Recall that we showed: for a smooth hypersurface

$X = \{F=0\} \subset \mathbb{P}^n$,

$H^{p,q}(X) = H^{p,q}(\mathbb{P}^n)$ if $p+q \neq n-1$.

because: for $p+q \leq n-2$, $j^*: H^{p,q}(\mathbb{P}^n) \to H^{p,q}(X)$ is isomorphism. by Lefschetz's Hyperplane Thm

for $p+q \geq n$,

$$\begin{array}{ccc} H^{p,q}(X)^* & \longleftarrow & H^{p+1,q+1}(\mathbb{P}^n)^* \\ \wr\wr & & \wr\wr \\ H^{n-1-p,n-1-q}(X) & \underset{\cong}{\xleftarrow{j^*}} & H^{n-1-p,n-1-q}(\mathbb{P}^n) \end{array}$$

$n-1-p+n-1-q \leq 2n-2-n = n-2$

$\Rightarrow H^{p,q}(X) \xrightarrow{j_*} H^{p+1,q+1}(\mathbb{P}^n)$ is isomorphism

So for $p+q \neq n-1$ and $0 \leq p+q \leq 2n-2$, $h^{p,q}(X) = \begin{cases} h^{p,q}(\mathbb{P}^n) & p+q \leq n-2 \\ h^{p+1,q+1}(\mathbb{P}^n) & p+q \geq n \end{cases}$

$= \delta_{pq}$

- Just need to consider $p+q = n-1$

Then considering the exact sequence:

$$0 \to \Omega^{p+1}_{\mathbb{P}} \to \Omega^{p+1}_{\mathbb{P}}\ X) \xrightarrow{Res} \Omega^p_X \to 0$$

leads us to the isomorphism:

$$H^q(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}}(\log X)) \cong \mathrm{Ker}\left\{ H^q(X, \Omega^p_X) \to H^{q+1}(\mathbb{P}, \Omega^{p+1}_{\mathbb{P}}) \right\}$$

$$\begin{tikzcd} H^q(X,\Omega^p_X) \arrow[r] \arrow[d, equal] & H^{q+1}(\mathbb{P},\Omega^{p+1}_{\mathbb{P}}) \arrow[d, equal] & \\ H^{p,q}(X) \arrow[r, "j_*"] \arrow[dr, "L"] & H^{p+1,q+1}(\mathbb{P}) \arrow[d, "j^*"', "\cong"] \arrow[dr, "\cong"] & \\ H^{p,q}(\mathbb{P}) \arrow[u, "j^*"] & H^{p+1,q+1}(X) \arrow[r, "j_*"] & H^{p+2,q+2}(\mathbb{P}) \end{tikzcd}$$

$p+2+q+2 = n+3 > 2n \Leftrightarrow n \leq 2$

$\mathrm{Ker}(j_*) = \mathrm{Ker}(L) = \mathrm{Prim}^{p,q}_X$

$$\Rightarrow \quad \mathrm{Prim}_X^{p,q} = H^q(\mathbb{P}^n, \Omega_{\mathbb{P}}^{p+1}(\log X)).$$

To calculate the right side, use the exact sequence: $\qquad n-(q+1)=p$

$$0 \to \Omega_{\mathbb{P}}^{p+1}(\log X) \to \Omega^{p+1}(X) \xrightarrow{d_1} \Omega^{p+2}(2X)/\Omega^{p+1}(X) \to \cdots \xrightarrow{d_q} \frac{\Omega^n((q+1)X)}{\Omega^n(qX)} \to 0$$

This means $\Omega_{\mathbb{P}}^{p+1}(\log X)$ is quasi-isomorphic to

$$C_\cdot : \ 0 \to \Omega^{p+1}(X) \xrightarrow{d_1} \Omega^{p+2}(2X)/\Omega^{p+2}(X) \to \cdots \xrightarrow{d_q} \frac{\Omega^n((q+1)X)}{\Omega^n(qX)} \to 0$$

$$\Rightarrow \quad H^q(\Omega_{\mathbb{P}}^{p+1}(\log X)) \cong \mathbb{H}^q(C_\cdot)$$

