Kodaira – Nakano vanishing:

X cpt. complex mfd.

$\begin{array}{c} L \\ \downarrow \\ X \end{array}$ a positive line bundle : $\exists$ hermitian metric h s.t. its Chern curvature $\frac{\sqrt{-1}}{2\pi} F_h = \frac{\sqrt{-1}}{2\pi} \bar\partial\partial \log h > 0$ i.e. $(g_{i\bar j}) > 0$ positive definite

$\frac{\sqrt{-1}}{2\pi} \bar\partial\partial \log h = \sqrt{-1} \sum_{i,j} g_{i\bar j} dz_i \wedge d\bar z_j$

Then $H^q(X, \Omega^p \otimes L) = 0$ if $p+q > n$.

Pf: By Dolbeault Thm, $H^q(X, \Omega^p \otimes L) \cong H^{p,q}_{\bar\partial}(X, L)$.

By Hodge Thm, $H^{p,q}_{\bar\partial}(X, L) = \mathcal{H}^{p,q}_{\Delta_{\bar\partial}}(L)$.

$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$

hermitian metric $h \rightsquigarrow$ Chern connection $D: A(L) \to A^1(L)$

Decomposition $D = D' + D'' = D' + \bar\partial$ where

$D': A(L) \to A(T^{*(1,0)}X \otimes L)$, $D'' = \bar\partial: A(L) \to A(T^{*(0,1)} \otimes L)$

$\rightsquigarrow D': A^{p,q}(L) \to A^{p+1,q}(L)$, $D'' = \bar\partial: A^{p,q}(L) \to A^{p,q+1}(L)$

$\eta \otimes s \mapsto \partial\eta \otimes s + (-1)^{\deg \eta} D's$, $\quad \eta \otimes s \mapsto \bar\partial\eta \otimes s + (-1)^{\deg\eta} \bar\partial s$

Curvature: $F_h = D^2 = (D' + \bar\partial)(D' + \bar\partial) = D'^2 + (D'\bar\partial + \bar\partial D') + \underset{0}{\underset{\shortparallel}{\bar\partial^2}}$

Under holomorphic trivialization $F_h = d\theta - \theta\wedge\theta = \bar\partial(\partial h \cdot h^{-1}) \in A^{1,1}(\mathrm{End}\, L) \cong A^{1,1}$

$\Rightarrow D'^2 = 0 \Rightarrow F_h = D'\bar\partial + \bar\partial D'$.

$\frac{\sqrt{-1}}{2\pi} F_h = \omega > 0$ is a Kähler metric.

$\rightsquigarrow L^2$ inner product on $A^{p,q}(L)$

Lefschetz operator $L = \omega \wedge \cdot$, adjoint $\Lambda = L^*: A^{p,q}(L) \to A^{p-1,q-1}(L)$

Claim: Generalized Kähler identities for holomorphic vector bundle

$$[\Lambda, \bar\partial] = -\sqrt{-1}\, D'^*, \quad [\Lambda, D'] = \sqrt{-1}\, \bar\partial^*.$$

$$[L, \bar\partial^*] = -\sqrt{-1}\, D', \quad [\Lambda, D'^*] = \sqrt{-1}\, \bar\partial.$$

<u>Pf</u>: Just need to prove $[\Lambda, \bar\partial] = -\sqrt{-1}\, D'^*$:

choose local frames s_i (not necessarily holomorphic)

$$\Lambda \bar\partial \sum_i \eta_i s_i = \Lambda \sum_i \bar\partial \eta_i \cdot s_i + \theta'' \wedge \eta_i \wedge s_i$$

$$= \sum_i \Lambda \bar\partial \eta_i \cdot s_i + \Lambda(\theta'' \cdot \eta_i \cdot s_i)$$

$$\bar\partial \Lambda \sum_i \eta_i s_i = \bar\partial \sum_i \Lambda \eta_i \cdot s_i$$

$$= \sum_i \bar\partial \Lambda \eta_i \cdot s_i + \sum_i \theta'' \cdot \Lambda \eta_i s_i$$

