X compact Kähler $\quad H^k(X,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$

Hodge decomposition: $\quad H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$

where $H^{p,q} \cong H^{p,q}_{\bar\partial}(X)$ and satisfies $\overline{H^{p,q}} = H^{q,p}$.

$\Rightarrow \quad \underbrace{b_k}_{\dim H^k(X,\mathbb{C})} = \sum_{p+q=k} \underbrace{h^{p,q}}_{\dim H^{p,q}}$

$\Rightarrow \quad b_{2m+1}$ is even: $\quad b_{2m+1} = 2\cdot \sum_{p\le m} h^{p,2m+1-p}$

- Proof of Hodge decomposition: Hodge Thm + Kähler identities.

Hodge Thm: $\quad H^k(X,\mathbb{C}) \cong \mathcal{H}^k_{\Delta_d} = \{\varphi\in A^k : \Delta_d\varphi = 0\}$, $\quad \Delta_d = dd^* + d^*d$

$$H^{p,q}_{\bar\partial}(X,\mathbb{C}) \cong \mathcal{H}^k_{\Delta_{\bar\partial}} = \{\varphi\in A^{p,q} : \Delta_{\bar\partial}\varphi = 0\}, \quad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

Kähler identities: $\quad [\Lambda, \bar\partial] = -\sqrt{-1}\,\partial^*, \quad [\Lambda, \partial] = \sqrt{-1}\,\bar\partial^*$

$$[L, \bar\partial^*] = -\sqrt{-1}\,\partial, \quad [L, \partial^*] = \sqrt{-1}\,\bar\partial$$

Claim: Only need to prove these for $(\mathbb{C}^n, \sqrt{-1}\sum_i dz_i\wedge d\bar z_i)$

This is because Kähler $\Rightarrow_{\forall p}$ $\exists$ holomorphic coordinates $\{z_i\}$ s.t. $g_{i\bar j} = \delta_{ij} + O(|z|^2)$, $z_i(p)=0$.

$\Rightarrow$ the error term compared with the flat case vanish at any point.

On $\mathbb{C}^n$: $\quad L = \sqrt{-1} \sum_i dz_i \wedge d\bar{z}_i \wedge$

$$\Rightarrow \quad \Lambda = -\sqrt{-1} \sum_i (d\bar{z}_i \wedge)^* \cdot (dz_i \wedge)^*$$

$$= -\sqrt{-1} \sum_i \iota_{\partial_{\bar{i}}} \iota_{\partial_i}$$

$$\langle d\bar{z}_i \wedge dz_I \wedge d\bar{z}_J, dz_K \wedge d\bar{z}_L \rangle = \langle dz_I \wedge d\bar{z}_J, \underbrace{\pm dz_K \wedge d\bar{z}_{L \setminus \{i\}}}_{= \iota_{\partial_i} dz_K \wedge d\bar{z}_L} \rangle$$

$$\langle d\bar{z}_i \wedge dz_I \wedge d\bar{z}_J, dz_K \wedge d\bar{z}_L \rangle = (-1)^{|I|} \delta_{IK} \delta_{\{i,J\},L}$$

$$\langle dz_I \wedge d\bar{z}_J, \pm dz_K \wedge d\bar{z}_{L \setminus \{i\}} \rangle = \delta_{IK} \cdot \delta_{J (L \setminus \{i\})}$$

$$\bar{\partial} = \sum_i d\bar{z}_i \wedge \partial_{\bar{i}} \Rightarrow \bar{\partial}^* = \sum_i (\partial_{\bar{i}})^* (d\bar{z}_i \wedge)^*$$

$$= -\sum_i \partial_i \cdot \iota_{\partial_{\bar{i}}}$$

$$[L, \bar{\partial}^*] = \sqrt{-1} \left[\sum_i dz_i \wedge d\bar{z}_i \cdot \left(-\sum_j \partial_j \iota_{\partial_{\bar{j}}} \right) + \sum_j \partial_j \iota_{\partial_{\bar{j}}} \sum_i dz_i \wedge d\bar{z}_i \right]$$

$$= -\sqrt{-1} \sum_i \partial_i \, dz_i = -\sqrt{-1} \partial .$$

$\rightsquigarrow$ take conjugation and adjoint to get other identities.

Kähler identities $\Rightarrow \bar{\partial} \partial^* + \partial^* \bar{\partial} = 0$ and

$$\Delta_d = 2 \Delta_{\bar{\partial}} = 2 \Delta_\partial$$

$$\rightsquigarrow \quad \mathcal{H}^k_{\Delta_d} = \bigoplus_{p+q} \mathcal{H}^{p,q}_{\Delta_{\bar{\partial}}} \Longrightarrow \text{Hodge decomposition.}$$

- Some Important facts on compact Kähler mflds.

Prop: X cpt. Kähler. Then any holomorphic p-form is closed.

Pf: φ holomorphic p-form. $\Rightarrow \varphi$ is a $(p,0)$ form satisfying $\bar{\partial}\varphi = \bar{\partial}^* \varphi = 0 \Rightarrow \varphi \in \mathrm{Ker}(\Delta_{\bar{\partial}}) = \mathrm{Ker}(\Delta_d)$ $\Rightarrow d\varphi = 0$.