$\uparrow$ hypercohomology

$\mathbb{H}^q(C_\cdot)$ = cohomology of the double complex

$$C_\cdot^{r,s} = C^s(\mathcal{F}^r)$$ where C^s is the Čech complex of $\mathcal{F}^r$

$$\mathcal{F}^r = \begin{cases} \Omega_{\mathbb{P}}^{p+1}(X) & r=1 \\ \dfrac{\Omega^{p+r}(rX)}{\Omega^{p+r}((r-1)X)} & 2 \le r \le q+1 \end{cases}$$

Use spectral sequence:

$$E_1^{r,s} = H^s(\mathcal{F}^r) = \begin{cases} H^s(\Omega_{\mathbb{P}}^{p+1}(X)), & r=1 \\ H^s\left(\dfrac{\Omega^{p+r}(rX)}{\Omega^{p+r}((r-1)X)}\right) & 2 \le r \le q+1 \end{cases}$$

Bott Vanishing: $H^i(\mathbb{P}^n, \Omega^j(kX)) = 0$, if $i>0$ and $k>0$.

$$\Rightarrow \quad H^s(\mathcal{F}^r) = 0 \text{ for } s>0$$

indeed: when $r=1$, $H^s(\Omega_{\mathbb{P}}^{p+1}(X)) = 0$.

when $r \ge 2$ $\qquad 0 \to \Omega^{p+r}((r-1)X) \to \mathcal{F}^r \to \Omega^{p+r}(rX) \to 0$

$$H^s(\Omega^{p+r}((r-1)X)) = 0 = H^s(\Omega^{p+r}(rX)) \implies H^s(F^r) = 0. \text{ for } s > a$$

$$\implies \mathbb{H}^q(C^\bullet) = \operatorname{Coker}\left(H^0(F^q) \xrightarrow{d} H^0(F^{q+1})\right)$$

$$H^0(F^q) = H^0\left(\frac{\Omega^{n-1}(qX)}{\Omega^{n-1}((q-1)X)}\right) \cong \frac{H^0(\Omega^{n-1}(qX))}{H^0(\Omega^{n-1}((q-1)X))}$$

$$H^0(F^{q+1}) = H^0\left(\frac{\Omega^n((q+1)X)}{\Omega^n(qX)}\right) \cong \frac{H^0(\Omega^n((q+1)X))}{H^0(\Omega^n(qX))}$$

$\boxed{\text{Bott Vanishing: } H^1(\Omega^{n-1}((q-1)X)) = 0 \text{ if } n > 2}$ because

$$\implies H^q(\Omega^{p+1}_{\mathbb{P}}(\log X)) \cong \mathbb{H}^q(C^\bullet) \cong \frac{H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}}((q+1)X))}{H^0(\Omega^n_{\mathbb{P}}(qX)) + dH^0(\Omega^{n-1}_{\mathbb{P}}(qX))}.$$

- Alternatively, one can split the resolution of $\Omega^{p+1}(\log X)$ to get:

$$0 \to \Omega^{p+1}(\log X) \to \Omega^{p+1}(X) \to \ker d_2 \to 0$$

$$\cdots$$

$$0 \to \ker(d_k) \to \frac{\Omega^{p+k}(kX)}{\Omega^{p+k}((k-1)X)} \to \ker d_{k+1} \to 0$$

$$\cdots$$

$$0 \to \ker(d_q) \to \frac{\Omega^{n-1}(qX)}{\Omega^{n-1}((q-1)X)} \to \frac{\Omega^n((q+1)X)}{\Omega^n(qX)} \to 0.$$

Use Bott vanishing to get

$$H^q(\Omega^{p+1}(\log X)) = H^{q-1}(\ker d_2) \quad (\text{if } q \geq 2)$$

$$\begin{aligned} &\qquad\qquad\quad \text{\rotatebox{90}{$\cong$}} \\ H^0(\ker d_q) &= \cdots = H^{q-2}(\ker d_3) \\ \| & \\ \operatorname{Coker}&\left(H^0\left(\frac{\Omega^{n-1}(qX)}{\Omega^{n-1}((q-1)X)}\right) \longrightarrow H^0\left(\frac{\Omega^n((q+1)X)}{\Omega^n(qX)}\right)\right) \\ \| & \\ &\frac{H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}}((q+1)X))}{H^0(\Omega^n_{\mathbb{P}}(qX)) + dH^0(\Omega^{n-1}_{\mathbb{P}}(qX))}. \quad \left(\begin{array}{l}\text{true also} \\ \text{when } q=0,1\end{array}\right) \end{aligned}$$

