Lefschetz Pencil.

Assume $X \hookrightarrow \mathbb{P}^N$ satisfies $H^0(\mathbb{P}^N, \mathcal{O}(H)) \xrightarrow{\cong} H^0(X, \mathcal{O}(H|_X))$

for example $X \hookrightarrow \mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{O}(H|_X))^*)$ is a Kodaira embedding.

- A pencil of the linear system $|H| = \mathbb{P}(H^0(X, \mathcal{O}(H|_X)))$ is a 2-dim subspace $V = \mathrm{span}\{s_0, s_1\}$ of $H^0(X, \mathcal{O}(H|_X)) \cong H^0(\mathbb{P}^N, \mathcal{O}(H))$.

equivalently a projective line $L \subset |H|$ which is considered as a $\mathbb{P}^1$-family of hyperplanes in $\mathbb{P}^N = \mathbb{P}(H^0(X, H)^*)$.

$\rightsquigarrow$ $\mathbb{P}^1$-family of hyperplane sections $X_{[u]} = \{ u_0 s_0 + u_1 s_1 = 0 \}$, $[u] = [u_0, u_1] \in \mathbb{P}^1$

- A pencil is called a Lefschetz pencil if the following 2 conditions are satisfied:

1. $\{ s_0 = s_1 = 0 \} \cong \mathbb{P}^{N-2}$ intersects X transversally.

   Set $B = \{ s_0 = s_1 = 0 \} \cap X$. Then B is smooth.

   $\Rightarrow$ hypersurfaces X_t are smooth along B.

2. $\forall t \in \mathbb{P}^1$, X_t is smooth or has at most one ordinary double point as singularity.

   ODP: ordinary double point $\cong$ $z_1^2 + z_2^2 + \cdots + z_n^2 = 0$.

<u>Thm</u>: There always exists a Lefschetz pencil.

<u>Pf</u>: Consider the incidence variety

$$\begin{tikzcd} Z = \{(x, H) \in X \times (\mathbb{P}^N)^* : T_x X \subseteq H\} \subseteq X \times (\mathbb{P}^N)^* \arrow[d, "p_1"'] \arrow[dr, "p_2"] & \\ X & (\mathbb{P}^N)^* \end{tikzcd}$$

Then $\begin{array}{c} Z \\ \downarrow p_1 \\ X \end{array}$ is a smooth fibration with fiber $\cong \underset{\{H \in \mathbb{P}^{N*} : T_x X \subset H\}}{\overset{\shortparallel}{\mathbb{P}^{N-n-1}}}$

$\Rightarrow \dim Z = n + N-n-1 = N-1$

$X^\vee = p_2(Z) = \{ H \in (\mathbb{P}^N)^* : H \text{ is tangent to } X \text{ at some point}\}$
$= \{ H \in (\mathbb{P}^N)^* : H \cap X \text{ is singular}\}$

$\Rightarrow \dim X^\vee \leq N-1$.

- if $\dim X^\vee < N-1$, then for a generic line $L \subset (\mathbb{P}^N)^*$, $L \cap X^\vee = \phi$
  $\Rightarrow X \cap H$ is smooth $\forall H \in L$
- if $\dim X^\vee = N-1$, then for a generic line $L \subset (\mathbb{P}^N)^*$, $L \cap X^\vee$ is a finite set contained in the smooth locus of $X^\vee$.

Need: if $H \in (X^\vee)_{sm.}$ and $H \supseteq T_x X$ with $x \in X$ then $H \cap X$ has exactly one ordinary double point.

Choose nbhds of $(x, H) \in X \times (\mathbb{P}^N)^*$ and identify points of $(\mathbb{P}^N)^*$ with linear functions on nbhd of $x \in X$.

Then $Z = \{ (z, H) \in X \times (\mathbb{P}^N)^* : \underset{z \in X \cap H}{\underset{\uparrow}{H(z) = 0}},\ \underset{T_z X \subset H}{\underset{\uparrow}{dH(z) = 0}} \}$ $\quad \frac{\partial H}{\partial z_i} = 0$

$T_{(x,H)} Z = \{ (v, k) \in T_x X \times T_H (\mathbb{P}^N)^* : \begin{array}{l} (d_v H) + k(x) = 0 \\ d_v \frac{\partial H}{\partial z_i} + \frac{\partial k}{\partial z_i}(x) = 0 \end{array} \}$

$\Rightarrow \operatorname{Ker}((p_2)_* : T_{(x,H)} Z \to T_H (\mathbb{P}^N)^*)$

$\shortparallel$

$\{ (v, 0) \in T_z X \times T_H (\mathbb{P}^N)^* : \underbrace{d_v H(x) = 0}_{\text{satisfied because } (x,H) \in Z},\ d_v \frac{\partial H}{\partial z_i} = 0 \}$

$\shortparallel 2$

$\{ (v, 0) \in T_z X : d_v \frac{\partial H}{\partial z_i} = 0 \}$ $\quad \frac{\partial^2 H}{\partial z_i \partial z_j} v_j$

$\shortparallel$

$\operatorname{Ker}\left(\operatorname{Hess}(H) = \left(\frac{\partial H}{\partial z_i \partial z_j} \right) : \mathbb{C}^n \to \mathbb{C}^n \right)$

If $H \in (X^\vee)_{sm}$ and $\dim X^\vee = N-1$, then $\ker (P_2)_* = 0$

$\Rightarrow$ $\ker(\mathrm{Hess} H) = 0$ $\Rightarrow$ x is a nondegenerate critical point of the function H

$\Rightarrow$ $\exists$ holomorphic coordinate s.t. $H = z_1^2 + \cdots + z_n^2$ and $x \in H \cap X$ is an ODP.