X cpt. Kähler

Prop ($\partial\bar\partial$-Lemma) Let φ be a (p,q)-form that is d-closed and d-exact. Then $\varphi = \partial\bar\partial\eta$ for some $(p-1,q-1)$-form

Pf: $\varphi = H\varphi + dd^*G\varphi + d^*dG\varphi$

φ d-exact $\Rightarrow$ $\underline{\varphi = dd^*G\varphi \in (\mathcal{H}^k)^\perp \cap A^{p,q} = (\mathcal{H}^{p,q})^\perp}$

$\Rightarrow \varphi = \bar\partial\bar\partial^*G\varphi + \bar\partial^*\bar\partial G\varphi$

φ is d-closed $\Rightarrow$ φ is $\bar\partial$-closed $\Rightarrow$ $\varphi = \bar\partial\underset{\gamma}{\underset{\shortparallel}{\bar\partial^*G\varphi}}$.
(p,q)

$\gamma = H\gamma + \partial\beta + \partial^*\beta' \Rightarrow \varphi = \bar\partial\partial\beta + \bar\partial\partial^*\beta'$

$\Rightarrow 0 = \partial\varphi = \underset{-\partial\partial^*\bar\partial\beta'}{\underset{\shortparallel}{\partial\bar\partial\partial^*\beta'}} \Rightarrow \underset{-\bar\partial\partial^*\beta' = 0}{\underset{\shortparallel}{\partial^*\bar\partial\beta'}} = 0.$

$\Rightarrow \varphi = \partial\bar\partial\beta$

Rmk: One can replace the condition of d-exact by $\bar\partial$-exact or ∂-exact with similar proof.

(Deligne–Griffiths–Morgan–Sullivan).

Rmk: $\partial\bar\partial$-Lemma holds $\Longleftrightarrow$ existence of Hodge decomp and $b_k = \sum_{p+q} h^{p,q}$

- Lefschetz Decomposition.

$sl_2(\mathbb{C})$ action on $\underbrace{\bigoplus_{k=0}^{n} \Lambda^k V_{\mathbb{C}}}_{= A}$. V a Hermitian vector space with o.n.b. $\{e_i\}$

$$sl_2(\mathbb{C}) = \mathrm{Span}\left\{ \underset{X}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{Y}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{H}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \right\}$$

$[X,Y]=H$, $[H,X]=2X$, $[H,Y]=-2Y$

Lefschetz operator: $L = \sqrt{-1}\sum_{i=1}^{n} e_i \wedge \overline{e_i}$

Adjoint of L: $\Lambda = -\sqrt{-1}\sum_{i=1}^{n} \iota_{\overline{e_i}}\, \iota_{e_i}$

$\Lambda^{p,q} V_{\mathbb{C}}$ is generated by $e_I \wedge \overline{e_J}$ with $|I|=p$, $|J|=q$

Assume $I \cap J = K$, $e_I \wedge \overline{e_J} = \pm e_K \wedge \overline{e_K} \wedge e_{I'} \wedge \overline{e_{J'}}$

$$L\Lambda(\underbrace{e_K \wedge \overline{e_K} \wedge e_{I'} \wedge \overline{e_{J'}}}_{\varphi}) = |K|\,\varphi + \sum_{\substack{l \in K \\ m \notin I \cup J \cup K}} e_{K \setminus l} \wedge \overline{e_{K\setminus l}} \wedge e_m \wedge \overline{e_m} \wedge e_{I'} \wedge \overline{e_{J'}}$$

$$\Lambda L(e_K \wedge \overline{e_K} \wedge e_{I'} \wedge \overline{e_{J'}}) = (n - |I'| - |J'| - |K|)\,\varphi + \sum_{\substack{l \in K \\ m \notin I \cup J \cup K}} e_{K \setminus l} \wedge \overline{e_{K\setminus l}} \wedge e_m \wedge \overline{e_m} \wedge e_{I'} \wedge \overline{e_{J'}}$$

$$\Rightarrow [\Lambda, L]\varphi = (n - |I'| - |J'| - 2|K|)\,\varphi = (n - \deg\varphi)\,\varphi.$$

$$H := \prod_{k=0}^{2n} (n-k)\cdot \pi_k \qquad \pi_k : \underset{\bigoplus_k A^k}{A} \to A^k \text{ projection}$$

$$[H, \Lambda] = 2\Lambda, \quad [H, L] = -2L$$

Then the representation is given by

$$X \to \Lambda, \quad Y \to L, \quad H \to H.$$

Finite dimensional Irreducible representation of $sl_2(\mathbb{C})$.

$$\rho : sl_2(\mathbb{C}) \longrightarrow \mathrm{End}(E).$$

- Can assume that it comes from a unitary representation of $su(2)$.

