Continuing the proof of Kodaira Embedding Thm.

Need to prove the surjectivity of maps when $m \gg 1$

- $ev_{x,y} : H^0(X, L^{\otimes m}) \longrightarrow L_x^{\otimes m} \oplus L_y^{\otimes m}$, $\forall x, y \in X$
- $d_x : H^0(X, L^{\otimes m} \otimes \mathcal{I}_x) \to L_x^{\otimes m} \otimes T_x^* X$, $\forall x \in X$.

- Use blow up construction: $\underset{\mu_x}{\overset{}{\mu}} : \widetilde{X} = Bl_x X \longrightarrow X$ with exceptional divisor E_x

to get:

$$L_x^{\otimes m} \oplus L_y^{\otimes m} \cong H^0(E_x, \mu^* L^{\otimes m}) \oplus H^0(E_y, \mu^* L^{\otimes m})$$

$$L_x^{\otimes m} \otimes T_x^* X \cong H^0(E_x, \mu^* L^{\otimes m} \otimes [-E_x]).$$

Blow up $0 \in \Delta^n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_i| < 1\}$.

$$\widetilde{\Delta}^n = Bl_0 \Delta^n = \{(z, [w]) \in \Delta^n \times \mathbb{P}^{n-1} : z \in \mathbb{C} \cdot w\}$$

$\cap$

total space of the tautological line bundle

$$\mathbb{C}^n \xleftarrow{\pi_1} \cdot \xrightarrow{\pi_2} \mathbb{P}^{n-1}.$$

$$\mu^{-1}(z) = \begin{cases} (z, [z]) & \text{if } z \neq 0 \\ \{(0, [w])\} = E \cong \mathbb{P}^{n-1} & \text{if } z = 0 \end{cases}$$

lines passing through 0.

$$\widetilde{X} = Bl_x X = (X \setminus \Delta^n_{\frac{1}{2}}) \cup Bl_0 \Delta^n$$

Similarly for $Bl_{x,y} X$, ...

• $H^0(E, \mu^* L^{\otimes m}|_E) = L_x^{\otimes m}$ because $\mu^* L^{\otimes m}|_E = \mu^* L_x$ is trivial and E is compact.

commutative diagram:

$$
\begin{tikzcd}
H^0(X, L^{\otimes m}) \arrow[r] \arrow[d, equal] & L_x^{\otimes m} \oplus L_y^{\otimes m} \arrow[d, equal] \\
H^0(\tilde{X}, \mu^* L^{\otimes m}) \arrow[r] & H^0(E_x, \pi^* L^{\otimes m}) \oplus H^0(E_y, \pi^* L^{\otimes m}) \arrow[d] \\
 & H^1(\tilde{X}, \mu^* L^{\otimes m} \otimes [-E_x] \otimes [-E_y])
\end{tikzcd}
$$

associated to

sheaf of hol. sections vanishing along $E_x \cup E_y$

$$0 \to \mathcal{O}(\mu^* L^{\otimes m})[-E_x - E_y] \to \mathcal{O}(\mu^* L^{\otimes m}) \to \mathcal{O}(\mu^* L^{\otimes m})|_{E_x} \oplus \mathcal{O}(\mu^* L^{\otimes m})|_{E_y} \to 0$$

$\Rightarrow$ sufficient to prove vanishing: $H^1(\tilde{X}, \mu^* L^{\otimes m} \otimes [-E_x] \otimes [-E_y]) = 0$

Want to use Kodaira's Vanishing Thm:

If A is ample, then $H^q(\tilde{X}, K_{\tilde{X}} \otimes A) = 0$ for $q > 0$.

Write:

$$\mu^* L^{\otimes m} \otimes [-E_x] \otimes [-E_y]$$
$$= K_{\tilde{X}} \otimes K_{\tilde{X}}^{-1} \otimes \mu^* L^{\otimes m} \otimes [-E_x] \otimes [-E_y]$$

(Assume the canonical bundle formula:
$$K_{\tilde{X}} = \mu^* K_X \otimes [E]^{\otimes n-1}$$
)

$$= K_{\tilde{X}} \otimes \mu^*(K_X \otimes L^m) \otimes [-E_x]^{\otimes n} \otimes [-E_y]^{\otimes n}$$

$$= K_{\tilde{X}} \otimes \mu^*(K_X \otimes L^{m_1}) \otimes (\mu^* L^{m-m_1} \otimes [-E_x]^n \otimes [-E_y]^{\otimes n}).$$

just need to show that $\mu^* L^m \otimes [-E_x] \otimes [-E_y]$ is positive for $m \gg 1$.

- pf of canonical bundle formula

Assume X has a meromorphic n form ω which is locally given by $\omega = f(z)\, dz_1 \wedge \cdots \wedge dz_n$.

