Divisor $\longrightarrow$ line bundle

$D \longrightarrow [D]$ together with a meromorphic section s_D

divisor associated to s: $\mathrm{div}(s)$ $\longleftarrow$ meromorphic section s of line bundle

$\Rightarrow$ if $[D_1]=[D_2]$, then $f=\frac{s_{D_2}}{s_{D_1}}$ is a globally defined function

and $D_2 = D_1 + \mathrm{div}(f)$.

$$\rightsquigarrow \quad H^0(X, \mathcal{M}^*) \to H^0(X, \mathcal{M}^*/\mathcal{O}^*) \to H^1(X, \mathcal{O}^*).$$

$D|_{U_\alpha} = \mathrm{div}(f_\alpha)$ for $f_\alpha \in \mathcal{M}^*(U_\alpha)$

$\Rightarrow$ $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ is the transition function of $[D]$

$\Rightarrow$ $g_{\alpha\beta} = \frac{s_\beta}{s_\alpha}$ where s_α is a nonvanishing section of $[D]|_{U_\alpha}$

Example: tautological line bundle of $\mathbb{P}^n = \{\underbrace{[z_0, \cdots, z_n]}_{= [z]} : z \neq 0\}$

$$L = \{([z], v) : v \in \mathbb{C} z\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$$

Over $U_0 = \{z_0 \neq 0\}$, coordinates $w_1 = \frac{z_1}{z_0}$, $w_2 = \frac{z_2}{z_0}$, $\cdots$, $w_n = \frac{z_n}{z_0}$

nonvanishing holomorphic section: $s_0(w) = (1, w) \in \mathbb{C}z$ with $[1,w]=[z]$

Over $U_1 = \{z_1 \neq 0\}$, coordinates $u_1 = \frac{z_0}{z_1}$, $u_2 = \frac{z_2}{z_1}$, $\cdots$, $u_n = \frac{z_n}{z_1}$

nonvanishing holomorphic section: $s_1(u) = (u_1, 1, u_2, \cdots, u_n) \in \mathbb{C}\cdot z$

coordinate change: $w_1 = \frac{1}{u_1}$, $w_2 = \frac{u_2}{u_1}$, $\cdots$, $w_n = \frac{u_n}{u_1}$

s_0 extend to become a meromorphic section of L:

$$s_0|_{U_1}(u) = (1, w_1, \cdots, w_n) = \left(1, \frac{1}{u_1}, \frac{u_2}{u_1}, \cdots, \frac{u_n}{u_1}\right)$$
$$= \frac{1}{u_1}(u_1, 1, u_2, \cdots, u_n) = \frac{1}{u_1} s_1$$

$\Rightarrow$ $\operatorname{div}(s_0)|_{U_1} = -(u_1 = 0) = -H|_{U_1}$ where $H = \{z_0 = 0\} \underset{\shortparallel}{} \mathbb{P}^n \setminus \mathbb{C}^n$

Similarly $\operatorname{div}(s_0)|_{\underset{\{z_i \neq 0\}}{\overset{\shortparallel}{U_i}}} = -H|_{U_i} \Rightarrow \operatorname{div}(s_0) = -H$

$\Rightarrow$ $L = [-H]$.

- canonical bundle of X : $K_X = \Lambda^n T^* X$

Calculation of $K_{\mathbb{P}^n}$: Over $U_0 = \mathbb{C}^n$, nonvanishing section:

$$\tilde{s}_0 = dw_1 \wedge dw_2 \wedge \cdots \wedge dw_n$$

extend to meromorphic section with

$$\tilde{s}_0|_{U_1} = d\left(\frac{1}{u_1}\right) \wedge d\left(\frac{u_2}{u_1}\right) \wedge \cdots \wedge d\left(\frac{u_n}{u_1}\right)$$
$$= -\frac{du_1}{u_1^2} \wedge \frac{du_2}{u_1} \wedge \cdots \wedge \frac{du_n}{u_1} = -\frac{1}{u_1^{n+1}} \underbrace{du_1 \wedge du_2 \wedge \cdots \wedge du_n}_{\text{nonvanishing over } U_1}$$

$\Rightarrow$ $\operatorname{div}(\tilde{s}_0)|_{U_1} = -(n+1)H|_{U_1}$

similarly $\operatorname{div}(\tilde{s}_0)|_{U_i} = -(n+1)H|_{U_i}$ $\Rightarrow$ $\operatorname{div}(\tilde{s}_0) = -(n+1)H$

$\Rightarrow$ $K_{\mathbb{P}^n} = [-(n+1)H]$

anticanonical bundle: $K_X^{-1} = \Lambda^n TX = (\Lambda^n T^*X)^*$

$K_{\mathbb{P}^n}^{-1} = [(n+1)H]$.

- $Y \subset X$ a smooth hypersurface.

$Y \cap U_\alpha = \{f_\alpha = 0\} \Rightarrow [Y] = \{\underbrace{\tfrac{f_\alpha}{f_\beta}}_{= g_{\alpha\beta}}\} \in H^1(X, \mathcal{O}^*)$.

