On the volume of Kähler-Einstein Fano varieties

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Volume comparison in Riemannian geometry

The bigger the curvature, the smaller the volume.

- Bishop: $Ric(g) \geq (n-1)g \Longrightarrow \mathrm{Vol}(X,g) \leq \mathrm{Vol}(\mathbb{S}^m, g_0)$.
- Bishop-Gromov: Relative comparison:

$$r \mapsto \frac{\mathrm{Vol}(B(x,r))}{\mathrm{Vol}(B(p_0,r))} \quad \left(\geq \frac{\mathrm{Vol}(X,g)}{\mathrm{Vol}(S^m,g_0)}\right)$$

is non-increasing on $(0,\infty)$.

- Volume comparison of Heintz-Karcher, Gray: bound the volume of tube around a submanifold by the volume of corresponding volume of model space.

Kähler manifold

- X: a complex manifold $\{U, (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n\}$.

Kähler metric: Hermitian $g = \sum_{i,j} g_{i\bar{j}}(z) dz_i \otimes d\bar{z}_j$ satisfying:

$$\nabla_{\partial_{z_i}} \partial_{z_j} = \sum_k \Gamma_{ij}^k \partial_{z_k} + 0 \cdot \partial_{\bar{z}_k}.$$

Equivalent characterization: The following $(1,1)$-form is closed

$$\text{Kähler form} \quad \omega = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

- Ricci-form of a Kähler metric: $Ric(\omega) = \sqrt{-1} \sum_{i,j} R_{i\bar{j}} dz_i \wedge d\bar{z}_j$:

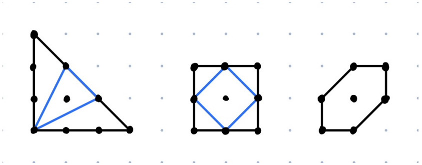
$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g_{k\bar{l}}).$$

The Ricci form $Ric(\omega)$ is a closed $(1,1)$-form that represents the 1st Chern class of $-K_X := \wedge^n T_{hol} X$.

Fano manifolds

Complex manifold X is Fano: $\exists$Kähler metric ω with $Ric(\omega) > 0$.

1. $\dim_{\mathbb{C}} = 1$: $\mathbb{P}^1 = \mathbb{S}^2$.
2. $\dim_{\mathbb{C}} = 2$: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2 \sharp k \overline{\mathbb{P}^2}$, $1 \leq k \leq 8$.
3. Hypersurface $\{F_d(Z_0, \dots, Z_n) = 0\} \subset \mathbb{P}^n$ of degree $d \leq n$
4. Toric Fano $\Longleftrightarrow$ reflexive lattice polytopes.



$\mathbb{P}^2(/\mathbb{Z}_3)$ $\mathbb{P}^1 \times \mathbb{P}^1(/\mathbb{Z}_2)$ $\mathbb{P}^2 \sharp 3 \overline{\mathbb{P}^2}$

Volume of Fano manifolds

Positive holomorphic line bundle $L = -K_X > 0$, $\omega \in c_1(L)$.

$$\mathrm{Vol}(X) = \int_X \omega^n = \lim_{m \to +\infty} \frac{h^0(X, L^{\otimes m})}{m^n/n!} = \mathrm{Vol}(-K_X).$$

Ex. $\mathrm{Vol}(\mathbb{P}^n) = (n+1)^n$
$> \ \mathrm{Vol}(\mathbb{P}^r \times \mathbb{P}^{n-r}) = \binom{n}{r}(r+1)^r(n-r+1)^{n-r}$.

Ex. $\mathrm{Vol}(Q^n) = 2n^n = \mathrm{Vol}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$ where

$$Q^n = \{Z_0^2 + Z_1^2 + \cdots + Z_{n+1}^2 = 0\} \subset \mathbb{P}^{n+1}.$$

Ex. For $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(n-1))$,

$$\mathrm{Vol}(X) = \frac{(2n-1)^n - 1}{n-1} \sim \frac{2^n e^{-3/2}}{n}(n+1)^n.$$

Sharp volume bounds of KE manifolds

Kähler-Einstein (KE) metric: $Ric(\omega) = \omega$.

Theorem (Fujita '15)

If X admits KE , then $\mathrm{Vol}(X) \leq (n+1)^n$ with equality iff $X \cong \mathbb{P}^n$.

