Lefschetz (1,1) - Theorem:

X projective manifold.

$$\{\text{divisors with } \mathbb{Q}\text{-coefficients}\} \xrightarrow{cl.} H^2(X,\mathbb{Q})$$

$$\sum_i a_i D_i \longmapsto \sum_i a_i [D_i] \in H^2(X,\mathbb{Q})$$

$$\mathrm{Im}(cl.) = H^2(X,\mathbb{Q}) \cap H^{1,1}(X) \subset H^2(X,\mathbb{C}) = H^2(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$ Hodge Decomposition

space of Hodge classes.

Hodge Conjecture: $H^{p,p}(X,\mathbb{Q}) := H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X) \subset H^{2p}(X,\mathbb{C})$

is spanned by classes of codimension p analytic cycles.

Want to use induction: Let $Y = X \cap H$ be a hyperplane section.

Then by Lefschetz Hyperplane Thm. $j^*: H^k(X,\mathbb{Q}) \to H^k(Y,\mathbb{Q})$ is an isomorphism if $k \le n-2$; is injective if $k = n-1$.

Use Poincaré duality. $j_*: H^{\ell-2}(Y,\mathbb{Q}) \to H^{\ell}(X,\mathbb{Q})$ is an isomorphism if $\ell \ge n+2 = 2n-(n-2)$; surjective if $\ell = n+1 = 2n-(n-1)$

j^*, j_* map Hodge classes to Hodge classes.

So if $x \in H^{p,p}(X,\mathbb{Q})$, with $2p \ge n+1$, $\exists\, y \in H^{p-1,p-1}(Y,\mathbb{Q})$ s.t. $j_*(y) = x$

HC for $Y \Rightarrow \exists$ codim $(p-1)$-analytic cycle Z_Y in Y with $[Z_Y] = y$

$\Rightarrow$ as codim p-analytic cycle $Z_X = j(Z_Y)$ in X, $[Z_X] = x$.

If $2p \le n-1$, consider a complete intersection of X with $n-2p$ hyperplanes, $X \cap H_1 \cap \cdots \cap H_{n-2p} = Y \leftarrow \dim = 2p$

$$\xi \in H^{p,p}(X,\mathbb{Q}) \to j^*\xi \in H^{p,p}(Y,\mathbb{Q}) \text{ is class of of } _{\text{0-cycle}}$$

Consider Y as a general member of an $(n-2p)$-dim. family of complete intersection subvarieties covering X and apply Hilbert scheme arguments, one can show that there is traced out an algebraic cycle class $\mu \in H^{p,p}(X,\mathbb{Q})$ s.t. $j^*(\xi) = j^*(\mu)$.

Lefschetz Thm $\Rightarrow$ $j^* : H^{p,p}(X,\mathbb{Q}) \to H^{p,p}(Y,\mathbb{Q})$ is injection

$\Rightarrow$ $\xi = \mu$ is algebraic.

So we focus on $H^{m,m}(X,\mathbb{Q}) = \mathrm{Prim}^{m,m} \oplus j_* H^{m-1,m-1}(Y,\mathbb{Q})$

We will sketch Lefschetz original proof for surfaces using the theory of <u>Lefschetz Pencils:</u> A $\mathbb{P}^1$-family of Hyperplane Sections

$$\{X_t = X \cap H_t\}_{t \in \mathbb{P}^1} \text{ where } \mathbb{P}^1 \text{ is a line in } (\mathbb{P}^N)^\vee = \mathbb{P}(H^0(X,H)).$$

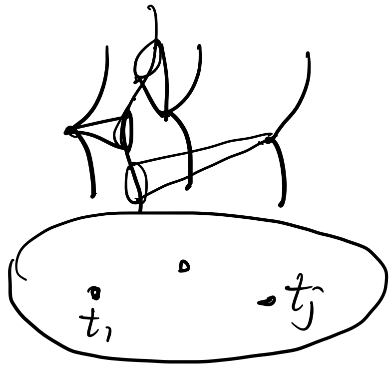
satisfying (1) The base locus $D = \bigcap_{t \in \mathbb{P}^1} X_t$ is smooth.

(2) X_t is smooth or at exactly one Ordinary Double Point (ODP)

$$\cong \{z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}$$

$\tilde{X} = Bl_D X$ blow up of base locus.

$$\rightsquigarrow \quad \begin{array}{ccc} \tilde{X} & \supset & X_t \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \ni & t \end{array}$$



away from critical points $\{t_1, \dots, t_r\}$

X_t is smooth, all diffeomorphic to each other.