- Decomposition of E into weight spaces:

$$E = \bigoplus_{\lambda} E_\lambda. \quad E_\lambda = \{ v \in E : Hv = \lambda \cdot v \}.$$

- $X : E_\lambda \to E_{\lambda+2}, \quad Y : E_\lambda \to E_{\lambda-2}$

$$\begin{aligned} HXv &= XHv + 2Xv \\ &= (\lambda+2) Xv. \end{aligned}$$

- Set $\lambda_1 = \max\{\lambda : E_\lambda \neq 0\}$. Choose $v \in E_m$.

v generates a chain:

$$\begin{tikzcd}
0 & Y^k v \arrow[l, "Y"'] & \cdots \arrow[l] & Y^2 v \arrow[l] & Yv \arrow[l] & v \arrow[l, "Y"'] \\
 & E_{\lambda_1 - 2k} \arrow[u, phantom, "\in" rotate=90] & & & E_{\lambda_1 - 2} \arrow[u, phantom, "\in" rotate=90] & E_{\lambda_1} \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

$$X\cdot Y^p v = [X, Y^p]v + \underbrace{Y^p X v}_{\shortparallel \atop 0}$$

$$= \sum_{i=0}^{p-1} Y^i [X,Y] Y^{p-1-i} v$$

$$= \sum_{i=0}^{p-1} \left(\lambda_1 - 2(p-1-i)\right) Y^{p-1} v$$

$$= \left(\lambda_1 - 2(p-1)\right) + 2\cdot \frac{p(p-1)}{2}\Big] Y^{p-1} v$$

$$= \cdot(\lambda - p + 1)\cdot Y^{p-1} v$$

$$0 = X\cdot Y^{k+1} v = (k+1)\cdot(\lambda_1 - k) = 0 \Rightarrow \lambda_1 = k \in \mathbb{Z}_{\geq 0}$$

$\Rightarrow$ Symmetric chain:

$$\begin{array}{ccccc} Y^k v & - & Y^{k-1} v & - \cdots - & v \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ E_{-k} & & & & E_k \end{array}$$

$$\text{Span}\{v, Yv, \dots, Y^k v\} = E \quad \left(\begin{array}{c}\text{since } E \text{ is} \\ \text{irreducible}\end{array}\right)$$

<u>Def</u>: $w \in E$ is called primitive if $Xv = 0$.

So any irreducible representation of E is generated by a primitive vector.

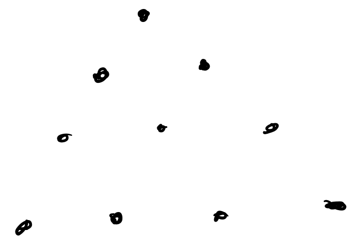
$\dim E = k+1$

1

2

3

4



Concrete representation of dim $2k+1$

$$E_k = \mathrm{Sym}^k(\mathbb{C}^2) = \{z_1^k, z_1^{k-1} z_2, \dots, z_1 z_2^{k-1}, z_2^k\}$$

$\mathfrak{sl}_2(\mathbb{C}) \circlearrowright E_k$ by:

$$X\cdot(z_1^i z_2^{k-i}) = i\cdot z_1^{i-1}\cdot z_2^{k-i+1}$$

$$Y\cdot(z_1^i z_2^{k-i}) = (k-i)\cdot z_1^{i+1}\cdot z_2^{k-i-1}$$

$$[X,Y](z_1^i\cdot z_2^{k-i}) = (k-i)\cdot(i+1) - i\cdot(k-i+1)$$

$$\begin{array}{ccl} \| & & \\ H\cdot z_1^i z_2^{k-i} & = & k-i-i = (k-2i) \end{array}$$

Any representation of $\mathfrak{sl}_2(\mathbb{C})$ decomposes into irreducible ones: For example:

$$\bigoplus_{k=0}^{n} \Lambda^k V_{\mathbb{C}} = \bigoplus_i E_i$$

Any $w \in E_i$ is equal to $Y^p v_i = L^p v_i$

where v_i is primitive.

So we arrive at the Lefschetz decomposition

Any $\varphi \in \Lambda^k V_{\mathbb{C}}$ can be decomposed as:

$$\varphi = \sum_{r \geq 0} L^r v_r$$

where $v_r \in \Lambda^{k-2r}$ is primitive (i.e. $X v_i = \Lambda v_i = 0$)

$$\left(\begin{array}{l} H \cdot v_r = (n-(k-2r))v_r = (n-k+2r)v_r \\ n-k+2r \geq 0 \Rightarrow r \geq \frac{k-n}{2} \end{array} \right)$$

(Ex) Such decomposition is unique.

Moreover: $L^p : \Lambda^{n-p} V_{\mathbb{C}} \to \Lambda^{n+p} V_{\mathbb{C}}$ is an isomorphism because $L^p = Y^p$ restricted to each irreducible summand is an isomorphism.

Applying the same argument to the action of $sl_2(\mathbb{C})$ on $H^*(X) \cong \mathcal{H}_d^*(X)$:

Thm (Hard Lefschetz Thm)

The map $L^k: H^{n-k}(X) \to H^{n+k}(X)$ is an isomorphism. If we define the primitive cohomology $P^{m-k}(X) = \ker \Lambda \cap H^{m-k}(X)$.

then $H^m(X) = \bigoplus_k L^k P^{m-2k}$.