$$\begin{array}{c} E \\ \downarrow \\ X \end{array} \quad \text{complex vector bundle}$$

$D: A(E) \longrightarrow A(T^*_{\mathbb{C}}X \otimes E)$ connection:

$$D(fs) = df \cdot s + f \cdot Ds \quad \forall f \in C^\infty(X),\ s \in A(E).$$

$\rightsquigarrow$ $D: A(\Lambda^k T^*_{\mathbb{C}} X \otimes E) \longrightarrow A(\Lambda^{k+1} T^*_{\mathbb{C}} X \otimes E)$

$$\eta \otimes s \xmapsto{D} d\eta \otimes s + (-1)^k \eta \wedge Ds$$

- Curvature of the connection D:

$$A(E) \xrightarrow{D} A(T^*_{\mathbb{C}} X \otimes E) \xrightarrow{D} A(\Lambda^2 T^*_{\mathbb{C}} X \otimes E)$$

D^2 is a tensor: $\quad D^2(fs) = f \cdot D^2 s \quad \forall f \in C^\infty(X),\ s \in A(E)$

$\Rightarrow \quad F_D := D^2 \in A^2(\Lambda^2 T^*_{\mathbb{C}} X \otimes \operatorname{End} E).$

- Local calculation: $\forall x \in X$, $\exists U \ni x$ s.t. $\pi^{-1}(U) \xrightarrow[\cong]{\varphi_U} U \times \mathbb{C}^r$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow[\cong]{\varphi_U} & U \times \mathbb{C}^r \\ \uparrow & & \updownarrow \\ U & & U \end{array}$$

$s_i(z) = \varphi_U^{-1}((z, e_i))$, $i = 1, \cdots, r$

local trivializing frames.

$Ds_i = \sum_j \theta_i^j s_j$ with $(\theta_i^j) \in \Lambda T^*_{\mathbb{C}} X \otimes \mathfrak{gl}(r)$.

$D^2 s_i = \sum_j \Omega_i^j s_j$ where $\Omega_i^j = d\theta_i^j - \theta_i^k \wedge \theta_k^j$. or $\Omega = d\theta - \theta \wedge \theta$

- Bianchi identity:

$$d\Omega = -d\theta \wedge \theta + \theta \wedge d\theta = -(\Omega + \theta \wedge \theta) \wedge \theta + \theta \wedge (\Omega + \theta \wedge \theta)$$

$$= -\Omega \wedge \theta + \theta \wedge \Omega \quad \text{or} \quad d\Omega - (\theta \wedge \Omega - \Omega \wedge \theta) = 0$$

$\Leftrightarrow$ $\boxed{d\Omega - [\theta, \Omega] = 0 \quad \text{Bianchi identity.}}$

$$\left(\Longleftrightarrow \quad F_D = D^2 \text{ satisfies } \quad \underset{\displaystyle [D, F_D]}{\overset{}{D^{\operatorname{End}E} F_D}} = 0 \right).$$

- Chern forms:

$$\det\Big(\mathrm{Id}_r + t\,\underbrace{\tfrac{\sqrt{-1}}{2\pi}F_D}_{R_D}\Big) = 1 + t\cdot \mathrm{Tr}(R_D) + t^2 \cdot \tfrac{1}{2}\big(\mathrm{Tr}(R_D)^2 - \mathrm{Tr}(R_D^2)\big) + \cdots + t^r \det(R_D)$$

$C_k(E,D) =$ coefficient of $t^k \in A(\Lambda^{2k} T^*_{\mathbb{C}} X)$.

<u>Thm</u>: $C_k(E,D)$ is a closed $(2k)$-form.

<u>Pf</u>: $C_k(E,D)$ is a polynomial of $\mathrm{Tr}(R_D), \mathrm{Tr}(R_D^2), \cdots, \mathrm{Tr}(R_D^r)$.

(<u>Fact</u>: {invariant polynomials on $\mathfrak{gl}(r)$} is generated by $\mathrm{Tr}(R^k), k=1,\cdots,r$
$P: \mathfrak{gl}(r) \to \mathbb{C}$ s.t. $P(gAg^{-1}) = P(A)$ $\forall g \in GL(r), A \in \mathfrak{gl}(r)$.
Compare: {symmetric polynomials in $\{\lambda_1, \cdots, \lambda_r\}$} is generated by $\{\sum_{i=1}^r \lambda_i^k,\ k=1,\cdots,r\}$)

So it is enough to prove that $\mathrm{Tr}(F_D^k)$ is closed. (the constant $\frac{\sqrt{-1}}{2\pi}$ does not affect closedness)

Locally: $d\,\mathrm{Tr}(F_D^k) = d\,\mathrm{Tr}(\Omega^k) = k\cdot \mathrm{Tr}(d\Omega \cdot \Omega^{k-1})$ ← use $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$= k\cdot \mathrm{Tr}([\theta, \Omega]\cdot \Omega^{k-1})$ Bianchi identity

$= \mathrm{Tr}([\theta, \Omega^k]) = 0$.