Theorem (L.-Miao '25)

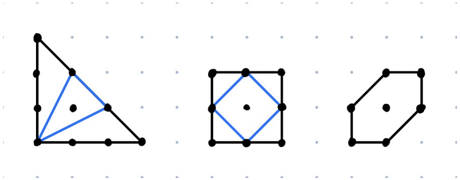
If X admits KE and $X \ncong \mathbb{P}^n$, then $\mathrm{Vol}(X) \leq 2n^n$ with equality true iff X is quadric hypersurface $Q^n \subset \mathbb{P}^{n+1}$ or the product $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.

- In general not true if X is not KE .
- There are versions for Kähler metrics with positive Ricci lower bound.
- Sharp volume comparison theorem for Kähler manifolds and currently no differential geometric proofs.

Application 1: A convex geometry result

Corollary (L.-Miao)

P: Reflexive lattice polytope with barycenter 0. If P is not isomorphic to the simplex $(n+1)\Delta_n$, then $\mathrm{Vol}(P) \le \frac{2n^n}{n!}$ and the equality holds iff P is isomorphic to $[0,2] \times (n\Delta_{n-1})$.



$P^\vee = \{y \in \mathbb{R}^n; \langle x, y\rangle \ge -1, \forall x \in P\}$ is again a lattice polytope.

Application: height of arithmetic Fano varieties

$\mathcal{L} = -K_{\mathcal{X}} \to \mathcal{X}$ an arithmetic Fano manifold overe $\mathbb{Z}$. $e^{-\phi}$ a continuous plurisubhmarmonic metric on $\mathcal{L} \to X$.

$$h_\phi(\mathcal{L}) = \lim_{k\to+\infty} \log \mathrm{Vol}\left\{ s \in H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R} : \sup_X \|s\|_\phi \le 1 \right\}.$$

Conjecture (Andreasson-Berman)

$$\sup_\phi \left(h_\phi(-K_{\mathcal{X}}) + \frac{1}{2}\mathrm{Vol}(X) \log \int_X e^{-\phi} \right)$$

is maximal exactly when $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and ϕ is the KE metric.

Corollary (L.-Miao)

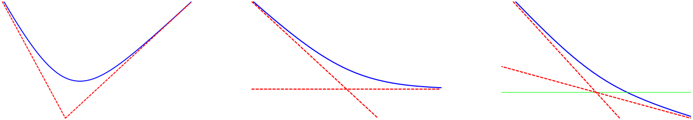
The conjecture is true for all toric Fano varieties.

Kähler-Einstein metric

Complex Monge-Ampère equation:

$$Ric(\omega_\varphi) = \omega_\varphi \Longleftrightarrow (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-\varphi}\Omega.$$

KE is a “convex” variational problem (Euler-Lagrangian equation).



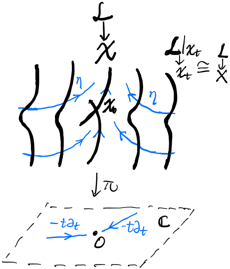
(a) Stable (b) Semistable (c) Unstable

Algebraic Obstructions:

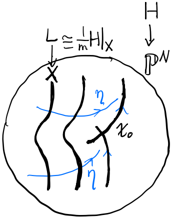
- $\mathrm{Aut}(X)$ is reductive.
- K-(poly)stability: for toric, this means barycenter is 0

K-stability (Tian, Donaldson)

A test configuration (TC): $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$:
a $\mathbb{C}^*$-equivariant family with $(\mathcal{X}_t, \mathcal{L}_t) \cong (X, -K_X)$ for $t \neq 0$.



Intrinsic view



Extrinsic view

K-polystable: (i) $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \mathrm{K}_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{L}^{\mathrm{n}} + \frac{\mathrm{n}}{\mathrm{n}+1}\mathcal{L}^{\mathrm{n}+1} \geq 0$
(K-semistable);

(ii) $\mathrm{Fut} = 0$ iff $(\mathcal{X}, \mathcal{L}) \cong (X, L) \times \mathbb{C}$, $\mathbb{C}^*$-equivariantly.

Yau-Tian-Donaldson Conjecture

Theorem (Chen-Donaldson-Sun, Tian; Berman)

A smooth Fano manifold X admits a KE metric if and only if X is K-polystable.

Theorem (L.-Tian-Wang + Liu-Xu-Zhuang)

A singular Fano variety admits KE if and only if X is K-polystable.

Corollary

A toric Fano variety admits KE iff the barycenter of polytope is 0.