$$\Rightarrow [\Lambda, \bar\partial] \sum_i \eta_i s_i = \sum_i [\Lambda, \bar\partial] \eta_i \cdot s_i + [\Lambda, \theta''] \eta_i s_i$$

$$= \sum_i (-\sqrt{-1}\, \partial^* \eta_i) \cdot s_i + [\Lambda, \theta''] \eta_i \cdot s_i$$

$$D'^* \sum_i \eta_i s_i = \sum_i \partial^* \eta_i \cdot s_i + \theta'^* \eta_i s_i$$

$$\Rightarrow [\Lambda, \bar\partial] + \sqrt{-1}\, D'^* = [\Lambda, \theta''] + \sqrt{-1}\, \theta'^*$$

is globally defined independent of (s_i)

choose s_i s.t. $\theta(p) = 0 \Rightarrow$ right-hand-side $= 0$.

Compare $\Delta_{\bar\partial}$ and $\Delta_{D'}$:

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$
$$= \bar\partial \cdot \frac{[\Lambda, D']}{\sqrt{-1}} + \frac{[\Lambda, D']}{\sqrt{-1}}\bar\partial = \frac{1}{\sqrt{-1}}\left(\bar\partial\Lambda D' - \bar\partial D'\Lambda + \Lambda D'\bar\partial - D'\Lambda\bar\partial\right)$$

$$\Delta_{D'} = D'D'^* + D'^*D'$$
$$= D'\frac{[\Lambda, \bar\partial]}{-\sqrt{-1}} + \frac{[\Lambda, \bar\partial]}{-\sqrt{-1}}D' = \frac{1}{\sqrt{-1}}\left(-D'\Lambda\bar\partial + D'\bar\partial\Lambda - \Lambda\bar\partial D' + \bar\partial\Lambda D'\right)$$

$$\Rightarrow \Delta_{\bar\partial} - \Delta_{D'} = \frac{1}{\sqrt{-1}}\left(\Lambda(D'\bar\partial + \bar\partial D') - (\bar\partial D' + D'\bar\partial)\Lambda\right)$$
$$= \frac{1}{\sqrt{-1}}\left(\Lambda \cdot \frac{2\pi}{\sqrt{-1}}\omega\wedge - \frac{2\pi}{\sqrt{-1}}\omega\wedge \cdot \Lambda\right) = 2\pi \cdot \bigoplus_{p,q}(p+q-n)\Big|_{A^{p,q}(L)}$$
$$= -2\pi \cdot [\Lambda, L] = -2\pi \bigoplus_{p,q}(n-(p+q))\Big|_{A^{p,q}(L)}$$

Let $\varphi \in \mathcal{H}^{p,q}_{\Delta_{\bar\partial}}$. then

$$0 = \Delta_{\bar\partial}\varphi = \Delta_{D'}\varphi + (p+q-n)\varphi$$
$$\Rightarrow \quad 0 = (\Delta_{D'}\varphi, \varphi) + (p+q-n)(\varphi, \varphi)$$
$$= \|D'\varphi\|^2 + \|D'^*\varphi\|^2 + (p+q-n)\|\varphi\|^2$$

$$\Rightarrow \varphi = 0 \quad \text{if} \quad p+q > n \Rightarrow H^q(X, \Omega^p \otimes L) \cong \mathcal{H}^{p,q}_{\Delta_{\bar\partial}} = 0.$$

Cor: $L \to X$ positive. Then

$$H^q(X, \Omega^p \otimes L^{-1}) = 0 \quad \text{if} \quad p+q < n$$

Pf: $H^q(X, \Omega^p \otimes L^{-1}) \cong H^{n-q}(X, \Omega^{n-p} \otimes L)$

$n-q+n-p > n \iff p+q < n$.