Assume that X_∞ is smooth

$t_j \rightsquigarrow X_{t_j}$ has an ODP

$\rightsquigarrow$ vanishing sphere $S_j^{n-1} \rightsquigarrow \delta_j \in H^{n-1}(X_\infty)$

Vanishing cone $C(S_j^{n-1}) = B_j^n \rightsquigarrow \Delta_j \in H^n(X, X_\infty)$.

<u>Fact</u>: $\mathrm{Prim}^n(X) = \left\{ \sum_j \lambda_j \Delta_j + C : \begin{array}{l} C \text{ is a chain in } X_\infty \\ \sum_j \lambda_j \delta_j = 0 \in H^{n-1}(X_\infty) \end{array} \right\}$

$$\overset{n=2m}{\Rightarrow} \quad \mathrm{Prim}^{m,m} = \Big\{ \overbrace{\sum_j \lambda_j \Delta_j + C}^{\Gamma} \in \mathrm{Prim}^n(X) : \int_\Gamma \eta = 0,\ \forall \eta \in F^{m+1} H^{2m} \Big\}$$

$\Downarrow$ if $n=2$

$$\int_\Gamma \eta = 0,\ \forall \eta \in H^0(X, \Omega_X^2)$$

- Abel-Jacobi map for $Y^{n'}$

Let Z_Y be an analytic cycle that is cohomologous to 0
codim p

$\Rightarrow$ $Z_Y = \partial C$ for a chain C of dimension $2(n'-p)+1$

$$AJ(Z_Y) \in \left(F^{n'-p+1} H^{2n'-2p+1}\right)^\vee / H_{2n'-2p+1}(Y, \mathbb{Z})$$

$$\cong$$

$$\frac{H^{2p-1}(Y, \mathbb{C})}{F^p H^{2p-1} + H^{2p-1}(Y, \mathbb{Z})}$$

$$\|$$

$J^p(Y)$ p-th intermediate Jacobian

$$\left(\begin{array}{l} (F^p H^{2p-1})^\vee = F^{n'-p+1} H^{2n'-2p+1} \\ F^p H^r = \bigoplus_{k \geq p} H^{k, r-k} \end{array} \right)$$

$$AJ(Z_Y)(\phi)=\int_C \phi \quad \text{for } [\phi]\in F^{n-p+1}H^{2n-2p+1}$$

well-defined: $[\phi_1]=[\phi_2] \Rightarrow \phi_2-\phi_1=\partial\bar\partial\eta = d(\partial\eta)$, $\quad \eta = \sum_k \eta^{k,l}$ with $k+l=2n'-2p-1$, $k\geq n'-p$

$$\int_C \phi_2-\int_C\phi_1=\int_C d(\partial\eta)=\int_{\partial C}\partial\eta=\underbrace{\int_Z \partial\eta}_{=0}$$

$\partial\eta$: $k+1\geq n'-p$

curve case: $n'=1$, $p=1$

$$AJ(\underset{\parallel}{Z})(\phi)=\int_\Gamma\phi \qquad [\phi]\in F^1H^1=H^{1,0}=H^0(\Omega^1_Y)$$

$$P_1+\cdots+P_r-(Q_1+\cdots+Q_r)=\partial\Gamma$$

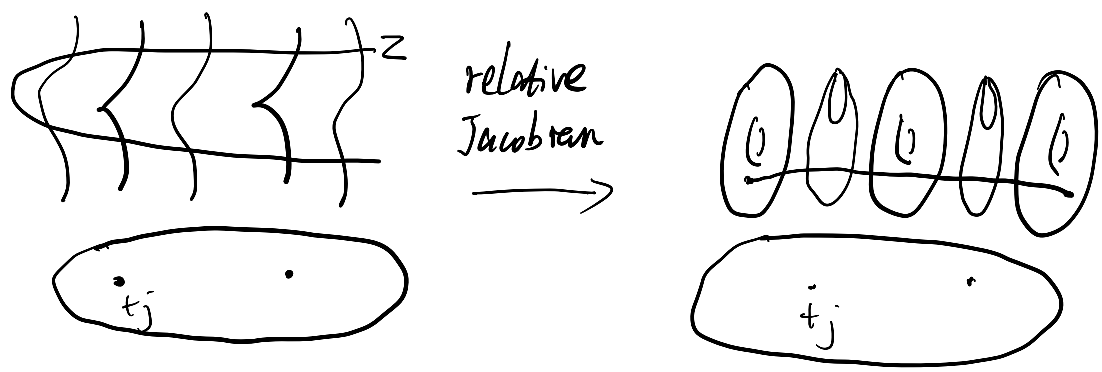
Equivalently, choose a basis $\{\omega_1,\dots,\omega_g\}$ of $H^0(\Omega^1_Y)$