∎

Thm: $[C_k(E,D)]$ does not depend on the choice of the connection D

$\Uparrow$

$H^{2k}_{dR}(X)$ $\qquad$ $C_k(E) := C_k(E,D)$ is called the k-th Chern classe of E

Pf: For any 2 connections D_0, D_1. connect them by a path

$$D_t = (1-t)D_0 + t\cdot D_1 \Rightarrow \dot{D}_t = \frac{d}{dt}D_t = D_1 - D_0 =: B$$

$\Uparrow$ tensor

$A(T^*_{\mathbb{C}}X \otimes \mathrm{End}\,E)$

induced connection on End E

$$\Rightarrow \frac{d}{dt}F_{D_t} = \frac{d}{dt}D_t^2 = D_t\cdot\dot{D}_t + \dot{D}_t D_t = \tilde{D}_t B \in A(\Lambda^2 T^*_{\mathbb{C}}X \otimes \mathrm{End}\,E)$$

$$\Rightarrow \frac{d}{dt}\mathrm{Tr}(F^k_{D_t}) = k\cdot\mathrm{Tr}(\dot{F}_{D_t}\cdot F^{k-1}_{D_t}) = k\cdot\mathrm{Tr}(\tilde{D}_t B\cdot F^{k-1}_{D_t})$$

$$\parallel$$

$$d[k\cdot\mathrm{Tr}(B\cdot F^{k-1}_{D_t})] = k\cdot\mathrm{Tr}(\tilde{D}_t(B\cdot F^{k-1}_{D_t})) = k\cdot\mathrm{Tr}(\tilde{D}_t B\cdot F^{k-1}_{D_t} + B\cdot \underbrace{\tilde{D}_t F^{k-1}_{D_t}}_{=0})$$

Ex Verify this (locally) $\qquad$ ∎ $\qquad$ by Bianchi

- Hermitian metric on E: $h = \{$Hermitian metric on $E_x : x \in X\}$

Smoothness: w.r.t. to (any) local trivialization $\pi^{-1}(U) \cong U \times \mathbb{C}^r$.

$\uparrow s_i$

U

$h_{i\bar{j}} = (s_i, s_j)$ is smooth in $z \in U$.

There are a lot of smooth Hermitian metric on E (using partition of unity).

Def: a connection D is compatible with the Hermitian metric

if $\quad d(s, s') = (Ds, s') + (s, Ds')$ for any $\underset{\substack{\cap \\ A(E)}}{s, s'}$.

D is then called a metric connection.

(Ex) There are a lot of metric connections. (use partition of unity)

For any Hermitian metric h, one can find local orthonormal frames $\{s_i\}$

s.t. $\quad (s_i, s_j) = \delta_{ij}$. Then

$$\theta_i^j + \overline{\theta_j^i} = 0$$

$$0 = d(s_i, s_j) = (Ds_i, s_j) + (s_i, Ds_j)$$

$$= (\theta_i^k s_k, s_j) + (s_i, \theta_j^k s_k) = \theta_i^k \delta_{kj} + \delta_{ik} \overline{\theta_j^k}$$

$\Rightarrow \quad \theta + \theta^* = 0$ where $\theta^* = \overline{\theta^t}$.

$\Rightarrow \quad \Omega + \Omega^* = 0 \Rightarrow \left(\frac{\sqrt{-1}}{2\pi}\Omega\right)^* = \frac{\sqrt{-1}}{2\pi}\Omega$

$\Rightarrow \quad \mathrm{Tr}\left(\left(\frac{\sqrt{-1}}{2\pi}\Omega\right)^k\right)$ is a real $(2k)$-form $\forall k \geq 0$.

$\Rightarrow \quad C_k(E, D)$ is a closed real $(2k)$-form.

$\Rightarrow \quad [C_k(E)] \in H^{2k}(X, \mathbb{R})$.

- Additive property of Chern classes.

If $0 \to E \to F \to G \to 0$ is an exact sequence of complex vector bundles

($E \subset F$ is a subbundle
$G = E/F$ is the quotient bundle)

Then as smooth vector bundles we have an isomorphism:

$$F \cong E \oplus G.$$

Pick connections D^E, D^G on E and G

$\Rightarrow$ $D^F = \begin{pmatrix} D^E & \\ & D^G \end{pmatrix}$ is a connection on F

$\Rightarrow$ $R_{D^F} = \begin{pmatrix} R_{D^E} & 0 \\ 0 & R_{D^G} \end{pmatrix}$

$$\Rightarrow \quad \underset{\displaystyle \sum_{k=0}^{r_F} c_k(F) \cdot t^k}{\underset{\|}{\det(Id + t \cdot R_{D^F})}} = \det(Id + t \cdot R_{D^E}) \cdot \det(Id + t \cdot R_{D^G})$$

$$\Rightarrow \begin{cases} c_1(F) = c_1(E) + c_1(G) \\ c_2(F) = c_2(E) + c_1(E) \cdot c_1(G) + c_2(G) \\ \cdots \\ c_{r_F}(F) = c_{r_E}(E) \cdot c_{r_G}(G) \end{cases}$$

$1+\sum_{k=1}^{r_E} c_k(E) \in H^{even}(X,\mathbb{R})$ also called the ~~total~~ Chern class of E.

