<u>Goal</u>: Understand the Hodge structure of sm. hypersurfaces in $\mathbb{P}^n$.

- cohomology of $\mathbb{P}^n$: $H^{p,q}(\mathbb{P}^n) = \begin{cases} \mathbb{C} & 0 \le p = q \le n \\ 0 & \text{otherwise} \end{cases}$

- $X = \{F = 0\}$ $F = F(z_0, z_1, \cdots, z_n)$ is a homogeneous polynomial of degree d.

$\rightsquigarrow$ F defines a holomorphic section of $H^{\otimes d}$ where H is the hyperplane bundle.

Indeed, H = dual of the $\underbrace{\text{tautological line bundle}}_{\{([z], \mathbb{C}\cdot z) : [z] \in \mathbb{P}^n\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}}$

The fiber $H_z^{\otimes d} = \{$polynomials of degree d on $\mathbb{C}\cdot z \cong \mathbb{C}\}$

$F \longrightarrow F|_{\mathbb{C}(z)} \in H_z^{\otimes d}$ is thus a holomorphic section. s_F

(In fact, $H^0(\mathbb{P}^n, H^{\otimes d}) = \{$homogeneous polynomials of degree d on $z_0, \cdots, z_n\}$)

$\Rightarrow$ $X = \{s_F = 0\}$ is a divisor whose associated line bundle is $H^{\otimes d}$ which is a positive line bundle.

By Lefschetz Hyperplane Thm and Hodge Decomposition, the map

$$j^*: H^{p,q}(\mathbb{P}^n) \longrightarrow H^{p,q}(X)$$

is an isomorphism when $p+q \le n-2$.

injective when $p+q = n-1$.

Dually, $j_*: H^{r,s}(X) \longrightarrow H^{r+1,s+1}(\mathbb{P}^n)$

is an isomorphism when $r+s \ge 2(n-1)-(n-2) = n$

surjective when $r+s = 2(n-1)-(n-1) = n-1$.

So if $p+q\neq n-1$, then $h^{p,q}(X)=\delta_{pq}$

$0\leq p+q\leq 2(n-1)$

So we can focus on $H^{p,q}(X)$ when $p+q=n-1$.

Let $\mathrm{Inv}^{p,q}=\mathrm{Im}(j^*: H^{p,q}(\mathbb{P}^n)\to H^{p,q}(X))$.

$\mathrm{Van}^{p,q}=\mathrm{Ker}(j_*: H^{p,q}(X)\to H^{p+1,q+1}(\mathbb{P}^n))$.

Then $H^{p,q}(X)=\mathrm{Inv}^{p,q}\oplus\mathrm{Van}^{p,q}$.

Note · $\dim\mathrm{Inv}^{p,q}=\delta_{pq}$. $\quad p+q=n-1$

$$\begin{tikzcd}
H^{p,q}(X) \arrow[r, "j_*"] \arrow[dr, "L"'] & H^{p+1,q+1}(\mathbb{P}^n) \arrow[d, "\cong"] \arrow[dr, "L", "\cong"'] & \\
& H^{p+1,q+1}(X) \arrow[r, "j_*", "\cong"'] & H^{p+2,q+2}(\mathbb{P}^n)
\end{tikzcd}$$

Lefschetz Hyperplane Thm. $\quad p+1+q+1 = n+1$

$\Rightarrow \mathrm{Van}^{p,q}=\ker(L: H^{p,q}(X)\to H^{p+1,q+1}(X))=P^{p,q}$

$(p+q=n-1=\dim X)$ $\qquad$ primitive cohomology

$\Rightarrow h^{p,q}=\delta_{pq}+h_0^{p,q}$ where $h_0^{p,q}=\dim P^{p,q}$. $\quad p+q=n-1$

This is also true if $p+q\neq n-1$:

$h^{p,q}=\delta_{pq}+h_0^{p,q}$ with $h_0^{p,q}=0$.

