Lefschetz hyperplane thm. by Morse theory.

$Y^{n-1} \subset X^n$ cpt. $\quad [Y]$ is positive
$$\overset{\|}{L}$$

$\rightsquigarrow$ Hermitian metric h on L, holomorphic section s_Y of L.

Consider the function $f = \log |s_Y|_h^2$.

At critical point $p \in X \setminus Y$ of f: $df(p) = 0$

complex Hessian matrix: $\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)_p < 0$ negative definite.

Real Hessian: $\quad \text{Hess} f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_i \partial x_j} & \frac{\partial^2 f}{\partial x_i \partial y_j} \\ \frac{\partial^2 f}{\partial y_i \partial x_j} & \frac{\partial^2 f}{\partial y_i \partial y_j} \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$

$$\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \overset{z_i = x_i + \sqrt{-1} y_i}{=\!=\!=} \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)\frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right) f$$
$$= \frac{1}{4}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial^2 f}{\partial y_i \partial y_j} + \sqrt{-1}\left(\frac{\partial^2 f}{\partial x_i \partial y_j} - \frac{\partial^2 f}{\partial y_i \partial x_j}\right)\right)$$
$$= \frac{1}{4}\left(A_{ij} + C_{ij} + \sqrt{-1}\cdot(B_{ij} - B_{ji})\right)$$

$\frac{1}{4} M$
$$\overset{\|}{\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)} = \frac{1}{4}\cdot\left(A + C + \sqrt{-1}(B - B^T)\right) \text{ negative definite}$$

$\Rightarrow \exists\ v_i = \xi_i + \sqrt{-1}\,\eta_i,\ i = 1, \cdots, n$ linearly independent over $\mathbb{C}$ s.t.
$$\updownarrow$$
$$\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \in \mathbb{R}^{2n}$$

$$v_i^* M v_i = (\xi_i^T - \sqrt{-1}\,\eta_i^T)\cdot(A + C + \sqrt{-1}(B - B^T))(\xi_i + \sqrt{-1}\,\eta_i)$$

$$= \xi_i^T (A+C)\xi_i + \eta_i^T (A+C)\eta_i - \xi_i^T (B-B^T)\eta_i + \eta_i^T (B-B^T)\xi_i$$

$$= \xi_i^T A \xi_i - \xi_i^T B \eta_i - \eta_i^T B^T \xi_i + \eta_i^T C \eta_i + \eta_i^T A \eta_i + \eta_i^T B \xi_i + \xi_i^T B^T \eta_i + \xi_i^T C \xi_i$$

$$= \mathrm{Hess}(f)(u_i, u_i) + \mathrm{Hess}(f)(w_i, w_i) < 0$$

where $u_i = \begin{pmatrix} \xi_i \\ -\eta_i \end{pmatrix}$, $w_i = \begin{pmatrix} \eta_i \\ \xi_i \end{pmatrix} \in \mathbb{R}^{2n}$

$\Longrightarrow$ $\mathrm{Hess}(f)(u_i, u_i) < 0$ or $\mathrm{Hess}(f)(w_i, w_i) < 0$

Note that $\{u_i, w_i ; i = 1, \dots, n\}$ are linearly independent

$\Rightarrow$ $\mathrm{Hess}(f)$ has at least n negative eigenvalues

$\Rightarrow$ X is obtained from Y by attaching cells of dimension $\geq n$ (Morse theory)

$\Longrightarrow$ Lefschetz hyperplane Thm with $\mathbb{Z}$-coeff.

Lefschetz (1,1)-Thm.

X projective manifold

cycle class map $\eta : \{\text{divisors}\} \longrightarrow H^2(X,\mathbb{Z})$

$$D = \sum_i a_i D_i \longmapsto \sum_i a_i \eta_{D_i}$$

<u>Thm</u>: $\mathrm{Im}(\eta) = H^2(X,\mathbb{Z}) \cap \partial^{-1}(H^{1,1})$

Here
$$\begin{tikzcd} H^2(X,\mathbb{Z}) \arrow[r] \arrow[dr, "\partial"'] & H^2(X,\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \arrow[d, "\pi^{1,1}"] \\ & H^{1,1} \end{tikzcd}$$

<u>Pf</u>: Facts:

- $$\left.\begin{tikzcd} D \arrow[r, mapsto] \arrow[d, squiggly] & \eta_D \in H^2(X,\mathbb{Z}) \arrow[d, equal] \arrow[d] \\ {[D]} \arrow[r, mapsto] & c_1([D]) \in H^2(X,\mathbb{R}) \end{tikzcd}\right\}$$

- $\{\eta_D : D \text{ divisors}\} = \{c_1(L) : L \text{ holomorphic line bdle}\}$

any holomorphic line bundle has a meromorphic section

$\rightsquigarrow$ $L = [D]$ for some divisor D.

(depends on the projectivity of X $\rightsquigarrow$ positive line bundle $H|_X$ restriction of hyperplane line bundle.)