• <u>Griffiths</u>: There is an isomorphism

$$\frac{H^0(\mathbb{P}^n, \Omega^n((q+1)X))}{H^0(\Omega^n(qX)) + dH^0(\Omega^{n-1}(qX))} \cong S_k / J_F \cap S_k$$

with

$$k = (q+1)\deg F - (n+1)$$

$$S_k = \left\{ \begin{array}{l} \text{homogeneous polynomials of} \\ \text{degree } k \text{ on } z_0, \cdots, z_n \end{array} \right\}$$

There is a map $S_k \xrightarrow{\bar{\Phi}} H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((q+1)X))$

$$A \longmapsto \omega_A = \frac{A \cdot \eta}{F^{q+1}}$$

when $\eta = \sum_{i=0}^{n} (-1)^i z_i \, dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n$

and $\deg A + n+1 = (q+1) \cdot \deg F$. (rescaling inv.)

$J_F =$ ideal generated by $\left\{ \frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n} \right\}$
|| Jacobian ideal

Fact: • $\bar{\Phi}$ is an isomorphism

(meromorphic n-forms are rational n-forms
a consequence of Chow's Thm)

• $\bar{\Phi}(J_F \cap S_k)$

$$= H^0(\mathbb{P}^n, \Omega^n(qX)) + d\, H^0(\mathbb{P}^n, \Omega^{n-1}(qX))$$

For example, Verify one direction: $\subseteq$

$$A = B\frac{\partial F}{\partial z_0} \Rightarrow \omega_A = B\cdot\frac{\partial F}{\partial z_0}\cdot \eta \Big/ F^{q+1}$$

$$-d\left(\frac{1}{q}\frac{1}{F^q}\cdot\sum_{j\geq 1}(-1)^j z_j\, \widehat{dz_0}\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\right)$$

$$\text{|||} \leftarrow \text{mod out } H^0(\Omega^n(qX))$$

$$+\frac{B}{F^{q+1}}\left(\sum_{r=0}^{n}\frac{\partial F}{\partial z_r}dz_r\right)\wedge\sum_{j\geq 1}(-1)^j z_j\, \widehat{dz_0}\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots$$

$$\text{||}$$

$$+\frac{B}{F^{q+1}}\Bigg(\sum_{j\geq 1}(-1)^j\frac{\partial F}{\partial z_0}z_j\, dz_0\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots$$

$$+\sum_{r\geq 1}\left(\sum_{j\geq 1}\frac{\partial F}{\partial z_j}z_j\right)\widehat{dz_0}\wedge dz_1\wedge\cdots\wedge dz_n\Bigg)$$

$$(-1)^j\cdot(-1)^{j-1}$$

$$\text{||}$$

$$F - z_0\frac{\partial F}{\partial z_0}$$

$$= +\frac{B}{F^{q+1}}\frac{\partial F}{\partial z_0}\sum_{j=0}^{n}(-1)^j z_j\cdots\wedge\widehat{dz_j}\cdots\wedge\cdots)$$

$$= +B\cdot\frac{\partial F}{\partial z_0}\eta \Big/ F^{q+1} \qquad \checkmark$$

Bott Vanishing:

$$H^i(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(kH)) = 0 \quad \text{for } i>0,\ k>0$$

$$\left(\text{nonzero only when: } \begin{cases} i=j\geq 0, & k=0 \\ i=0 & k>j \\ i=n, & -k>n-j \\ & (k<-n+j) \end{cases} \right)$$

Idea: Use Euler's exact sequence:

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus(n+1)} \to T\mathbb{P}^n \to 0$$

$$\rightsquigarrow \quad 0 \leftarrow \mathcal{O} \leftarrow \mathcal{O}(-1)^{\oplus n+1} \leftarrow T^*\mathbb{P}^n \leftarrow 0$$

$$\rightsquigarrow \quad 0 \leftarrow \Omega^{p-1} \leftarrow \mathcal{O}(-p)^{\binom{n+1}{p}} \leftarrow \Omega^p \leftarrow 0$$

Use associated long exact sequence and use induction.