To calculate $h^{p,q} = \dim H^q(\Omega_X^p)$, one can use the Hirzebruch-Riemann-Roch formula:

$$\sum_{s=0}^{n-1} (-1)^s h^{p,s}(X) = \chi(\Omega_X^p) = \int_X ch(\Omega_X^p) \cdot Td(TX)$$

$$\text{left hand side} = \sum_{s=0}^{n-1} (-1)^s \left(h_0^{p,s} + \delta_{ps}\right)$$

$$= (-1)^{n-1-p} h_0^{p,n-1-p} + (-1)^p = \chi(\Omega_X^p)$$

$$\Rightarrow h_0^{p,n-1-p} = (-1)^{n-1-p} \chi(\Omega_X^p) + (-1)^n.$$

- To calculate $\chi(\Omega_X^p)$, we use the short exact sequence

$$0 \to TX \to T\mathbb{P}^n|_X \to N_X \to 0$$

$$\rightsquigarrow \quad 0 \leftarrow T^*X \leftarrow T^*\mathbb{P}^n|_X \leftarrow N_X^* \leftarrow 0$$

$$\rightsquigarrow \quad 0 \leftarrow \Omega_X^p \leftarrow \Omega_{\mathbb{P}^n}^p|_X \leftarrow \Omega_X^{p-1}(-d) \leftarrow 0$$

$$0 \leftarrow \Omega_X^p(i) \leftarrow \Omega_{\mathbb{P}^n}^p(i)|_X \leftarrow \Omega_X^{p-1}(i-d) \leftarrow 0$$

$$\Rightarrow \chi(\Omega_X^p(i)) = \underbrace{\chi(\Omega_{\mathbb{P}^n}^p(i)|_X)}_{\substack{\parallel \\ \chi(\Omega_{\mathbb{P}^n}^p(i)) - \Omega_{\mathbb{P}^n}^p(i-d)}} - \chi(\Omega_X^{p-1}(i-d))$$

because

$$\left(0 \to \Omega^p_{\mathbb{P}^n}(i-d) \longrightarrow \Omega^p_{\mathbb{P}^n}(i) \longrightarrow \Omega^p_{\mathbb{P}^n}(i)|_X \to 0\right)$$

$$\Rightarrow \chi(\Omega^p_X(i)) = \chi(\Omega^p_{\mathbb{P}^n}(i)) - \chi(\Omega^p_{\mathbb{P}^n}(i-d)) - \chi(\Omega^{p-1}_X(i-d)).$$

$\rightsquigarrow$ One can then inductively calculates $h_0^{p,q}$ (and hence $h^{p,q}$), which depend only on d an n, and are given by polynomials in these variables.

- A close form formula for the generating function

$$H(d) = \sum_{p,q=0}^{\infty} h_0^{p,q} x^p y^q$$

Thm (Hirzebruch):

$$H(d) = \frac{(1+y)^{d-1} - (1+x)^{d-1}}{(1+x)^d y - (1+y)^d \cdot x}.$$

Next, we want to understand $P^{p,q}$ = primitive part of

$$H^{p,q}(X) \cong H^q(X, \Omega^p).$$

To do this, define the sheaf:

$$\Omega_{\mathbb{P}}^{p+1}(\log X) = \text{sheaf of meromorphic } (p+1)\text{-forms with logarithmic poles along } X$$

$$= \mathcal{O}_{\mathbb{P}}\text{-module generated by } \Big\{ \overbrace{\frac{dz_1}{z_1}}^{d\log z_1}, dz_2, \dots, dz_n \Big\}$$

where $X = \{z_1 = 0\}$ locally.

$$= \{\omega \in \Omega_{\mathbb{P}}^{p+1}(X) : d\omega \text{ has a pole of order} \leq 1 \text{ along } X\}$$

$$= \ker\{d : \Omega_{\mathbb{P}}^{p+1}(X) \to \Omega_{\mathbb{P}}^{p+2}(2X)/\Omega_{\mathbb{P}}^{p+2}(X)\}.$$

$$\Omega_{\mathbb{P}}^{p+1}(X) = \mathcal{O}_{\mathbb{P}}\Big\{ \frac{dz_1}{z_1}, \frac{dz_2}{z_1}, \dots, \frac{dz_n}{z_1} \Big\}$$

Then there is an exact sequence:

$$0 \to \Omega_{\mathbb{P}^n}^{p+1} \to \Omega_{\mathbb{P}}^{p+1}(\log X) \xrightarrow{\text{Res}} \Omega_X^p \to 0$$

$$\text{Res}\Big(\frac{dz_1}{z_1} \wedge \sum_{\substack{|I|=p \\ 1 \notin I}} a_I \, dz_I + \sum_{\substack{|J|=p+1 \\ 1 \notin J}} b_J \, dz_J \Big) = \sum_{\substack{|I|=p \\ 1 \notin I}} a_I \, dz_I$$

is the Poincaré residue map.

