$$\{\text{Kähler manifolds}\} \subsetneq \{\text{complex manifolds}\}$$
$$\cup\!\!\!\!\!/\ \ (\supsetneq)$$
$$\{\text{projective manifolds}\}$$

$$g^* = g \Leftrightarrow \overline{g_{i\bar{j}}} = g_{j\bar{i}},\ \forall i,j$$

X a complex mfd, $g = \sum_{i,j} g_{i\bar{j}}(dz_i \otimes d\bar{z}_j + d\bar{z}_j \otimes dz_i)$ Hermitian metric

$\Updownarrow$

$\omega = \sqrt{-1} \sum_{i,j} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$ is a real 2-form:

$$\bar{\omega} = -\sqrt{-1} \sum_{i,j} \overline{g_{i\bar{j}}}\, d\bar{z}_i \wedge dz_j = \sqrt{-1} \sum_{j,i} g_{j\bar{i}}\, dz_j \wedge d\bar{z}_i = \omega.$$

g is called a Kähler metric or ω is called a Kähler form if $d\omega = 0$.

X is Kähler if it admits a Kähler metric.

$$d\omega = \sqrt{-1} \sum_{i,j} \left(\frac{\partial g_{i\bar{j}}}{\partial z_k} dz_k + \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} d\bar{z}_l \right) \wedge dz_i \wedge d\bar{z}_j$$
$$= \sqrt{-1} \sum_{i,j} \frac{1}{2} \left(\frac{\partial g_{i\bar{j}}}{\partial z_k} - \frac{\partial g_{k\bar{j}}}{\partial z_i} \right) dz_k \wedge dz_i \wedge d\bar{z}_j + \text{complex conjugate}.$$

So $d\omega = 0 \Leftrightarrow \dfrac{\partial g_{i\bar{j}}}{\partial z_k} = \dfrac{\partial g_{k\bar{j}}}{\partial z_i}$, $\forall i,j,k$. at any point

- Equivalent characterization using Chern connection D of (TX, g).

Torsion of D: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

$$T(\partial_i, \partial_j) = \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i - [\partial_i, \partial_j] = \theta_j^k(\partial_i) - \theta_i^k(\partial_j)$$
$$= g^{k\bar{l}} \partial_i g_{j\bar{l}} - g^{k\bar{l}} \partial_j g_{i\bar{l}}.$$

($\theta = \partial g \cdot g^{-1}$ is the connection form of Chern connection)

So ω is Kähler $\Leftrightarrow$ the Chern connection is torsion free

$\Updownarrow$

Chern connection $=$ Levi-Civita connection (complexified).

• Proposition: g is Kähler if $\forall p \in X$, $\exists$ holomorphic coordinates $\{z_i\}$ centered at P (i.e. $z_i(P)=0$, $i=1,\dots,n$) and

$$g_{i\bar{j}} = \delta_{ij} + O(|z|^2) \quad \forall i,j.$$

Proof: By a linear transformation, it is easy to find holomorphic coordinates $\{w_i\}$ s.t.

$$g_{i\bar{j}} = \delta_{ij} + O(|w|) = \delta_{ij} + \sum_k b_{i\bar{j}k} w_k + b_{i\bar{j}\bar{k}} \bar{w}_k + O(|w|^2)$$

Since $\overline{g_{i\bar{j}}} = g_{j\bar{i}}$, we have $\overline{b_{i\bar{j}\bar{k}}} = b_{j\bar{i}k}$ or equiv.

$$b_{i\bar{j}\bar{k}} = \overline{b_{j\bar{i}k}}$$

Kähler condition: $\dfrac{\partial g_{i\bar{j}}}{\partial w_k} = \dfrac{\partial g_{k\bar{j}}}{\partial w_i} \Rightarrow \boxed{b_{i\bar{j}k} = b_{k\bar{j}i}}$ (*)

consider the transformation. $w_k = z_k + \sum_{ij} a_{kij} z_i z_j$ with $\boxed{a_{kij} = a_{kji}}$ (**)

Calculate:

$$\omega = \sqrt{-1}\, g_{i\bar{j}}\, dw_i \wedge d\bar{w}_j = \sqrt{-1}\,\Big(\delta_{ij} + \sum_k b_{i\bar{j}k}(z_k + O(|z|^2)) + b_{i\bar{j}\bar{k}}(\bar{z}_k + O(|z|^2))\Big).$$

$$\Big(dz_i + \sum_{pq} a_{ipq}(z_p dz_q + z_q dz_p)\Big)$$

$$\Big(d\bar{z}_j + \sum_{rs} \overline{a_{jrs}}(\bar{z}_r d\bar{z}_s + \bar{z}_s d\bar{z}_r)\Big)$$

use $a_{ipq} = a_{iqp}$

$$= \sqrt{-1}\Big[(\delta_{ij} + b_{i\bar{j}k} z_k + b_{i\bar{j}\bar{k}} \bar{z}_k) dz_i \wedge d\bar{z}_j + 2\, dz_j \wedge d\bar{z}_s \cdot \overline{a_{jrs}} \cdot \bar{z}_r + 2\, a_{ipq} z_p dz_q \wedge d\bar{z}_i\Big] + O(|z|)^2$$

So we want
$$\begin{cases} b_{i\bar{j}k} + 2 a_{jki} = 0 \Rightarrow \boxed{a_{jki} = -\frac{1}{2} b_{i\bar{j}k}} \\ b_{i\bar{j}\bar{k}} + 2\overline{a_{ikj}} = 0 \Leftarrow \overline{a_{ikj}} = -\frac{1}{2}\overline{b_{j\bar{i}k}} = -\frac{1}{2} b_{i\bar{j}\bar{k}} \end{cases}$$

Kähler $\Downarrow$ (*) (**) compatible

(The converse direction is easy).