Serre duality: $H^q(X, \Omega^p \otimes L^{-1}) \times H^{n-q}(X, \Omega^{n-p} \otimes L) \to \mathbb{C}$

$$(\varphi, \psi) \mapsto \int \varphi \wedge \psi$$

is nondegenerate. because

$$(\varphi, *\overline{\varphi}) \mapsto \int \varphi \wedge *\overline{\varphi} = \|\varphi\|^2 > 0.$$

$$\overline{*}: A^{p,q}(L) \to A^{n-p,n-q}(L^{-1})$$

$$\varphi \mapsto *\overline{\varphi}$$

conjugate complex linear.

$\Rightarrow$ $\overline{*}: H^q(X, \Omega^p \otimes L^{-1}) \to H^{n-q}(X, \Omega^{n-p} \otimes L)$

is conjugate complex linear isomorphism.

<u>Cor</u> (Kodaira Vanishing) If L is positive

$$H^q(X, K_X \otimes L) = 0 \text{ for } q > 0.$$

(because Ω^n_X is the sheaf of holomorphic sections of $K_X = \Lambda^n T^* X$)

Lefschetz Hyperplane Thm.

X cpt. complex. $\iota: Y \subset X$ sm. hypersurface.

If the associated line bundle $[Y]$ is positive,

then $H^k(X, \mathbb{Q}) \xrightarrow{\iota^*} H^k(Y, \mathbb{Q})$ is

an isomorphism if $k \leq n-2$ and

injective if $k = n-1$.

(Dually: $\overset{\parallel m+2}{H^{2n-k}(X, \mathbb{Q})} \xleftarrow{\iota_*} \overset{\cong m}{H^{2(n-1)-k}(Y, \mathbb{Q})}$

is an isomorphism if $m \geq n$ and surjective if $m = n-1$.)

<u>Application</u>: $\alpha \in H^n(X)$ is primitive

$$\Leftrightarrow L\alpha = 0 \in H^{n+2}(X) \cong H^n(Y)$$

If $\alpha = \eta_V$ for an analytic cycle V, then

α is primitive $\Leftrightarrow \eta_V \cap \alpha = 0$ in

$$H^{n+2}(X) \cong H^n(Y).$$

$$\Leftrightarrow [V \cap \alpha] = 0 \text{ in } H_{n-1}(Y)$$

<u>Pf</u>: By Universal Coeff. Thm, just need to prove the statement for cohomology with $\mathbb{C}$-coeff.

Then we have the Hodge decomposition:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

$$H^k(Y, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(Y)$$

Just need to prove:

$$H^q(\Omega_X^p) \longrightarrow H^q(\Omega_Y^p)$$

is isomorphism if $p+q \leq n-2$ and injective if $p+q = n-1$.

The map $\Omega_X^p \to \Omega_Y^p$ is the composition:

$$\Omega_X^p \to \Omega_X^p|_Y \to \Omega_Y^p$$

- short exact sequences:

$$0 \to \Omega_X^p(-Y) \to \Omega_X^p \to \Omega_X^p|_Y \to 0$$

$$0 \to \Omega_Y^{p-1} \otimes N_Y^* \to \Omega_X^p|_Y \to \Omega_Y^p \to 0$$

(come from $0 \to N_Y^* \to T^*X \to T^*Y \to 0$)

$\rightsquigarrow$ long exact sequence

$\Rightarrow$ $H^q(\Omega_X^p) \to H^q(\Omega_X^p|_Y)$ are
$H^q(\Omega_X^p|_Y) \to H^q(\Omega_Y^p)$ isomorphisms

if $H^q(\Omega_X^p(-Y)) = 0 = H^{q+1}(\Omega_X^p(-Y))$

<u>and</u> $H^q(\Omega_Y^{p-1}(-Y)) = 0 = H^{q+1}(\Omega_Y^{p-1}(-Y))$

Because $[Y]$ is positive, these are true

if $p+q+1 < n$ (same as $q+1+(p-1) < n-1$).

Similarly, we get injectivity if

$p+q = n-1$ (same as $p-1+q = n-2$)

(while $H^{q+1}(\Omega_X^p(-Y))$ may not vanish).