Exact sequence of holomorphic vector bundles:

$$0 \to TY \to TX|_Y \to \underset{\text{normal bundle}}{N_Y} \to 0$$

dual: $$0 \leftarrow T^*Y \leftarrow T^*X|_Y \leftarrow \underset{\text{conormal bundle}}{N_Y^*} \leftarrow 0$$

$N_Y^*|_{U_\alpha}$ has a nonvanishing section $df_\alpha|_Y =: s_\alpha$

transition function of N_Y^* from U_β to U_α is equal to $\frac{s_\beta}{s_\alpha}$:

$$f_\alpha = \frac{f_\alpha}{f_\beta} \cdot f_\beta \Rightarrow \underset{s_\alpha}{\underset{\parallel}{df_\alpha|_Y}} = \frac{f_\alpha}{f_\beta} \cdot \underset{s_\beta}{\underset{\parallel}{df_\beta|_Y}} \quad \text{since } f_\beta|_Y = 0$$

$$\Rightarrow \frac{s_\beta}{s_\alpha} = \frac{f_\beta}{f_\alpha} = g_{\alpha\beta}^{-1} \Rightarrow N_Y^* = [-Y]|_Y$$

$$\Rightarrow N_Y = [Y]|_Y.$$

- $\Lambda^n T^*X|_Y \cong \Lambda^n T^*Y \otimes N_Y^*$

$$\Rightarrow K_X|_Y = K_Y \otimes [-Y]_Y \Leftrightarrow K_X \otimes [Y]|_Y = K_Y$$

(adjunction formula).

- A divisor D is called principal if $D = \mathrm{div}(f)$ for a global meromorphic function $f \in \mathcal{M}^*(X)$.

The map $\dfrac{\{\text{divisors}\}}{\{\text{principal divisors}\}} \longrightarrow \left\{\begin{matrix}\text{holomorphic line}\\ \text{bundles}\end{matrix}\right\}$

is injective. When is it surjective?

<u>Prop1</u>: If X is a projective mfd, then the map is surjective.

To prove this, we need to construct meromorphic sections for any line bundle L.

Let H be the restriction of hyperplane bundle $H = [z_0 = 0]$ on X. It is enough to construct holomorphic sections of $L \otimes H^m$ and H^m for $m \gg 1$. Since $\frac{s'}{s}$ with $s' \in H^0(L \otimes H^m)$, $s \in H^0(H^m)$ would be a meromorphic section of L.

There are a lot of holomorphic sections for $L\otimes H^m$, H^m because H is positive:

<u>Def</u>: A line bundle L is positive if there exists a Hermitian metric h on L s.t. its normalized Chern curvature $R_h = \frac{\sqrt{-1}}{2\pi} F_h$ is a positive form.

More concretely, h is locally represented by positive function $|s|_h^2$ with s a nonvanishing hol. section

$$\Rightarrow \quad F_h = \bar{\partial}\left(\frac{\partial |s|_h^2}{|s|_h^2}\right) = \bar{\partial}\partial \log |s|_h^2 =: \bar{\partial}\partial \log h$$

$$= \sum_{i,j} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j \quad \text{with } g_{i\bar{j}} := -\frac{\partial^2 \log h}{\partial z_i \partial \bar{z}_j}.$$

Then positivity means that:

$(g_{i\bar{j}}) > 0$ i.e. a positive definite matrix at any point.

Ex: There is a natural Hermitian metric on the tautological line bundle on $\mathbb{P}^n$:

$$\underset{\substack{\shortparallel \\ (1,w)}}{\|s_0\|_h^2} = 1+|w|^2$$

$$\Rightarrow F_h = \bar\partial\partial \log(1+|w|^2) = -\partial\bar\partial \log(1+|w|^2)$$

$$\left(\frac{\partial^2}{\partial w_i \partial \bar w_j} \log(1+|w|^2)\right) = \left(\frac{\delta_{ij}}{1+|w|^2} - \frac{w_i \bar w_j}{(1+|w|^2)^2}\right) \text{ positive definite}$$

$\Rightarrow [-H]$ is negative and

$[H]$ is positive.

$$c_1([H]) = \left[\frac{\sqrt{-1}}{2\pi} F_{-h}\right] = \underset{\substack{\cap \\ H^2(X,\mathbb{R})}}{\left[\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log(1+|w|^2)\right]}$$

- $Y \subset X$, $L \to X$ positive $\Rightarrow L|_Y$ is positive

In particular, the restriction of hyperplane bundle to a projective manifold $H|_X$ is a positive line bundle over X.

- Kodaira–Nakano Vanishing Theorem:
If $A \to X$ is a positive line bundle over a compact complex manifold, then

$$H^q(X, \Omega^p \otimes A) = 0 \quad \text{for } p+q>n.$$

sheaf of holomorphic p-forms — sheaf of holomorphic sections of L

$\Rightarrow$ In particular $H^q(X, K_X \otimes A) = 0$ for $q>0$.

since Ω^n is the sheaf of holomorphic sections of K_X.

- If A is positive, then $L \otimes A^m$ is positive for $m \gg 1$. (for any fixed line bundle L) $\Rightarrow$

$$H^q(X, L \otimes A^m) = H^q(X, K_X \otimes K_X^{-1} \otimes L \otimes A^m) = 0$$

for $m \gg 1$.

- By Hirzebruch-Riemann-Roch thm:

$$\sum_{q=0}^{n} (-1)^q h^0(X, L\otimes A^m) = \int_X ch(L\otimes A^m)\, Td(TX)$$

$$= \int_X e^{c_1(L)} \cdot e^{m\cdot c_1(A)} \cdot Td(TX), \qquad Td(TX) = \left(1+\frac{c_1}{2}+\frac{c_1^2+c_2}{12}+\cdots\right)$$

$$= \frac{m^n}{n!} \underset{\vee \atop 0}{c_1(A)^n} + O(m^{n-1}).$$

$$\Rightarrow \quad h^0(X, L\otimes A^m) = \underbrace{\frac{m^n}{n!} c_1(A)^n + O(m^{n-1})}_{\gg 1}$$

$\Rightarrow$ $\exists$ (a lot of) sections of $L\otimes A^m$

$\Rightarrow$ Proposition 1: $\{\text{divisors}\} \to \{\text{line bundles}\}$ surjective for projective manifolds.