Examples: • Any 1-$\dim_{\mathbb{C}}$ complex mfd. is Kähler:

$$\omega = \sqrt{-1}\, g\, dz \wedge d\bar{z} \Rightarrow d\omega = 0, \quad z \in \mathbb{C}.$$

• $\mathbb{P}^N$ is Kähler with a canonical Kähler metric

Fubini-Study Kähler metric: under coordinate $z_i = \frac{Z_i}{Z_0}$, $i = 1, \cdots, N$

$$\omega_{FS} = \sqrt{-1}\,\partial\bar{\partial} \log\left(1 + \sum_{i=1}^{N} |z_i|^2\right) = \sqrt{-1}\left(\frac{\sum_i dz_i \wedge d\bar{z}_i}{1+|z|^2} - \frac{\sum_{i,j} \bar{z}_i dz_i \wedge z_j d\bar{z}_j}{(1+|z|^2)^2}\right)$$

When $N=1$, this is the round metric on Riemann sphere:

$$\omega_{S^2} = \sqrt{-1} \cdot \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$$

• If X is Kähler, $Y \subset X$ is a complex submfd, then Y is also Kähler: ω Kähler on $X \Rightarrow \omega|_Y$ Kähler on Y

In particular, any projective mfd. is a Kähler manifold

• Non-Kähler example.

Hopf surface: Fix $\lambda \in \mathbb{R}_{>0}$, $0 < \lambda < 1$.

$X = (\mathbb{C}^2 - \{0\})/\mathbb{Z}$. where $\mathbb{Z}$ acts on $\mathbb{C}^2 - \{0\}$ by $m \cdot (z_1, z_2) = (\lambda^m z_1, \lambda^m z_2)$

(Ex) X is diffeomorphic to $S^3 \times S^1$.

X is non-Kähler because of the

<u>Prop</u>: If a compact complex manifold X^n is Kähler, then $b_{2k}(X) > 0$. $\forall\ 0 \leq k \leq n$.

<u>Proof</u>: $d\omega = 0 \Rightarrow [\omega] \in H^2(X, \mathbb{R})$

$$\int_X [\omega]^n = \int_X \omega^n = \underbrace{\int_X \det(g_{i\bar{j}})\, dz \wedge d\bar{z}\, c_n}_{= \mathrm{Vol}(X,\omega)} > 0$$

$\Rightarrow [\omega]^k \neq 0 \in H^{2k}(X, \mathbb{R})$.

- Fact: For cpt complex surfaces, Kähler $\Leftrightarrow$ b_1 is even.

(Kodaira, Siu, ...)

- Hodge theory on compact Kähler mfds.

  - Hodge Decomposition:

$$
\begin{array}{ccc}
H^k(X,\mathbb{C}) & \cong \bigoplus_{p+q=k} & H^{p,q}_{\bar\partial}(X) \\
\text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\
\mathcal{H}^k & \bigoplus_{p+q=k} & \mathcal{H}^{p,q} \\
\| & & \| \\
\{\eta : \Delta_d \eta = 0\} & \bigoplus & \{\varphi \in A^{p,q} : \Delta_{\bar\partial}\varphi = 0\} \\
\cap & & \\
A^k & &
\end{array}
$$

$$\Delta_d = dd^* + d^*d \qquad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

The decomposition follows from the identity

$$\Delta_d = 2\Delta_{\bar\partial} = 2\Delta_\partial$$

$$\left(\text{where } \Delta_\partial = \partial\partial^* + \partial^*\partial\right)$$

as operators on $A^k(X) = \bigoplus_{p+q=k} A^{p,q}(X)$.

It in turn follows from the important Kähler identities.

Let $L: A^k \to A^{k+2}$ be the Lefscletz operator defined by $\eta \mapsto \omega \wedge \eta$

$\Lambda = L^*: A^k \to A^{k-2}$ is the adjoint of L

Then Kähler identities:

$$[\Lambda, \bar\partial] = -\sqrt{-1}\,\partial^*, \quad [\Lambda, \partial] = \sqrt{-1}\,\bar\partial^*$$

$$[L, \bar\partial^*] = \sqrt{-1}\,\partial, \quad [L, \partial^*] = -\sqrt{-1}\,\bar\partial.$$

(one implies the others).

Assuming the Kähler identities, it is straight forward to prove:

$$\bar\partial^*\partial + \partial\bar\partial^* = 0 \quad \text{and} \quad \Delta_\partial = \Delta_{\bar\partial}$$

which easily implies $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$.

For example: $\Delta_\partial = \partial\partial^* + \partial^*\partial = \partial\dfrac{[\Lambda,\bar\partial]}{-\sqrt{-1}} + \dfrac{[\Lambda,\bar\partial]}{-\sqrt{-1}}\partial$

$$= \sqrt{-1}\left(\partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial\right)$$

while $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial = \bar\partial\cdot\dfrac{[\Lambda,\partial]}{\sqrt{-1}} + \dfrac{[\Lambda,\partial]}{\sqrt{-1}}\bar\partial$

$$= -\sqrt{-1}\cdot\left(\bar\partial\Lambda\partial - \bar\partial\partial\Lambda + \Lambda\partial\bar\partial - \partial\Lambda\bar\partial\right)$$

$$= \sqrt{-1}\cdot\left(-\bar\partial\Lambda\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial + \partial\Lambda\bar\partial\right) = \Delta_\partial$$