Lefschetz Decomposition:

X cpt. Kähler

$$H^k(X,\mathbb{C}) = \bigoplus_{r\geq 0} L^r P^{k-2r} \quad \text{where } P^{k-2r} = \operatorname{Ker}(\Lambda)\cap H^{k-2r}$$

Primitive cohomology.

$\rightsquigarrow$ representation of $sl_2(\mathbb{C})$ on $\bigoplus_{k=0}^{2n} H^k(X,\mathbb{C}) = V$:

$$sl_2(\mathbb{C}) = \left\{ \underset{X}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{Y}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{H}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \right\}$$

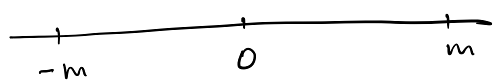
$$X \longmapsto \Lambda$$
$$Y \longmapsto L$$
$$H \longmapsto \bigoplus_{k=0}^{2n} (n-k)\pi_k$$

$V = \bigoplus_i V_i$. V_i irreducible

$V_i = \operatorname{span}\{v, Yv, \dots, Y^m v\}$ s.t. $Xv = 0$

$$0 \leftarrow \underset{V_i(\lambda-2m)}{Y^m v} \leftarrow \cdots \leftarrow Y^k v \cdots \leftarrow \underset{V_i(\lambda-2)}{Yv} \leftarrow \underset{V_i(\lambda)}{v}$$

$\lambda - 2m = -\lambda \Rightarrow \lambda = m \geq 0$



$L^k = Y^k : V_i(k) \longrightarrow V_i(-k)$ is an isomorphism for each irred. representation V_i.

$\rightsquigarrow$ Hard Lefschetz Thm:

$$L^k : \underset{\substack{\text{weight } k \\ = n-(n-k)}}{H^{n-k}(X)} \longrightarrow \underset{\text{weight } -k}{H^{n+k}(X)} \quad \text{is an isomorphism.}$$

$L^{n-i} \cong H^{2n-i}$

Corollary: For $i \leq n$, $L : H^i \to H^{i+2}$ is injective.

$\Rightarrow b_i \leq b_{i+2}$.

- Hodge-Riemann Bilinear relation.

Bilinear form: $B: H^k(X) \times H^k(X) \to \mathbb{C}$

$$(\eta, \varphi) \mapsto \int_X \eta \wedge \varphi \wedge \omega^{n-k}$$

Let $P^{r,s} = \operatorname{Ker}(\Lambda) \cap H^{r,s}$ for $s = k-r$.

Then
- $B(P^{r,s}, P^{r',s'}) = 0$ if $r+r' \neq k$.
- $\exists C_r$, s.t. $C_r \cdot B(P^{r,s}, \overline{P^{r,s}}) > 0$ (positive definite)

(this follows from the identity: for $\varphi \in P^{r,s} = P^k \cap H^{r,s}$

$$\underbrace{\ast \underset{\substack{\cap \\ 2m+k}}{L^m \varphi}}_{\substack{\cap \\ 2n-(2m+k)}} = C'_{m,r} L^{n-m-k} \varphi$$

In particular $\ast \varphi = C'_{0,r} L^{n-k} \varphi$

$$\Rightarrow C'_{0,r} \int_X \varphi \wedge \overline{\varphi} \wedge \omega^{n-k} = \int_X \varphi \wedge \ast \overline{\varphi} = \|\varphi\|^2 > 0.$$
)

X complex mfd.

- A divisor $D = \sum_i a_i D_i$ is a locally finite formal linear combinations of analytic hypersurfaces. with $a_i \in \mathbb{Z}$.

If $X = \bigcup_\alpha U_\alpha$ and $D_i \cap U_\alpha = \{f_{i\alpha} = 0\}$.

Then there is an associated line bundle $[D]$ with transition function given by $g_{\alpha\beta} = \prod_i \left(\frac{f_{i\alpha}}{f_{i\beta}}\right)^{a_i}$.

sheaf theoretic description:

$\mathcal{M}^* =$ sheaf of meromorphic functions that is not identically 0

$\mathcal{O}^* =$ sheaf of non-vanishing holomorphic functions.

$$\{\text{Divisors}\} \cong H^0(X, \mathcal{M}^*/\mathcal{O}^*)$$

Short exact sequence of sheaves $0 \to \mathcal{O}^* \to \mathcal{M}^* \longrightarrow \mathcal{M}^*/\mathcal{O}^* \longrightarrow 0$

$\rightsquigarrow$ long exact sequence of cohomology gps:

$$H^0(X, \mathcal{M}^*) \to H^0(X, \mathcal{M}^*/\mathcal{O}^*) \xrightarrow{\delta} H^1(X, \mathcal{O}^*)$$

$$\prod_i f_{i\alpha}^{a_i} \qquad\qquad \| \qquad \{\text{line bundles}\}$$

$$\| $$

$$\delta\,\{f_\alpha\} = \left\{ \frac{f_\alpha}{f_\beta} = g_{\alpha\beta} \right\}$$

L

$\|$

Cocycle $[\{g_{\alpha\beta}\}] \in H^1(X, \mathcal{O}^*)$: $\quad g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$.

$[D]$ is trivial $\iff$ $\exists$ nonvanishing functions $h_\alpha \in \mathcal{O}^*(U_\alpha)$ s.t. $h_\alpha g_{\alpha\beta} h_\beta^{-1} = 1$

$\iff \quad \dfrac{f_\alpha}{f_\beta} = \dfrac{h_\beta}{h_\alpha}$ i.e. $f_\alpha h_\alpha = f_\beta h_\beta = f \in \mathcal{M}^*(X)$

$\iff$ $D =$ divisor associated to a globally defined meromorphic function $f \in \mathcal{M}^*(X)$.

(if $f = \frac{a_\alpha}{b_\alpha}$ with $a_\alpha, b_\alpha \in \mathcal{O}(U_\alpha)$, then $D = (f) := (a_\alpha = 0) - (b_\alpha = 0)$)

- meromorphic section of L $\leftrightarrow$ meromorphic functions $\{a_\alpha\}$ satisfying $a_\beta = g_{\alpha\beta} \cdot a_\alpha$

because $f_\alpha = g_{\alpha\beta} f_\beta$. so D determines a meromorphic section s_D of $[D]$.

D is called effective if $a_i \geq 0$. this happens iff f_α is holomorphic for each α
$\shortparallel$
$\sum_i a_i D_i$

iff s_D is a holomorphic section of $[D]$.

$$
\begin{array}{ccccc}
D = \sum_i a_i D_i & \longrightarrow & [D] \in H^1(X, \mathcal{O}^*) & & 0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \\
\downarrow & & \downarrow \qquad \downarrow \searrow^{\delta} & H^2(X, \mathbb{Z}) & \\
\eta_D = \sum_i a_i \eta_{D_i} & = & c_1([D]) \in H^2(X, \mathbb{R}) \swarrow & &
\end{array}
$$

cycle class associated to D

$\eta_D \in H^2(X, \mathbb{R}) \cong H^{2n-2}(X, \mathbb{R})^*$

$\eta_D(\psi) = \sum_i a_i \int_{D_i} \psi$ for any $\psi \in A^{2n-2}(X)$ with $d\psi = 0$.