$\rightsquigarrow$ Long exact sequence:

$$h_X^{p,q-1} \geq \delta_{p,q-1} = h^{p+1,q}$$

$\Rightarrow$ ∂_* is surjective hence isom.

$p+1+q = n = (n-1)+1$

$(n-1)-1 = p+q-1 = n-2$

$$\begin{tikzcd}
 & H^{p+1,q}(X) & \\
H^{p,q-1}(X) \arrow[r, "\partial_*"] \arrow[ur, "L \cong"] \arrow[d, equal] & H^{p+1,q}(\mathbb{P}) \arrow[u, "j^*"'] \arrow[d, equal] & \\
H^{q-1}(X, \Omega_X^p) \arrow[r] & H^q(\mathbb{P}, \Omega_{\mathbb{P}}^{p+1}) \arrow[r] & H^q(\mathbb{P}, \Omega_{\mathbb{P}}^{p+1}(\log X)) \arrow[d] \\
 & H^{q+1}(\mathbb{P}, \Omega_{\mathbb{P}}^{p+1}) & H^q(X, \Omega_X^p) \arrow[l, "G"'] \arrow[d, equal] \\
 & H^{p+1,q+1}(\mathbb{P}) \arrow[u, "\cong"] \arrow[dl, "L"', "\cong"] \arrow[d, "j^*", "\cong"'] & H^{p,q}(X) \arrow[l, "\partial_*"'] \arrow[dl, "L"] \\
H^{p+2,q+2}(\mathbb{P}) & H^{p+1,q+1}(X) \arrow[l, "\partial_*"', "\cong"] &
\end{tikzcd}$$

$\Rightarrow$ ∂_* is injective

$p+1+q+1 \geq n+1$, Lefschetz Hyperplane Thm.

$$\Rightarrow \operatorname{Ker}(G) = \operatorname{Ker}\left(H^{p,q}(X) \to H^{p+1,q+1}(X)\right)$$
$$= P^{p,q}.$$

Because $j_*: H^{p,q-1}(X) \to H^{p+1,q}(\mathbb{P})$ is injective, $\mathrm{Ker}(G) \cong H^q(\mathbb{P}, \Omega_{\mathbb{P}}^{p+1}(\log X))$

So we get an isomorphism:

$$P^{p,q} \cong H^q(\mathbb{P}, \Omega_{\mathbb{P}}^{p+1}(\log X)) \quad \text{when } \underset{n-1}{\overset{p+q}{\shortparallel}}.$$

To calculate the right hand, we use the

exact sequence: with $p+q=n-1$ i.e. $q=n-1-p$

$$0 \to \Omega_{\mathbb{P}}^{p+1}(\log X) \to \Omega_{\mathbb{P}}^{p+1}(X) \xrightarrow{d_1} \Omega_{\mathbb{P}}^{p+2}(2X)/\Omega_{\mathbb{P}}^{p+2}(X) \xrightarrow{d_2} \frac{\Omega_{\mathbb{P}}^{p+3}(3X)}{\Omega_{\mathbb{P}}^{p+3}(2X)}$$
$$\cdots \longrightarrow \cdots \xrightarrow{d_q} \Omega_{\mathbb{P}}^{n}((q+1)X)/\Omega_{\mathbb{P}}^{n}(qX) \to 0.$$

We will use this exact sequence and Bott's vanishing thm to prove:

$$\boxed{H^q(\Omega_{\mathbb{P}^n}^{p+1}(\log X)) = \frac{H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n}((q+1)X))}{H^0(\mathbb{P}^n, \Omega_{\mathbb{P}}^{n}(qX)) + d\,H^0(\mathbb{P}^n, \Omega^{n-1}(qX))}.}$$

Refs: · Arapura: Algebraic Geometry over The complex Numbers.

· Lewis: A survey of the Hodge Conjecture.