$$\begin{array}{ll} E & \text{holomorphic vector bundle} \\ \downarrow & \\ X & h: \text{Hermitian metric on } E \end{array}$$

$g$ : Hermitian metric on $TX$ $\rightsquigarrow$ Hermitian metric on $\Lambda^{p,q}T^*X$

$= \sum_{\alpha,\beta} g_{\alpha\bar\beta}\, dz_\alpha \otimes d\bar z_\beta$

$\Lambda^{p,q}T^*X = \Lambda^p T^{*(1,0)}X \otimes \Lambda^q T^{*(0,1)}X = \mathrm{Span}\{ dz_I \wedge d\bar z_J : |I| = p,\ |J| = q \}$

If $\left\{ \frac{\partial}{\partial z_\alpha}\Big|_p \right\}$ is an orthonormal basis of $g|_{TX}$

then $\{ dz_I \wedge d\bar z_J \}$ is an orthonormal basis of induced metric on $\Lambda^{p,q}T^*X$.

$$dvol_g = \det(g_{\alpha\bar\beta})\, dz_1 \wedge \cdots \wedge dz_n \wedge d\bar z_1 \wedge \cdots \wedge d\bar z_n \cdot \underbrace{(\sqrt{-1})^n (\sqrt{-1})^{(n-1)+\cdots+1}}_{(\sqrt{-1})^{n+\frac{(n-1)\cdot n}{2}} = (\sqrt{-1})^{\frac{n^2+n}{2}}}$$

Hodge $*$-operator: $\varphi, \psi \in \Lambda^{p,q}T^*X$, then

$$*: \Lambda^{p,q} \longrightarrow \Lambda^{n-q,n-p} \qquad \boxed{\langle \varphi, \psi \rangle_p = \frac{\varphi \wedge *\bar\psi}{dvol_g}}$$

$$\left( \begin{array}{c} \text{Can Extend to be an operator which is conjugate } \mathbb{C}\text{-linear} \\ \bar *_h : \Lambda^{p,q} \otimes E \longrightarrow \Lambda^{n-p,n-q} \otimes E^* \\ \eta \otimes s \longmapsto \bar * \eta \otimes (\cdot, s)_h \end{array} \right)$$

From now on, assume that $X$ is a compact complex manifold.

- Inner product on $A^{p,q}(E) = \{\text{smooth sections on } \Lambda^{p,q}T^*X \otimes E\}$.

$$\underset{\text{E-valued (p,q)-form}}{(\varphi, \psi)} = \int_X \langle \varphi, \psi \rangle_h\, dvol_g = \int \varphi \wedge \bar *_h \psi = \int \langle \varphi \wedge * \bar\psi \rangle_h$$

- formal adjoint of $\bar\partial : A^{p,q-1} \to A^{p,q}$:

$$(\bar\partial \eta, \psi) = (\eta, \bar\partial^* \psi) \quad \forall \eta \in A^{p,q-1}(E)$$

compatible with the Hermitian metric

$$\bar\partial\eta\wedge\bar{*}_h\psi = \langle\bar\partial\eta\wedge*\bar\psi\rangle_h = \bar\partial\langle\eta\wedge*\bar\psi\rangle_h \pm(-1)^{\deg\eta}\langle\eta\wedge\overline{D'*\psi}\rangle_h$$

$$= d\langle\eta\wedge*\bar\psi\rangle_h \pm(-1)^{\deg\eta}\langle\eta\wedge*\overline{*D'*\psi}\rangle_h$$

$$\Rightarrow \bar\partial^*\psi = \pm *D'*\psi.$$

$D = D' + \bar\partial$ Chern connection.

