Kodaira Embedding Theorem

Thm: X cpt. complex mfd.

X is a projective manifold (i.e. $\exists$ an embedding $\iota: X \hookrightarrow \mathbb{P}^N$)

if and only if $\exists$ a positive line bundle L over X.

- L positive means: $\exists$ Hermitian metric h s.t. $\sqrt{-1}\bar\partial\partial\log h > 0$.
- "only if" is immediate: $H \to \mathbb{P}^N$ the hyperplane bundle over $\mathbb{P}^N$ is positive: H is dual to the tautological line bundle $H^* = \{[z], \mathbb{C}z\}$

Hermitian metric h_1 on H^*: $s(z) = (1, z) \in H^*|_{\mathbb{C}^n}$ local trivialization

$|s|_{h_1}^2 = 1+|z|^2 \Rightarrow \sqrt{-1}\bar\partial\partial\log h_1 = \sqrt{-1}\bar\partial\partial\log(1+|z|^2) = -\sqrt{-1}\partial\bar\partial\log(1+|z|^2) < 0$

$\Rightarrow$ (induced) dual metric h on H has positive curvature

$\Rightarrow$ $H|_X \to X$ is a positive line bundle.

- Chow's thm: Any closed analytic subsets of $\mathbb{P}^N$ is algebraic. (i.e. defined as zero sets of a family of polynomials)

Serre's GAGA principle: over projective manifolds.

$\{$ meromorphic functions $\} \longleftrightarrow \{$ rational functions (i.e. quotients of polynomials) $\}$

$\{$ holomorphic vector bundles $\} \longleftrightarrow \{$ algebraic vector bundles $\}$

$\rightsquigarrow$ any global analytic object on a projective variety is algebraic.

$\rightsquigarrow$ can use analytic techniques to study algebraic varieties.

- Proof of the "if" direction: More precisely, Kodaira proved:

There exists $m \gg 1$, s.t. there is an embedding given by:

$$\begin{aligned} L: \quad X &\longrightarrow \mathbb{P}^N \\ p &\longmapsto [s_0(p), \cdots, s_N(p)] = \left[\frac{s_0}{e}(p), \cdots, \frac{s_N}{e}(p)\right] \end{aligned}$$

where $\{s_i\}_{i=0}^N$ is a basis of $H^0(X, L^{\otimes m})$, and e is a local holomorphic trivializing section of L near P.

Need to show: (1.) $Bs(L^{\otimes m}) = \bigcap_{i=0}^{N} \{s_i = 0\}$ is empty
$\uparrow$ base point locus (i.e. base point free)

(2). L is injective: $\forall x \neq y \in X$, $L(x) \neq L(y)$.

Both (1) and (2) are implied by the following condition: $\exists m$, s.t. $\forall x, y \in X$.

$$ev_{x,y}: H^0(X, L^{\otimes m}) \longrightarrow L_x^{\otimes m} \oplus L_y^{\otimes m}$$
$$s \mapsto (s(x), s(y))$$

To see this, note the following fact:

$L = L_{\{s_i\}}$ depends on the basis $\{s_i\}$, but if we choose a different basis $\{\tilde{s}_i\}$, there exists an invertible matrix $T \in GL(N+1, \mathbb{C})$ s.t. $(\tilde{s}_i) = (s_i) \cdot T$. This induces automorphism T of $\mathbb{P}^N$ s.t.

$$\begin{tikzcd} X \arrow[r, "L_{\{s_i\}}"] \arrow[dr, "L_{\{\tilde{s}_i\}}"'] & \mathbb{P}^N \arrow[d, "T"] \\ & \mathbb{P}^N \end{tikzcd}$$

So the condition of being an embedding does not depend on the choice of the basis $\{s_i\}$

Now if $(*)$ is surjective, then $\exists$ s_0 s.t. $s_0(x) \neq 0$, $s_0(y) = 0$
s_1 s.t. $s_1(x) = 0$, $s_1(y) \neq 0$

$\rightsquigarrow$ s_0, s_1 are linearly independent, and $Bs(L^{\otimes m}) = \phi$

$\rightsquigarrow$ extend to a basis $\{s_0, s_1, \cdots, s_N\}$ of $H^0(X, L^{\otimes m})$.

$$\rightsquigarrow \quad \underset{\overset{\|}{L_{\{s_i\}}(x)}}{L(x)} = [s_0(x), s_1(x), \dots, s_N(x)] = \left[\tfrac{s_0}{s_0}(x), \tfrac{s_1}{s_0}(x), \dots, \tfrac{s_N}{s_0}(x)\right] = [1, 0, \dots]$$

$\begin{pmatrix} s_0 : \text{trivializing near } x \\ s_1 : \quad - \quad - \quad y \end{pmatrix}$

$$L(y) = [s_0(y), s_1(y), \dots, s_N(y)] = \left[\tfrac{s_0}{s_1}(y), \tfrac{s_1}{s_1}(y), \dots, \tfrac{s_N}{s_1}(y)\right] = [0, 1, \dots,]$$

$$\Rightarrow \quad L(x) \neq L(y).$$

Now $ev_{x,y}$ is a morphism in the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{O}(L^{\otimes m}) \otimes \mathcal{I}_{x,y} \to \mathcal{O}(L^{\otimes m}) \longrightarrow L^m_x \oplus L^m_y \to 0$$

where $\mathcal{I}_{x,y}$ = sheaf of holomorphic functions vanishing at x and y

$L^m_x = L^{\otimes m}_x$ is the (skyscraper) sheaf that assigns:

$$L^{\otimes m}_x : \underset{\substack{\uparrow \\ \text{open set}}}{U} \longmapsto \begin{cases} L^{\otimes m}_x & x \in U \\ 0 & x \notin U. \end{cases}$$

$$\rightsquigarrow \quad H^0(\mathcal{O}(L^{\otimes m})) \to \underset{\substack{\cong \\ \mathbb{C} \oplus \mathbb{C}}}{L^{\otimes m}_x \oplus L^{\otimes m}_y} \xrightarrow{ev_{x,y}} H^1(\mathcal{O}(L^{\otimes m}) \otimes \mathcal{I}_{x,y})$$

