_{X_{a-1}} = \int_{X_a} -d_a\cdot\omega_{FS}|_{X_{a-1}}$$

So

$$\int_{X_a}\theta_v\omega_{FS}^{N-a} = d_a\int_{X_{a-1}}\theta_v\omega_{FS}^{N-a+1} + \frac{\sqrt{-1}}{2\pi}\int_{X_{a-1}}\theta_v\partial\bar{\partial}\log\frac{|F_a|^2}{(\sum_i|Z_i|^2)^d}\wedge\omega_{FS}^{N-a}$$

Using integration by parts, the second integral on the right equals

$$\begin{aligned}
\frac{\sqrt{-1}}{2\pi}\int_{X_{a-1}}\bar{\partial}\theta_v \quad\wedge\quad \partial\log\frac{|F_a|^2}{(\sum_i|Z_i|^2)^d}\wedge\omega_{FS}^{N-a} &= \int_{X_{a-1}} i_v\omega_{FS}\wedge\partial\log\frac{|F_a|^2}{(\sum_i|Z_i|^2)^{d_a}}\wedge\omega_{FS}^{N-a}\\
&= -\frac{1}{N-a+1}\int_{X_{a-1}} v(\log\frac{|F_a|^2}{(\sum_i|Z_i|^2)^{d_a}})\omega_{FS}^{N-a+1}\\
&= -\frac{1}{N-a+1}\int_{X_{a-1}}(\mu_a - d_a\frac{\sum_i\lambda_i|Z_i|^2}{\sum_i|Z_i|^2})\omega_{FS}^{N-a+1}\\
&= -\frac{1}{N-a+1}\mu_a\deg(X_{a-1}) + d_a\frac{1}{N-a+1}\int_{X_{a-1}}\theta_v\omega_{FS}^{N-a+1}
\end{aligned}$$

So

$$(N-a+1)\int_{X_a}\theta_v\omega_{FS}^{N-a} = -\mu_a\deg(X_{a-1}) + d_a(N-a+2)\int_{X_{a-1}}\theta_v\omega_{FS}^{N-a+1}$$

While

$$(N+1)\int_{X_0}\theta_v\omega_{FS}^N = (N+1)\int_{\mathbb{CP}^N}\frac{\sum_i\lambda_i|Z_i|^2}{\sum_i|Z_i|^2}\omega_{FS}^N = \sum_i\lambda_i = \lambda$$

By induction, we get

$$(N-r+1)\int_{X_r}\theta_v\omega_{FS}^{N-r} = -\prod_\alpha d_\alpha\sum_\beta\frac{\mu_\beta}{d_\beta} + \lambda\prod_\alpha d_\alpha = \prod_\alpha d_\alpha\left(-\sum_\beta\frac{\tilde{\mu}_\beta}{d_\beta} + (N+1-r)\frac{\lambda}{N+1}\right)$$

This is the same as $g(1)$, (9).

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