---

For each $[\varphi]\in H^{p,q}_{\bar\partial}(X,E)$, want to find a representative $\varphi$ that has the minimum norm.
$[\varphi] = [\varphi+\bar\partial\eta]$

$$\|\varphi\|^2 \le \|\varphi+\bar\partial\eta\|^2 = \|\varphi\|^2 + 2\,\mathrm{Re}(\varphi,\bar\partial\eta) + \|\bar\partial\eta\|^2 \quad \forall\eta$$

$$\overset{\text{test on } \pm\eta,\ \pm i\eta}{\Longleftrightarrow} \quad (\varphi,\bar\partial\eta) = (\bar\partial^*\varphi,\eta) = 0 \quad \forall\eta \iff \bar\partial^*\varphi = 0.$$

So we want to solve: $\boxed{\bar\partial\varphi = 0,\ \bar\partial^*\varphi = 0}$

Equivalently: $(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)\varphi = 0.$

( since $\left((\bar\partial^*\bar\partial+\bar\partial\bar\partial^*)\varphi,\varphi\right) = \|\bar\partial\varphi\|^2 + \|\bar\partial^*\varphi\|^2$ )

Set $\Delta = \Delta_E = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$ Laplacian operator

$$\mathcal{H}^{p,q}(E) = \{\varphi\in A^{p,q}(E) : \Delta\varphi = 0\}$$

space of harmonic $E$-valued $(p,q)$-forms.

Thm (Hodge Theorem for $\bar\partial$-operator). $X$ cpt. complex, $E$ holom. vector bundle.

1. $\dim \mathcal{H}^{p,q}(E) < +\infty$

2. $\exists$ orthogonal projection $H: A^{p,q} \to \mathcal{H}^{p,q}$ and Green's operator $G: A^{p,q} \to A^{p,q}$ s.t. $\bar\partial G = G \bar\partial$, $\bar\partial^* G = G \bar\partial^*$ and

$$G \Delta = \Delta \cdot G = Id - H.$$

Cor: $\mathcal{H}^{p,q}(E) \cong H^{p,q}_{\bar\partial}(E)$.

Indeed, $\forall [\varphi] \in H^{p,q}_{\bar\partial}(E)$,
$$\begin{aligned}\varphi &= H(\varphi) + \Delta G(\varphi) \\ &= H(\varphi) + \bar\partial(\bar\partial^* G(\varphi)) + \underbrace{\bar\partial^*(\bar\partial G(\varphi))}_{= \bar\partial^* G \bar\partial \varphi = 0} \\ &= H(\varphi) + \bar\partial(\bar\partial^* G(\varphi)).\end{aligned}$$

$\Rightarrow [\varphi] = [H(\varphi)]$.

Moreover:
$$\begin{aligned}A^{p,q}(E) &= \mathcal{H} \oplus \mathcal{H}^\perp \\ &= \mathcal{H} \oplus (Id - H)A \\ &= \mathcal{H} \oplus \Delta G A \\ &= \mathcal{H} \oplus \bar\partial A \oplus \bar\partial^* A.\end{aligned}$$
$A = A^{p,q}$

So $H^{p,q}_{\bar\partial}(E) \to \mathcal{H}^{p,q}(E)$ is an isomorphism.

$[\varphi] \mapsto [H(\varphi)]$.

To construct $H$ and $G$, we can to first construct the inverse to $I+\Delta$ which is positive definite.

Want to solve $(I+\Delta)\varphi = \psi \quad \forall \psi \in A^{p,q}$.

Because $\quad ((I+\Delta)\varphi, \eta) = (\psi, \eta)$ Dirichlet inner product.

$$\overset{\|}{(\varphi,\eta)} + (\bar\partial\varphi, \bar\partial\eta) + (\bar\partial^*\varphi, \bar\partial^*\eta) = \mathfrak{D}(\varphi,\eta).$$

To solve this, we want to apply Riesz representation thm. to the linear functional $\eta \mapsto (\psi,\eta)$ and show that

$$|(\psi,\eta)| \leq C\cdot \mathfrak{D}(\eta,\eta)^{\frac{1}{2}}.$$

Then we can solve $\varphi$ in an appropriate Sobolev space.

Then we will use compactness and regularity result to construct $H$ and $G$. (next lecture)