$\Rightarrow$ $ev_{x,y}$ is surjective if there is a vanishing:

$$H^1(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{I}_{x,y}) = 0 \quad (*)$$

(3) $\forall x \in X$, $d\iota : T_xX \to T_{\iota(x)}\mathbb{P}^N$ is injective

To translate this condition: Choose $s_0 \in H^0(X, L^{\otimes m})$ with $s_0(x) \neq 0$.

Choose basis for the codim 1 subspace $V = \mathrm{Ker}(H^0(X, L^{\otimes m}) \to L_x^{\otimes m})$

$\{s_1, \cdots, s_N\}$

$\rightsquigarrow$ a basis $\{s_0, s_1, \cdots, s_N\}$ of $H^0(X, L^{\otimes m})$

write $s_i(z) = s_0(z) \cdot f(z)$ with holomorphic functions f_i near x

$\Rightarrow$ ι is represented by vector valued holomorphic function:

$$\iota(z) = \underset{\mathbb{P}^N}{\underset{\pitchfork}{[1, f_1(z), \cdots, f_N(z)]}} = \underset{F(z)}{\underset{\shortparallel}{(f_1(z), \cdots, f_N(z))}} \in \mathbb{C}^N$$

Now $d\iota : \underset{T_0\mathbb{C}^n}{\underset{\cong}{T_xX}} \to \underset{T_0\mathbb{C}^N}{\underset{\cong}{T_{\iota(x)}\mathbb{P}^N}}$ is injective iff

$(d\iota)^* : T_0^*\mathbb{C}^N \to T_0^*\mathbb{C}^n$ is surjective.

$\qquad\qquad \shortparallel$

$\sum_i \lambda_i dw_i \mapsto \sum_i \lambda_i df_i|_0$

Consider a morphism: $H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{I}_x) \xrightarrow{d} T_x^*X \otimes L_x^{\otimes m}$

$$\underset{\substack{\shortparallel \\ f \cdot e^m \text{ with } f(0)=0}}{s} \longmapsto ds|_x := df|_{T_xX} \otimes e_x^m$$

$\uparrow$ local trivialization

This is well defined:

if $s = \tilde{f}\cdot\tilde{e}^m = f\cdot e^m \Rightarrow \tilde{f} = f\,\frac{e^m}{\tilde{e}^m} = f\cdot g$ with $\tilde{f}(0)=f(0)$ and $g \in \mathcal{O}^*$

$$\Rightarrow d\tilde{f}|_{T_xX} = df\cdot g + f\cdot dg|_{T_xX} = df|_{T_xX}\cdot g(x)$$

$$\Rightarrow d\tilde{f}|_{T_xX} \otimes \tilde{e}^m = df|_{T_xX}\, g(x) \otimes \tilde{e}^m = df|_{T_xX} \otimes e^m$$

Note that $H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{I}_x) = V = \ker(ev_x)$.

$$\cup \qquad\qquad \| $$

$$\mathrm{Span}\{s_1, \cdots, s_N\}.$$

$$s = \sum_{i=1}^N \lambda_i s_i \xmapsto{d} \sum_{i=1}^N \lambda_i\, df_i|_x \otimes s_0|_x$$

$$\| \qquad\qquad\qquad \|$$

$$\left(\sum_{i=1}^N \lambda_i \frac{s_i}{s_0}\right)\cdot s_0 \qquad (dl)^*\left(\sum_{i=1}^N \lambda_i w_i\right) \otimes s_0|_x$$

$$\|$$

$$\sum_{i=1}^N \lambda_i f_i \cdot s_0$$

So $(dl)^*$ is surjective iff

$$d: H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{I}_x) \to T_x^*X \otimes L_x^{\otimes m}$$

is surjective.

The morphism $d.$ is a morphism in the long exact sequence associated to the short exact sequence of sheaves:

$$0 \to \mathcal{O}(L^m) \otimes I_x^2 \longrightarrow \mathcal{O}(L^m) \otimes I_x \longrightarrow T_x^* X \otimes L_x^{\otimes m} \to 0$$

where I_x^2 is the sheaf of holomorphic functions that vanish at x to the second order:

$$I_x^2 = \{ f(0) = \frac{\partial f}{\partial z_i}(0) = 0 ,\ i = 1, \cdots, n \} \text{ where } z_i(x) = 0,\ i = 1, \cdots, n$$

$$\rightsquigarrow \quad H^0(\mathcal{O}(L^{\otimes m}) \otimes I_x) \xrightarrow{d.} T_x^* X \otimes L_x^m \to H^1(\mathcal{O}(L^m) \otimes I_x^2)$$

$\Rightarrow$ $d.$ is surjective if there is a vanishing:

$$H^1(X, \mathcal{O}(L^{\otimes m}) \otimes I_x^2) = 0 \qquad (**)$$

So the proof of Kodaira Embedding Thm is reduced to the proof of vanishing's $(*)$ and $(**)$, which needs the construction of blow up and Kodaira's Vanishing Thm.