$$Z\longmapsto \left(\int_\Gamma\omega_1,\dots,\int_\Gamma\omega_g\right)\in\mathbb{C}^n \text{ mod periods: } \left\{\begin{matrix}\left(\int_{\alpha_i}\omega_1,\dots,\int_{\alpha_i}\omega_g\right)\\ \left(\int_{\beta_i}\omega_1,\dots,\int_{\beta_i}\omega_g\right)\end{matrix}\right\}\cong\mathbb{Z}^{2n}$$

$$J^1(Y)=\text{Jacobian}=H^{1,0}(Y)^*/H_1(Y,\mathbb{Z})$$

$$=\frac{H^2(Y,\mathbb{C})}{H^{1,0}+H^1(Y,\mathbb{Z})}=H^{0,1}(Y)/H^1(Y,\mathbb{Z})\cap H^{0,1}(Y).$$

- Lefschetz Pencil



analytic cycle Z with primitive class $[Z] \in H^{m,m}(X)$ $\rightsquigarrow$ $AJ_{X_t}(Z \cap X_t) \in J^m(X_t)$

$$AJ_{X_t}(Z\cap X_t) = \nu_t$$

$$[Z] \in H^{m,m}(X) \xrightarrow{j^*} H^{m,m}(X_t), \qquad [Z] \xrightarrow{L} H^{m+1,m+1}(X)$$

$$0 = [Z \cap X_t] \in H^{m,m}(X_t) \underset{\cong}{\xrightarrow{j_*}} H^{m+1,m+1}(X)$$

$$H^{m,m}(X_t) \cong H^{n-m-1,n-m-1}(X_t)^*, \qquad H^{m+1,m+1}(X) \cong H^{n-m-1,n-m-1}(X)$$

because

$$2(n-m-1) = 2n - n - 2 = n-2$$

$$0 = L[Z] = j_* j^*[Z] \iff j^*[Z] = 0$$

$\rightsquigarrow$ holomorphic section $\nu = (\nu_t)_{t \in \mathbb{P}^1 \setminus \{t_1, \cdots, t_r\}}$

$\rightsquigarrow$ holomorphic section $\overline{\nu} = (\nu_t)_{t \in \mathbb{P}^1} \in H^0(\mathbb{P}^1, \mathcal{J})$

(called) normal function

sheaf of holomorphic sections of compactified relative Jacobian fibration.

- For a projective surface X with a Lefschetz Pencil, and for each primitive class $\xi \in \mathrm{Prim}^{1,1}(X,\mathbb{Q})$ Lefschetz constructs a normal function $\nu_\xi \in H^0(\mathbb{P}^1, \overline{\mathcal{J}})$ and then uses Jacobi Inversion Thm to construct an analytic cycle (a curve).

Explicitly, ν_ξ is given by the formula: $\xi = \sum_j \lambda_j \Delta_j + C$, Δ_j vanishing cone

$$\langle \nu_\xi, \omega(t)\rangle = \frac{1}{2\pi\sqrt{-1}} \cdot \sum_j \lambda_j \int_{t_*}^{t} \frac{\int_{\delta_j} \omega(s)}{s-t}\, ds$$

where $\omega(t) = \mathrm{Res}\left(\frac{\eta}{ts_0+s_1}\right)$ where $\eta \in H^0(X, \Omega^2(H))$

$s_0, s_1 \in H^0(X,H)$ spans the pencil

$t_* \in \mathbb{C}\setminus\{t_1,\cdots,t_r\}$

- $\langle \nu_\xi, \omega(t)\rangle$ has logarithmic singularity near t_j: (the meaning of being "normal")

$$\langle \nu_\xi, \omega(t)\rangle - \lambda_j \cdot \frac{1}{2\pi\sqrt{-1}} \int_{\delta_j} \omega(t) \cdot \log(t-t_j)$$ is holomorphic and single-valued.

- $\int_{\xi} \eta = 0 \quad \forall\ \eta \in H^0(\Omega_X^2) \Rightarrow$ ν_ξ extends across to be a holomorphic section near $\infty \in \mathbb{P}^1$.

- In higher dimension, it is still true that any primitive $\xi \in \mathrm{Prim}^{m,m}$ determines a normal function.

  But the Jacobi Inversion Thm in general does not hold.

  So other methods are needed to study the Hodge Conjecture.