- holomorphic vector bundle

$$\begin{array}{c} E \\ \downarrow \pi_E \\ X \end{array}$$

E is a complex mfd. with a holomorphic surj. map $\underset{\pi_E}{\overset{\pi}{\shortparallel}}$ to X

- $\forall x \in X$, $\exists$ an open subset $U \ni x$ s.t $\exists$ biholomorphic map

$$\varphi_U : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}^r \quad \text{(local trivialization)}.$$

- For two such open subsets $U, V \subset X$ with $U \cap V \neq \phi$

$$\varphi_{UV} := \varphi_U \circ \varphi_V^{-1} : (U \cap V) \times \mathbb{C}^r \to (U \cap V) \times \mathbb{C}^r$$

$$(p, v) \mapsto (p, \varphi_{UV}(p) \cdot v)$$

is linear for the variable v for any $p \in U \cap V$.

$\rightsquigarrow$ holomorphic map $\varphi_{UV} : U \cap V \longrightarrow GL(r, \mathbb{C})$

satisfying the cocycle condition:

$$(*) \quad \varphi_{UV} \circ \varphi_{VW} \circ \varphi_{WU} = Id_{\mathbb{C}^r} \text{ if } U \cap V \cap W \neq \phi.$$

Let $\mathcal{G}\mathcal{L}_r$ be sheaf of holomorphic maps from any open sets to $GL(r, \mathbb{C})$.

Then $\{\varphi_{UV}\}$ is a cocycle by $(*)$ and hence defines a cohomology class in $H^1(X, \mathcal{G}\mathcal{L}_r)$.

Conversely if we have the data $\{\varphi_{UV}\}$ satisfying $(*)$.

$\{U \times \mathbb{C}^r\}$ to get a holomorphic vector bundle: $E = \bigcup U \times \mathbb{C}^r / \sim$

$$(p, v) \sim (p, \varphi_{UV}(p) \cdot v), \ \forall p \in U \cap V, \ v \in \mathbb{C}^r$$

$\rightsquigarrow$ There is a one-to-one correspondence:

$$\{\text{isomorphism classes of holomorphic vector bundles}\}$$
$$\Updownarrow$$
$$H^1(X, \mathcal{G}\mathcal{L}_r).$$

<u>Ex</u>: X^n complex mfd.

$E_1 = T_{hol} X$: holomorphic tangent bundle.

$\forall\, x \in X$, $\exists$ coordinate chart $\{U, \{z_1, \cdots, z_n\}\}$, then

$$\pi_{E_1}^{-1}(U) = U \times \mathbb{C}\left\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\right\} \cong U \times \mathbb{C}^r.$$

Similarly, $\overset{E_2}{\overset{\shortparallel}{T^*_{hol}}} X$: holomorphic cotangent bundle

$$\pi_{E_2}^{-1}(U) = U \times \mathbb{C}\{dz_1, \cdots, dz_n\} \cong U \times \mathbb{C}^r.$$

- r is called the rank of E

If $r = 1$, then E is called a line bundle.

$GL_1 \cong \mathcal{O}^* \rightsquigarrow$ {isomorphism classes of holomorphic line bundles}

$$\Updownarrow$$

$$H^1(X, \mathcal{O}^*).$$

exponential sequence: $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$

$\rightsquigarrow$ long exact sequence of cohomology groups:

$$\cdots \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}) \to \cdots$$

$$\begin{array}{ccc} \cup\!\!\shortmid & & \\ [L] & \longmapsto & \delta([L]) =: c_1(L) \end{array}$$

The connecting morphism δ defines a cohomology class $\underset{H^2(X,\mathbb{Z})}{c_1(L)}$ called the 1st Chern class of the line bundle L.

For higher rank vector bundles, we will define Chern classes for any complex vector bundles using the Chern-Weil theory.

- Let E be any complex vector bundle
  (the transition functions $\{\varphi_{UV}\}$ are not assumed to be holomorphic)
  only smooth (and linear in $\mathbb{C}^r$)

<u>Def</u>: A connection on E is a $\mathbb{C}$-linear map

$$D : \mathcal{A}(E) \longrightarrow \mathcal{A}(T_{\mathbb{C}}^{*}X \otimes E)$$

$\mathcal{A}(E)$: space of smooth sections of E

$\mathcal{A}(T_{\mathbb{C}}^{*}X \otimes E) = (T_{\mathbb{R}}^{*}X \otimes_{\mathbb{R}} \mathbb{C}) \otimes E$: sm. sections of $T_{\mathbb{C}}^{*}X \otimes E$ = E-valued 1-forms

that satisfies the Leibniz rule:

$$D(f \cdot s) = df \cdot s + f \cdot Ds \quad \text{for any } f \in C^{\infty}(X, \mathbb{C}) \text{ and } s \in \mathcal{A}(E)$$