Moreover, if $x \in H \cap X$ is an ODP, then $\mathrm{Hess}(H)$ invertible

$$\underset{\displaystyle T_H X^\vee}{\underset{\cap}{\mathrm{Im}((P_2)_*)}} = \{ k \in T_H (\mathbb{P}^N)^* : k(x) = 0 \} \cong \mathbb{C}^{N-1}$$

If $x' \in H \cap X$ is another ODP, then $T_H (\mathbb{P}^N)^*$ contains another $\mathbb{C}^{N-1}$

$\Rightarrow$ H is not smooth on $X^\vee$.

$\rightsquigarrow$ A generic line L s.t. $L \cap X^\vee =$ finite set $\subset (X^\vee)_{sm}$ gives a Lefschetz pencil.

Assume $\{ X_t = H_t \cap X : t \in \mathbb{P}^1 \}$ is a Lefschetz pencil.

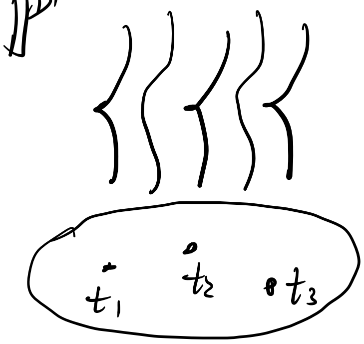
Blow up the base locus $B = \bigcap_{t \in \mathbb{P}^1} X_t$ to get a singular fibration

$$\begin{array}{ccccc} Bl_B X = & \widetilde{X} & \supset & \widetilde{X}_t \cong X_t \\ & \downarrow & & \downarrow \\ & \mathbb{P}^1 & \ni & t \end{array}$$

- Critical points $\{t_i : i=1,\dots,r\} \subset \mathbb{P}^1$

  X_{t_i} has one ODP.

  X_t is smooth if $t \notin \{t_i\}$.



Local geometry: $U = \{z \in \mathbb{C}^n : |z|^2 \leq \varepsilon^2\}$ $\quad z_i = x_i + \sqrt{-1}\, y_i$

$|z_1|^2 + \cdots + |z_n|^2 = |x|^2 + |y|^2$

$\downarrow \pi$

$\Delta = \{w : |w| \leq \rho\}$

$w = z_1^2 + \cdots + z_n^2 = |x|^2 - |y|^2 + 2\sqrt{-1}\,(x \cdot y)$, $\quad x \cdot y = \sum_i x_i y_i$

$t \in \mathbb{R}_{\geq 0}$

$U_t = \pi^{-1}(t) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x|^2 + |y|^2 \leq \varepsilon^2,\ |x|^2 - |y|^2 = t^2,\ x \cdot y = 0\}$

$\Downarrow$

$|x|^2 = |y|^2 + t^2 \geq t^2 \Rightarrow |x| \geq t > 0$

$2|y|^2 + t^2 \leq \varepsilon^2 \Rightarrow |y|^2 \leq \frac{\varepsilon^2 - t^2}{2}$

choose $\rho < \varepsilon$

$(v\sqrt{t^2+|w|^2}, w)$

$\uparrow$

(v, w) $\qquad \{(\frac{x}{|x|}, y)\} \in S^{n-1} \times B_{\sqrt{\frac{\varepsilon^2 - t^2}{2}}}$

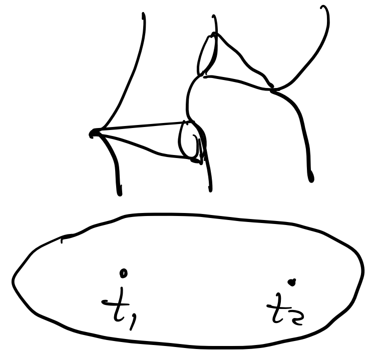
$\Rightarrow S_t^{n-1} \cong \{(x,0) \in \mathbb{R}^n \times \mathbb{R}^n : |x|^2 = t^2\}$ vanishing sphere.

$\bigcup_{0 \leq t \leq \rho} S_t^{n-1} = \{(x,0) \in \mathbb{R}^n \times \mathbb{R}^n : 0 \leq |x|^2 \leq \rho^2\} =$ Ball of radius ρ

$=$ cone over S_ρ^{n-1}

vanishing cone.

$$\begin{array}{ccccc} X_\infty & \subset & \tilde{X} & \supset & X_t \\ \downarrow & & \downarrow & & \downarrow \\ \{\infty\} & \in & \mathbb{P}^1 & \ni & t \end{array}$$



Assume X_∞, X_0 are smooth.

Thm: There exists a deformation contraction from $\tilde{X} \setminus X_\infty$ to a union of X_0 with vanishing cones (balls) $C_1, \cdots, C_r$ glued along vanishing spheres $S_1, \cdots, S_r$.

(complexified version of Morse theory.)

This will lead to Lefschetz Hyperplane Thm:

$$j^*: H^k(X, \mathbb{Z}) \to H^k(X_0, \mathbb{Z})$$

is isomorphism if $k \le n-2$ and injective if $k = n-1$.