Assume $0 = x \in \Delta^n$, over $\pi_2^{-1}\{w_1 \neq 0\} \cap \tilde{X} = U_0$

$$(z, [w]) = (z, [1, u_2, \cdots, u_n]) \in \tilde{X} \text{ iff } z \in \mathbb{C}(1, u)$$

$$u_2 = \frac{w_2}{w_1}, \quad u_n = \frac{w_n}{w_1}$$

$$z_2 = z_1 u_2,\ z_3 = z_1 u_3,\ \cdots,\ z_n = z_1 \cdot u_n$$

$\Rightarrow (z_1, u_2, \cdots, u_n)$ are coordinates of U_0

$$\Rightarrow \mu^* \omega = (\mu^* f) \cdot dz_1 \wedge d(z_1 u_2) \wedge \cdots \wedge d(z_1 u_n)$$

$$= (\mu^* f)\, dz_1 \wedge z_1\, du_2 \wedge \cdots \wedge z_1\, du_n$$

$$= \mu^* f\, z_1^{n-1}\, dz_1 \wedge du_2 \wedge \cdots \wedge du_n$$

$\Rightarrow$ over U_0, the divisor defined by the meromorphic

n-form $\mu^* \omega$ is equal to

$$\mathrm{div}(\mu^* f) + (n-1)\{z_1 = 0\} = \mathrm{div}(f^* \omega)$$

$$\mathrm{div}(\mu^* f) = \mathrm{div}(\omega) + (n-1)E.$$

(By Symmetry) This holds over the whole $\tilde{X}$

$$\Rightarrow \quad K_{\tilde{X}} = K_X \otimes [E]^{\otimes n}$$

$$K_{\tilde{X}} = [\mathrm{div}(f^*\omega)], \quad K_X = [\mathrm{div}(\omega)], \quad [E]^{\otimes n} = [(n-1)E]$$

- construction of Hermitian metric on $\mu^* L^{\otimes m} \otimes [-E]$ with positive Chern curvature.

Note that:

$$[E]|_E = N_{E|\tilde{\Delta}^n} = [-H_{\mathbb{P}^{n-1}}]$$

tautological line bundle

$$\Rightarrow [-E]|_E = N^*_{E|\tilde{\Delta}^n} = [H_{\mathbb{P}^{n-1}}]$$

Moreover

$$[E]|_{\tilde{\Delta}^n} = \pi_2^*[-H_{\mathbb{P}^{n-1}}] \Rightarrow [-E]|_{\tilde{\Delta}^n} = \pi^* H_{\mathbb{P}^{n-1}}$$

So over $\mu^{-1}(\Delta^n)$, the pullback of the canonical Hermitian metric on $H_{\mathbb{P}^{n-1}}$ is a Hermitian metric on $[-E]|_{\tilde{\Delta}^n}$ whose curvature is semipositive and strictly positive in the direction tangent to E.

By using partition of unity, this Hermitian metric extends to a Hermitian metric on $[-E]$ over $\tilde{X}$ with the same positivity property hold in a smaller nbhd. of E.

Because L is positive, $\mu^* L^{\otimes m}$ has a Herm. metric whose curvature is positive away from E and, near E, positive in the transverse direction to E.

By choosing $m \gg 1$, we can then get a Herm. metric on $\mu^* L^{\otimes m} \otimes [E]$ with positive curvature (because the negativity of $[-E]$ which is away from E can be dominated by the positivity of $L^{\otimes m}$ away from E)

Combining the above discussion (applied to $Bl_{x,y}X$)

We indeed get the surjectivity:

$$H^0(X, L^{\otimes m}) \to L_x^{\otimes m} \oplus L_y^{\oplus m}.$$

- For the 2nd surjectivity:

There is a morphism

$$L_x^{\otimes m} \otimes T_x^* \Delta^n \longrightarrow H^0(E, \overset{\overset{H_{\mathbb{P}^{n-1}}}{\cong}}{\mu^* L^{\otimes m}} \otimes [-E]) \cong \mathbb{C}^n$$

$$e \otimes \sum_i \lambda_i \, df_i \longmapsto e \otimes \sum_i \lambda_i \cdot w_i$$

which is an isomorphism

and commutative diagram:

$$\begin{tikzcd}
H^0(X, L^{\otimes m} \otimes I_x) \arrow[r] \arrow[d, equal] & L_x^{\otimes m} \otimes T_x^* X \arrow[d, "\cong"] \\
H^0(\widetilde{X}, \mu^* L^{\otimes m}(-E)) \arrow[r] & H^0(E, \mu^* L^{\otimes m} \otimes [-E]) \arrow[d] \\
& H^1(\widetilde{X}, \mu^* L^{\otimes m} \otimes [-2E])
\end{tikzcd}$$

associated to

$$0 \to \mathcal{O}(\mu^* L^{\otimes m})(-2E) \to \mathcal{O}(\mu^* L^{\otimes m}(-E)) \to \mathcal{O}(\mu^* L^{\otimes m}(-E))\big|_E \to 0$$

Sufficient to prove: for $m \gg 1$

$H^1(\tilde{X}, \mu^* L^{\otimes m}(-2E)) = 0$ which can be proved by the similar argument as before.

Finally because the 2 conditions of surjectivity (corresponding to injectivity & immersion) are open condition w.r.t. $x, y \in X$, there is an $m \gg 1$ such that they are satisfied by all $x, y \in X$ thanks to the compactness of X.