$\rightsquigarrow$ D induces maps

$$D : \mathcal{A}(\Lambda^k T_{\mathbb{C}}^{*}X \otimes E) \longrightarrow \mathcal{A}(\Lambda^{k+1} T_{\mathbb{C}}^{*}X \otimes E)$$

defined by $\quad \eta \otimes s \longmapsto D(\eta \cdot s) := d\eta \otimes s + (-1)^k \eta \wedge Ds$

$\eta \in \mathcal{A}(\Lambda^k T_{\mathbb{C}}^{*}X)$
$s \in \mathcal{A}(E)$

Consider: $\quad D^2 : \mathcal{A}(E) \longrightarrow \mathcal{A}(\Lambda^2 T_{\mathbb{C}}^{*}X \otimes E)$

$D^2 = D \circ D$

Then $D^2(fs) = D(df \otimes s + f \cdot Ds) \quad \forall f \in C^{\infty}(X, \mathbb{C}),\ s \in \mathcal{A}(E)$

$$= -df \wedge Ds + df \wedge Ds + f \cdot D^2 s = f D^2 s$$

$\Rightarrow$ D^2 is a tensor. i.e. $D^2 \in \mathcal{A}(\Lambda^2 T_{\mathbb{C}}^{*}X \otimes \operatorname{End} E)$.

$D^2 = F_D$ (curvature tensor of D)

- Local calculations:

Under local trivalization $\varphi_U : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}^r$

$s_i(p) := \varphi_U^{-1}(\{p\} \times e_i\}$ defines local sections s_i of E over U

$\forall p \in U$, $\{s_i(p) \, ; i = 1, \cdots, r\}$ for a basis for $E_p = \pi^{-1}(\{p\}) \cong \mathbb{C}^r$

$\{s_i\}$ are called local (trivializing) frames for E over U.

A connection D is determined locally by:

$$D s_i = \sum_{j=1}^{r} \theta_i^j s_j \quad \text{with} \quad \theta_i^j \in A(U, \Lambda^1 T_{\mathbb{C}}^* X \otimes \operatorname{End} E)$$

called connection forms. w.r.t. the frame $\{s_i\}$

$$F_D \cdot s_i = D^2 s_i = D\Big(\sum_j \theta_i^j s_j\Big) = \sum_j \Big(d\theta_i^j s_j - \sum_k \theta_i^j \wedge \theta_j^k s_k\Big)$$

$$= \sum_j \Big(d\theta_i^j - \sum_k \theta_i^k \wedge \theta_k^j\Big) s_j$$

$$= \sum_j \Omega_i^j s_j \qquad \Omega_i^j = d\theta_i^j - \sum_k \theta_i^k \wedge \theta_k^j$$

$$\Omega_i^j \in A(U, \Lambda^2 T_{\mathbb{C}}^* X \otimes GL(r)).$$

$$\Rightarrow F_D = \sum_{i,j=1}^{r} \Omega_i^j \, s_j \otimes s_i^* \in A(X, \Lambda^2 T_{\mathbb{C}}^* X \otimes \operatorname{End} E)$$

where $\{s_i^*\}$ is the dual basis of $\{s_i\}$

Ex Check the right hand side does not depend on the choice of $\{s_i\}$.

- Chern forms: Expand the characteristic polynomial:

$$\det\Big(\mathrm{Id}_r + t \cdot \underbrace{\frac{\sqrt{-1}}{2\pi} F_D}_{=: R_D}\Big) = 1 + t \cdot \operatorname{Tr}(R_D) + t^2 \frac{\operatorname{Tr}(R_D)^2 - \operatorname{Tr}(R_D^2)}{2} + \cdots + t^r \cdot \det(R_D)$$

$$= 1 + t \cdot c_1(R_D) + t^2 c_2(R_D) + \cdots + t^r c_r(R_D)$$

$c_k(R_D)$ is a polynomial in $\mathrm{Tr}(R_D), \mathrm{Tr}(\overset{R_D \wedge R_D}{\overset{\parallel}{R_D^2}}), \cdots, \mathrm{Tr}(R_D^r)$

an is a differential form of degree $2k$ on X.

<u>Thm:</u> ① $c_k(R_D)$ is a closed form and hence defines a cohomology class $[c_k(R_D)] \in H^{2k}(X, \mathbb{C})$.

② $[c_k(R_D)]$ does not depend on the choice of the connection D

③ $[c_k(R_D)] \in H^{2k}(X, \mathbb{R}) \hookrightarrow H^{2k}(X, \mathbb{C})$.

$c_k(E) := [c_k(R_D)]$ is called the k-th Chern class of E.