- From now on, assume that E is a holomorphic vector bundle over a complex manifold.

So $\forall x \in X$, $\exists\, U \ni x$ and local holomorphic trivialization:

$$\begin{tikzcd} \varphi_U : \pi^{-1}(U) \arrow[r] & U \times \mathbb{C}^r \\ U \arrow[u, "s_i"'] & U \arrow[u, "e_i"'] \end{tikzcd}$$

$\{ s_i(z) = \varphi_U^{-1}(z, e_i) \}$ form a local holomorphic frame.

Any section $s \in A(E)$ locally over U can be written as $s = \sum_i a_i s_i$ with a_i smooth function on U.

s is called <u>holomorphic</u> if a_i are holomorphic functions for all such holomorphic trivialization.

- There is a well-defined $\bar\partial$-operator:

$$\bar\partial : A(E) \longrightarrow A(T^{*(0,1)}X \otimes E)$$
$$s \longmapsto \sum_i \bar\partial a_i \otimes s_i$$

So a section $s \in A(E)$ is holomorphic if and only if $\bar\partial s = 0$.

· Let D be a connection on E,

$$A(E) \xrightarrow{D} A(T_{\mathbb{C}}^{*}X \otimes E) \begin{array}{l} \xrightarrow{\pi'} A(T_{\mathbb{C}}^{*(1,0)}X \otimes E) \\ \xrightarrow[\pi'']{} A(T_{\mathbb{C}}^{*(0,1)}X \otimes E) \end{array}$$

$\Rightarrow D$ decomposes into $\underset{\pi' \circ D}{\overset{\shortparallel}{D'}} + \underset{\pi'' \circ D}{\overset{\shortparallel}{D''}}$

<u>Def</u>: A connection D is compatible with the holomorphic structure if $D'' = \bar{\partial}$.

If s is holomorphic, then $Ds = D's \in A(T^{*(1,0)}X \otimes E)$.

Choose a local trivialization $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ with local holomorphic frames $\{s_i\}$.

Then the connection form defined by $Ds_i = \sum_j \theta_i^j s_j$ satisfies $\theta_i^j \in A(U, T^{*(1,0)}X)$.

(Chern)
<u>Thm</u>: Given a Hermitian metric h on a holomorphic vector bundle $E \to X$, there exists a unique connection D that is compatible with the Hermitian metric h and is compatible with the holomorphic structure.

(This connection is sometimes called the Chern connection)

<u>Pf</u>: Choose a holomorphic trivialization with holomorphic frames $\{s_i\}$. Then because D is compatible with h:

$$\begin{array}{ccc} d(s_i, s_j) & = & (Ds_i, s_j) + (s_i, Ds_j) \\ \| & & \| \\ dh_{i\bar{j}} & & \theta_i^k h_{k\bar{j}} + h_{i\bar{k}} \overline{\theta_j^k} \\ \| & & \\ \partial h_{i\bar{j}} + \bar{\partial} h_{i\bar{j}} & & \end{array}$$

If D is compatible with the holomorphic structure, then

$$\theta_i^k \in A(T^{*1,0}X) \Longrightarrow \partial h_{i\bar{j}} = \theta_i^k \cdot h_{k\bar{j}}$$

$$\Rightarrow \theta_i^k = \partial h_{i\bar{j}} (h^{-1})^{\bar{j}k}$$

So $\theta = \partial h \cdot h^{-1}$

$$\Rightarrow \Omega = d\theta - \theta\wedge\theta = d(\partial h\cdot h^{-1}) - \partial h\cdot h^{-1}\wedge\partial h\cdot h^{-1}$$

$$= \bar\partial(\partial h\cdot h^{-1}) - \partial h\cdot \underbrace{\partial h^{-1}}_{\substack{\| \\ -h^{-1}\cancel{\partial h}\cdot h^{-1}}} - \partial h\cdot \cancel{h^{-1}}\wedge\partial h\cdot h^{-1}$$

$$= \bar\partial(\partial h\cdot h^{-1}) \in A(T^{*(1,0)}\otimes T^{*(0,1)}\otimes \mathrm{End}(E))$$

$$\| $$

$$A(\Lambda^{1,1}T^*X\otimes \mathrm{End}E).$$

E holomorphic vector bundle

$\rightsquigarrow$ $\mathcal{O}(E)$ = sheaf of (local) holomorphic sections

Thm (Dolbeault thm for holomorphic vector bundles)

$$H^q(X,\mathcal{O}(E)) \cong H^{p,q}_{\bar\partial}(X,E)$$

$$\| $$

$$\frac{\ker\{\bar\partial: A^{p,q}(E)\to A^{p,q+1}(E)\}}{\mathrm{Im}\{\bar\partial: A^{p,q-1}(E)\to A^{p,q}(E)\}}$$