- $H^1(X, \mathcal{O}^*) = \{\text{holomorphic line bundles}\}$

$$\{ g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta) : g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1 \}$$

Consider the exponential sequence:

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

$\rightsquigarrow$

$$\begin{tikzcd}
H^1(X, \mathcal{O}_X^*) \arrow[r, "\delta"] & H^2(X, \mathbb{Z}) \arrow[r] \arrow[d, "\iota"] & H^2(X, \mathcal{O}_X) \arrow[d, phantom, "\cong \text{ Dolbeault}"] \\
 & H^2(X, \mathbb{R}) \arrow[d] \arrow[r, phantom, "\circlearrowright"] & H^{0,2}_{\bar\partial}(X) \\
 & H^2_{dR}(X, \mathbb{C}) \arrow[ur, "\pi^{0,2}"] &
\end{tikzcd}$$

$H^2(X,\mathbb{R}) \cong H^2_{dR}(X, \mathbb{R})$

$H^2_{dR}(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$

$\boxed{\text{Fact: } \iota \circ \delta = c_1}$

Note that $H^2(X,\mathbb{R}) \cap \ker \pi^{0,2} = H^{1,1}$:

$$\pi^{0,2}(\underset{\substack{\shortparallel \\ \varphi^{2,0}+\varphi^{1,1}+\varphi^{0,2}}}{\varphi}) = \varphi^{0,2} = 0 \Rightarrow \varphi^{2,0} = \overline{\varphi^{0,2}} = 0$$

$$\Rightarrow \varphi = \varphi^{1,1} \in H^{1,1}.$$

Just need to show that the diagram is commutative.

need to find the image under isomorphism element in sheaf cohomology: $H^2(X,\mathbb{R}) \cong H^2_{dR}(X)$

$a = \{a_{\alpha\beta\gamma} \in \mathbb{Z}\}$, $\delta a = 0$.

The isomorphism is obtained by using the resolution:

$$0 \to \mathbb{R} \to A^0 \to A^1 \to A^2 \to \cdots$$

Set $\mathcal{L}^i = \ker(A^i \to A^{i+1})$. Then:

$$0 \to \mathbb{R} \to \mathcal{A}^0 \to \mathcal{L}^1 \to 0$$

$$0 \to \mathcal{L}^1 \to \mathcal{A}^1 \to \mathcal{L}^2 \to 0$$

$\mathcal{A}^i$ is fine sheaf.

$$\rightsquigarrow \quad H^1(X,\mathcal{L}^1) \xrightarrow{\cong} H^2(X,\mathbb{R}) \xrightarrow{\mu} H^2(X,\mathcal{A}^0) = 0$$

$$H^0(X,\mathcal{A}^1) \xrightarrow{d} H^0(X,\mathcal{L}^2) \to H^1(X,\mathcal{L}^1) \to H^1(X,\mathcal{A}^1) = 0$$

$(a_{\alpha\beta\gamma}) \in H^2(X,\mathbb{R}) \Rightarrow \mu(a_{\alpha\beta\gamma}) = 0$ in $H^2(X,\mathcal{A}^0)$

$\Rightarrow \quad a_{\alpha\beta\gamma} = b_{\alpha\beta} - b_{\beta\gamma} + b_{\gamma\alpha}$ with $b_{\alpha\beta} \in \mathcal{A}^0(U_\alpha \cap U_\beta)$

$\Rightarrow \quad 0 = d a_{\alpha\beta\gamma} = db_{\alpha\beta} - db_{\beta\gamma} + db_{\gamma\alpha}$

$\Rightarrow \quad db_{\alpha\beta} \in Z^1(\mathcal{A}^1)$

$\Rightarrow$ because $H^1(\mathcal{A}^1) = 0$, $db_{\alpha\beta} = \omega_\alpha - \omega_\beta$
with $\omega_\alpha \in \mathcal{A}^1(U_\alpha)$

$\Rightarrow \quad 0 = d\omega_\alpha - d\omega_\beta \Rightarrow [d\omega_\alpha] \in \dfrac{H^0(X,\mathcal{L}^2)}{dH^0(X,\mathcal{A}^1)} \quad \left(\left(H^2_{dR}(X)\right)\right)$

Similarly, the image of $(a_{\alpha\beta\gamma}) \in H^2(X, \mathcal{O})$ under the isomorphism $H^2(X,\mathcal{O}) = H^{0,2}_{\bar\partial}(X)$ is obtained by:

$$a_{\alpha\beta\gamma} = b_{\alpha\beta} - b_{\beta\gamma} + b_{\gamma\alpha}, \quad b_{\alpha\beta} \in A^0(U_\alpha \cap U_\beta)$$

$$\Rightarrow \quad 0 = \bar\partial b_{\alpha\beta} - \bar\partial b_{\beta\gamma} + \bar\partial b_{\gamma\alpha}$$

$$\Rightarrow \quad \bar\partial b_{\alpha\beta} \in Z^1(A^{0,1})$$

$\Rightarrow$ because $H^1(A^{0,1}) = 0$, $\bar\partial b_{\alpha\beta} = \omega_\alpha^{0,1} - \omega_\beta^{0,1}$

with $\omega_\alpha^{0,1} \in A^{0,1}(U_\alpha)$

$$\Rightarrow \quad 0 = \bar\partial \omega_\alpha^{0,1} - \bar\partial \omega_\beta^{0,1}$$

$$\Rightarrow \quad [\bar\partial \omega_\alpha^{0,1}] \in \frac{\ker(\bar\partial : H^0(A^{0,2}) \to H^0(A^{0,3}))}{\bar\partial H^0(X, A^{0,1})}.$$

Now the commutativity follows:

$$\pi^{0,2}([d\omega_2]) = [\bar{\partial}\omega_2].$$