Generalizations: Kähler-Ricci solitons, Calabi-Yau cone metrics, constant scalar curvature Kähler metrics ...

Valuative criterion

Divisorial Valuation: $v = \mathrm{ord}_E : \mathbb{C}(X) \to \mathbb{R}$, $v(f) = \mathrm{ord}_E(\mu^* f)$.
$\mu : Y \to X$ is birational morphism and $E \subset Y$ a prime divisor.

- $v(f_1 + f_2) \geq \min\{v(f_1), v(f_2)\}$		weighted
- $v(f_1 \cdot f_2) = v(f_1) + v(f_2)$	$\leftrightsquigarrow$	vanishing
- $v(\mathbb{C}) = 0$.		order

Theorem (Fujita, L. '16)

X is K-semistable if and only if for any $v = \mathrm{ord}_D$,

$$\beta(v) = A_X(v) - \frac{1}{\mathrm{Vol}(X)} \int_0^{+\infty} \mathrm{vol}(\mu^*(-K_X) - tE)dt \geq 0.$$

$$\begin{aligned} A_X(v) &= \mathrm{ord}_F(K_{Y/X}) + 1 \quad \text{(log discrepancy} \leftrightsquigarrow \text{sum of weights)} \\ \mathrm{Vol}(D) &= \lim_{m \to +\infty} \frac{h^0(X, \mathcal{O}(mD))}{m^n/n!}. \end{aligned}$$

An Example

Smooth quadric hypersurface:

$$X = Q^3 = \{Z_0 Z_1 + Z_2 Z_3 + Z_4^2 = 0\} \subset \mathbb{P}^4.$$

$\mathbb{C}^*$-action: $t \circ (Z_0, Z_1, Z_2, Z_3, Z_4) = (Z_0, t^2 Z_1, Z_2, t^2 Z_3, tZ_4)$.

Fixed point sets: $C := [*, 0, *, 0, 0] \cong \mathbb{P}^1 \cong C2 = [0, *, 0, *, 0]$.

Normal bundle of C: $N_{C/X} = \mathcal{O} \oplus \mathcal{O}(1)$.

$\mathbb{C}^*$-action $\rightsquigarrow$ valuation $v = \mathrm{ord}_E$: $f(z) \in \mathbb{C}(X)$,

$$v(f(z)) = \min \left\{ 2k + \ell; \sum_{\alpha} c_{k\ell}(z_2) z_3^k z_4^\ell, c_{k\ell} \neq 0 \right\}.$$

Then $A(v) = 3$, $\beta(v) = \mathrm{Fut(v)} = 0$.

Rational Curves on Fano manifolds

- Mori's theory: $\exists$ rational curve $f : \mathbb{P}^1 \to X^n$ with
$$0 < \deg f = \int_{\mathbb{P}^1} f^* c_1(X) \leq n+1.$$
- Kollár-Miyaoka-Mori, Campana: X is rationally connected.

Minimal rational curves: $d = \deg(f) \in \{2, 3, \ldots, n+1\}$.

$$f^* TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{d-2} \oplus \mathcal{O}^{n-d+1}.$$

Fact: There are always minimal rational curves on X which sweep out a dense open set.

(minimal rational curves $\leftsquigarrow\!\!\rightsquigarrow$ minimal surface, closed geodesics)

Classification of Fano manifolds using rational curves

Theorem (Cho-Miyaoka-Shepherd-Barron)

If $\deg(f) \geq n+1$ *for any rational curve* f*, then* $X \cong \mathbb{P}^n$*.*

Set $d_{\min} = \min\{\deg f ; f \text{ is minimal}\} \in \{2, \ldots, n+1\}$.

Theorem

1. *(Miyaoka)* $d_{\min} = n+1 \iff X \cong \mathbb{P}^n$*.*
2. *(Casagrande-Druel)* $d_{\min} = n \iff X \cong Q^n$ *or blowup of* $\mathbb{P}^n$ *along a codim 2 subvariety contained in hyperplane.*

($\leftsquigarrow\!\rightsquigarrow$ index estimates for minimal surfaces/geodesics and sphere theorems)

Sketch of proof of volume bounds

- $d_{\min} = n$: verify directly: K-semistable $\Rightarrow V = \mathrm{Vol}(X) \leq 2n^n$.

- For $2 \leq d = d_{\min} \leq n-1$. Weighted blowup along a minimal rational curve $(1^{\oplus d-2}, 2^{\oplus n-d+1})$ with exceptional prime divisor E.

