- Proof of Hodge Theorem.

First construct the inverse of $(I+\Delta)^{-1} =: K$ as a bounded operator $K: W_0 \longrightarrow W_1$

where $W_0 \cong L^2$ and

W_1 is the Sobolev space that is the completion of $A^{p,q}$ w.r.t. the Dirichlet norm determined by the inner product:

$$\mathcal{D}(\varphi, \eta) = (\varphi, \eta)_{L^2} + (\bar\partial\varphi, \bar\partial\eta)_{L^2} + (\bar\partial^*\varphi, \bar\partial^*\eta)_{L^2}$$

Do this by using the Riesz representation thm: $\forall \varphi \in W_0$, want to find ψ s.t. $(I+\Delta)^{-1}\varphi = \psi \Leftrightarrow (I+\Delta)\psi = \varphi$

$$\Leftrightarrow \quad (\varphi, \eta)_{L^2} = ((I+\Delta)\psi, \eta) = \mathcal{D}(\psi, \eta)$$

So consider the linear functional $F: \eta \mapsto (\varphi, \eta)_{L^2}$.

Then $\quad |F(\eta)| \leq \|\varphi\|_{L^2} \cdot \|\eta\|_{L^2} \leqslant \|\varphi\|_{L^2} \cdot \|\mathcal{D}(\eta)\|$

$$\Rightarrow \quad \underset{\overset{\parallel}{(\varphi,\eta)}}{F(\eta)} = \mathcal{D}(\psi, \eta) \text{ for some } \psi \in W_1 \text{ which satisfies}$$

$$\underset{\overset{\parallel}{\|\psi\|^2_{W_1}}}{\mathcal{D}(\psi,\psi)} = (\varphi, \psi)_{L^2} \leq \|\varphi\|_{L^2} \cdot \|\psi\|_{L^2} \leq \epsilon^{-1}\|\varphi\|_{L^2} + \underset{\overset{\wedge}{\epsilon \cdot \|\psi\|_{W_1}}}{\underbrace{\epsilon \cdot \|\psi\|_{L^2}}}$$

$$\Rightarrow \quad \underset{\overset{\parallel}{K\varphi}}{\|\psi\|_{W_1}} \leqslant C \cdot \|\varphi\|_{L^2}$$

$\rightsquigarrow \quad K: \underset{\overset{\parallel}{L^2}}{W_0} \rightarrow W_1 \quad$ a bounded operator

$\rightsquigarrow \quad K: W_0 \to W_0$ is a compact operator

$$\begin{tikzcd} W_0 \arrow[r] \arrow[dr] & W_0 \\ & W_1 \arrow[u] \end{tikzcd}$$

$\leftarrow$ Rellich Lemma (compare with the Ascoli-Arzela thm.)

Ex Verify that K is selfadjoint.

- Fact from functional analysis: Spectral decomposition for self-adjoint compact operator:
$$W_0 = \bigoplus_m E(K, \rho_m)$$
where ρ_m are real eigenvalues of K and $E(K, \rho_m)$ are the finite-dim eigenspaces.

- Regularity Lemma: If $\Delta\psi = \varphi$ with $\varphi \in W_s$

$W_s = \{\varphi \in L^2 : D^k\varphi \in L^2, \ 0 \le k \le s\}$ as distributions

higher order Sobolev spaces

Then $\psi \in W_{s+2}$.

Sobolev Lemma: $W_{[\frac{2n}{2}]+1+s} \subset C^s(\Lambda^{p,q} \otimes E)$ and
$$A^{p,q}(E) = \bigcap_s W_s(E)$$

$(\rho_m \neq 0)$

Now $\varphi \in E(K, \rho_m) \Leftrightarrow (I+\Delta)^{-1}\varphi = \rho_m \varphi \Leftrightarrow (I+\Delta)\varphi = \rho_m^{-1}\varphi$

$\Rightarrow \varphi$ is smooth by the above 2 lemmas (Ex)

$\Rightarrow E(K, \rho_m) \subset A^{p,q}(E)$.

Note that $\Delta\varphi = (\rho_m^{-1} - 1)\varphi = \underbrace{\frac{1-\rho_m}{\rho_m}}_{=\lambda_m}\varphi$ $\quad \lambda_m \Leftrightarrow \rho_m = \frac{1}{1+\lambda_m}$

$E(\Delta, \lambda_m) = E(K, \rho_m)$

$\lambda_m = 0 \Leftrightarrow \rho_m = 1$

Define: $G|_{E(\Delta,\lambda_m)} : E(\Delta, \lambda_m) \to E(\Delta, \lambda_m)$
$$\varphi \mapsto \frac{1}{\lambda_m}\varphi$$

and $G|_{E(\Delta, 0)} = 0$.

Then (Ex) G satisfies all the requirements:

- $G \circ \Delta = \Delta \circ G = \mathrm{Id} - H$ (G is a parametrix for Δ)
- $G\bar{\partial} = \bar{\partial} G$, $G\bar{\partial}^* = \bar{\partial}^* G$.

Consequence of Hodge Thm: X cpt. E holomorphic

$$\mathcal{H}^{p,q}(E) \underset{\text{Hodge}}{\cong} H^{p,q}_{\bar{\partial}}(X,E) \underset{\text{Dolbeault}}{\cong} H^q(X, \Omega^p \otimes E)$$

are finite dimensional: $h^{p,q}(E) = \dim H^q(X, \Omega^p \otimes E) < +\infty$.

$h^{p,q}(E) = h^q(\Omega^p \otimes E)$

Define: $\chi(E) = \sum_{q=0}^{n} (-1)^q h^{0,q}(E)$.