- Estimate $\mathrm{vol}(\mu^* L - xE) \geq V - \phi_d(x)$ with $\phi_d(x)$ a piecewise polynomial which for for x small coincides with

$$(\mu^* L - xE)^n = 2^{-(n-d+1)} x^{n-1} (dn - (d-2)x).$$

- Valuative criterion $\rightsquigarrow V \leq \phi_d(T) < 2n^n$ where T is the solution to $\int_0^T \phi_d(t)dt = (T - 2n + d)\phi_d(T)$.
Check: $\phi_d(T) < 2n^n$ except for $d = 2$.

For $d = 2$, $\phi_d(x) = 2^{-(n-1)} 2n x^{n-1}$, $T = n$ and $\phi_d(T) = 2n^n$.

Equality Case when $d = 2$

Trivial Normal bundle:
$N_{C/X} = \mathcal{O}(1)^{\oplus(d-2)} \oplus \mathcal{O}^{\oplus(n-d+1)} = \mathcal{O}^{\oplus(n-1)}$.

Seshadri constant:

$$\begin{aligned}\epsilon(L, C) &= \max\{t \in \mathbb{R}_{\geq 0} : \mu^* L - tE \text{ is positive}\} \\ &= \max\{t \in \mathbb{R}_{\geq 0} : \mathrm{vol}(\mu^* L - tE) = (\mu^* L - tE)^n\} \\ &= n \quad \text{(if volume equality holds).}\end{aligned}$$

Rigidity theorem for Seshadri constant:

Proposition (Bauer-Szemberg, Liu-Zhuang; L.-Miao)

If $\epsilon(-K_X, C) > n - 1$, then $X \cong C \times \mathbb{P}^{n-1}$.

Klt singularities

Singularity: $x \in X$: normal, Gorenstein (i.e. $-K_X$ well-defined).

$\mu : (Y, \sum_i E_i) \to X$ a resolution of singularities (Hironaka).

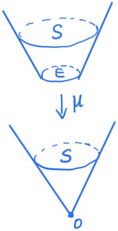
$$K_Y + \sum_i E_i = \mu^* K_X + \sum_i A_X(E_i) E_i$$

X is Klt if $A_X(E_i) = A_X(\mathrm{ord}_{E_i}) > 0$ for all i.

Analytically, this means that there is a no-where vanishing holomorphic volume form s near $x \in X$ such that the (n, n)-form $s \wedge \bar{s}$ is L^2-integrable on the regular locus near $x \in X$.

Examples:

- $\dim_{\mathbb{C}} X = 2$: quotient singularity $\mathbb{C}^2/G$



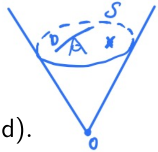
- Regular Fano cones $\mathcal{C} = C(S, r^{-1}K_X^{-1}) = C_r(S)$

$$\mathbb{C}^n = C(\mathbb{P}^{n-1}, \mathcal{O}(1)) = C_n(\mathbb{P}^{n-1})$$

- $X = C(\mathcal{S}, r^{-1}K_{\mathcal{S}}^{-1})$: $\mathcal{S} = (S, D)$ a Fano orbifold.

Ex: $A_{k-1} : X = \{z_1^2 + z_2^2 + z_3^2 + z_4^k = 0\} = C_r(\mathcal{S})$.

$$\begin{aligned} \mathcal{S} &= X /\!\!/ \mathbb{C}^* \text{with weights } (k, k, k, 2) \\ &= \big(\mathbb{P}^2, (1 - k^{-1})(\text{smooth conic})\big) \quad \text{(for k odd)}. \end{aligned}$$



Normalized volumes of valuations

$\mathrm{Val}_{X,x}$: space of real valuation (with center $x \in X$).

Definition (**L.**'15)

$$\begin{array}{rcl} \widehat{\mathrm{vol}} : \mathrm{Val}_{X,x} & \longrightarrow & \mathbb{R}_{>0} \cup \{+\infty\} \\ v & \mapsto & A_X(v)^n \cdot \mathrm{vol}(v). \end{array}$$

$$\widehat{\mathrm{vol}}(x, X) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v) > 0.$$

- $A_X(v)$: log disrepancy of v; X klt $\Longrightarrow A_X(v) > 0$.
- $\mathrm{vol}(v) = \lim_{m \to +\infty} \frac{\dim_{\mathbb{C}} \mathcal{O}_{X,x}/\{f; v(f) \geq m\}}{m^n/n!}$.