Thm (Hirzebruch-Riemann-Roch formula):

$$\chi(E) = \int_X [\underbrace{ch(E) \cdot Td(X)}_{H^{even}(X,\mathbb{R})\text{ cohomological ring}}]_{2n}$$

take the degree $2n$-component and integrate.

$$ch(E) = \sum_i e^{x_i} = \sum_i \left(1 + x_i + \frac{x_i^2}{2!} + \frac{x_i^3}{3!} + \cdots\right)$$

$$= rk(E) + \sum_i x_i + \frac{1}{2!}\sum_i x_i^2 + \frac{1}{3!}\sum_i x_i^3 + \cdots$$

$$= rk(E) + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - c_2(E)\right) + \frac{1}{6}\cdot[Tr(R_D^3)]$$

$\frac{1}{2}(c_1(E)^2 - c_2(E))$ here $= [Tr(R_D^2)]$

$$= rk(E) + \sum_{k=1}^{\infty} \frac{1}{k!}[Tr(R_D^k)].$$

$\{x_i\}$ are formal Chern roots which diagonalize R_D after certain ring extension: $R_D \sim \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_r \end{pmatrix}$

$$Td(X) = \prod_{\alpha=1}^{n} \frac{y_\alpha}{1-e^{-y_\alpha}}$$ where $\{y_\alpha\}$ are formal Chern roots of the Chern connection for (TX, g).

Expand:

$$\frac{y}{1-e^{-y}} = \frac{y}{\underbrace{1-\left(1-y+\frac{y^2}{2!}-\frac{y^3}{3!}+\frac{y^4}{4!}\right)}_{y-\frac{y^2}{2!}+\frac{y^3}{3!}-\frac{y^4}{4!}}} = \frac{1}{1-\left(\frac{y}{2}-\frac{y^2}{6}+\frac{y^3}{24}+\cdots\right)}$$

$$= 1+\left(\frac{y}{2}-\frac{y^2}{6}+\frac{y^3}{24}+\cdots\right)+\left(\frac{y}{2}-\frac{y^2}{6}+\frac{y^3}{24}+\cdots\right)^2+\left(\frac{y}{2}-\frac{y^2}{6}+\frac{y^3}{24}-\cdots\right)^3+\cdots$$

$$= 1+\frac{y}{2}+y^2\left(-\frac{1}{6}+\frac{1}{4}\right)+y^3\left(\frac{1}{24}-2\,\frac{1}{2\times 6}+\frac{1}{8}\right)+\cdots$$

$$= 1+\frac{y}{2}+\frac{1}{12}y^2+y^3\underbrace{\frac{(1-4+3)}{24}}_{0}+y^4(\cdots)+\cdots$$

$$\prod_\alpha \frac{y_\alpha}{1-e^{-y_\alpha}} = \prod_\alpha \left(1+\frac{y_\alpha}{2}+\frac{1}{12}y_\alpha^2+y_\alpha^4\cdots\right)$$

$$= 1+\frac{1}{2}\sum_\alpha y_\alpha+\frac{1}{4}\prod_{\alpha\neq\beta} y_\alpha y_\beta+\frac{1}{12}\sum_\alpha y_\alpha^2+\left(\sum_{\alpha\neq\beta} y_\alpha\cdot y_\beta^2\cdot\frac{1}{24}\right)+\cdots$$

$$\frac{1}{8}\sum_{\alpha<\beta<\gamma} y_\alpha y_\beta y_\gamma \; + \; \left(\sum_{\alpha\neq\beta} y_\alpha\cdot y_\beta^2\cdot\frac{1}{24}\right) = \frac{1}{24}\left(\prod_{\alpha\neq\beta} y_\alpha y_\beta\right)\cdot\left(\sum y_\gamma\right) = \frac{1}{24}c_1(X)\cdot c_2(X)$$

$$= 1+\frac{1}{2}c_1(X)+\frac{1}{12}\Big(\underbrace{\big(\sum_\alpha y_\alpha\big)^2+\prod_{\alpha\neq\beta} y_\alpha y_\beta}_{c_1^2(X)+c_2(X)}\Big)+$$

$$= 1+\frac{1}{2}c_1(X)+\frac{1}{12}\left(c_1^2(X)+c_2(X)\right)+\frac{1}{24}c_1(X)c_2(X)+\cdots$$

$$\in H^{even}(X,\mathbb{R})$$

$$\rightsquigarrow \chi(E) = \int_X ch(E)\cdot Td(X)$$

$$= \int_X \left(rk(E) + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - c_2(E)\right) + \cdots\right)$$
$$\left(1 + \frac{1}{2}c_1(X) + \frac{1}{12}\left(c_1^2(X) + c_2(X)\right) + \frac{1}{24}c_1(X)c_2(X)\right)$$

Special cases. • $\dim X = 1$, and $E = L$ is a holomorphic line bundle (i.e. $rk = 1$)

Then
$$\chi(L) = \int_X (1 + c_1(L))\cdot\left(1 + \frac{1}{2}c_1(X)\right)$$
$$= \int_X c_1(L) + \frac{1}{2}\int_X c_1(X)$$
$$= \deg(L) + \frac{2-2g}{2} = \deg(L) - g + 1$$

(Classical Riemann-Roch)

• $E = X \times \mathbb{C}$ trivial. Then $\mathcal{O}(E) = \mathcal{O}_X$

$$\chi(\mathcal{O}_X) = \sum_{q=0}^{n}(-1)^q h^q(\mathcal{O}_X)$$

$$= \begin{cases} \int \frac{c_1(X)}{2} = 1-g & \dim X = 1 \\ \int_X \frac{c_1(X)^2 + c_2(X)}{2} & 2 \\ \int_X \frac{c_1(X)\cdot c_2(X)}{24} & 3 \end{cases}$$