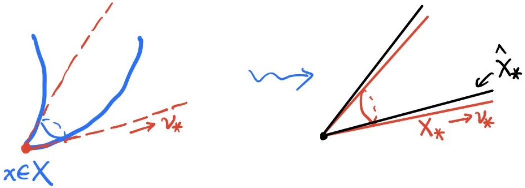
Ex.: $\widehat{\mathrm{vol}}(0, \mathbb{C}^n) = \inf_{a_i > 0} \left\{ \frac{(a_1 + a_2 + \cdots + a_n)^n}{a_1 \cdot a_2 \cdots a_n} \right\} = n^n$.

Stable degeneration conjecture (local stability theory)

Theorem (Blum, L., Liu, Xu, Zhuang; L.-Wang-Xu)

1. *$\exists$ a unique minimizer v_* s.t. $\widehat{\mathrm{vol}}(x, X) = \widehat{\mathrm{vol}}(v_*)$.*
2. *v_* is quasi-monomial and is finitely generated.*
3. *X degenerates to $X_* := \mathrm{Spec}(\mathrm{gr}_{v_*}\mathcal{O}_{x,X})$, which is a K-semistable Fano cone.*
4. *X_* degenerate uniquely to a K-polystable Fano cone $\hat{X}_*$.*

K-polystable Fano cone $\Longleftrightarrow$ Calabi-Yau cone metrics.



Example of normalized volumes

$$X = A_{k-1} : \{z_1^2 + z_2^2 + z_3^2 + z_4^k = 0\} \subset \mathbb{C}^4.$$

k	$1 \leq k \leq 3$	$k = 4$	$k \geq 5$
v_*	$(k, k, k, 2)$	$(4, 4, 4, 2)$	$(4, 4, 4, 2)$
X_*	X	X	$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$
$\hat{X}_*$	X	$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$	$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$
$\widehat{\mathrm{vol}}(x, X)$	$\frac{(k+2)^3}{k^2}$	$27/2$	$27/2$

Relation to volume bounds I

- Local-to-global inequality (Fujita, Liu):

$$X \text{ K-semistable} \quad \Longrightarrow \quad \frac{\mathrm{Vol}(X)}{\mathrm{Vol}(\mathbb{P}^n)} \leq \frac{\widehat{\mathrm{vol}}(x,X)}{\widehat{\mathrm{vol}}(0,\mathbb{C}^n)}.$$

This is a type of relative volume comparison.

Ex.: X KE $\Rightarrow \mathrm{Vol}(X) \leq \mathrm{Vol}(\mathbb{P}^n) = (n+1)^n$ (K. Fujita).

- Fact (Liu-Xu): $\widehat{\mathrm{vol}}(x,X) \leq \widehat{\mathrm{vol}}(0,\mathbb{C}^n)$; "=" iff $x \in X$ smooth.

ODP conjecture: $\widehat{\mathrm{vol}}(x,X) \leq 2(n-1)^n$ for singular klt

$\Longrightarrow \mathrm{Vol}(X) < 2n^n$ for singular K-semistable X.

Known cases: toric (Moraga-Süß); 3-dim. klt singularities (Liu-Xu).

Relation to volume bounds II

- Kobayashi-Ochiai: $-K_{Y^{n-1}} = L^{\otimes r} \Longrightarrow r \le n$; "$=$" iff $Y \cong \mathbb{P}^{n-1}$.

Corollary (ODP for regular K-semistable Fano cones)

$Y \ncong \mathbb{P}^{n-1}$ *K-semistable* $\Rightarrow \widehat{\mathrm{vol}}(x, C_r(Y)) = r \cdot \mathrm{Vol}(Y) \le 2(n-1)^n$.

- ODP conjecture can be reduced to quasi-regular Fano cones.

Conjecture (Shokurov, bound for minimal log discrepancy)

$\mathrm{mld}(x, X) = \min_{E_i} A_X(E_i) \le n$; *"$=$" iff* $x \in X$ *is smooth.*

Theorem (L.-Zhou '25)

$\mathrm{mld}(x, X) \le n$ *for any isolated quasi-regular Fano cone* (x, X).

$\rightsquigarrow$ To prove ODP conj., we need suitable bounds for KE Fano orbifolds (work in progress).

Thanks for your attention!