X: complex mfd. $Y^m \subseteq X^n$ closed subvariety

$\rightsquigarrow$ cycle class $[Y] \in H^{2n-2m}(X,\mathbb{R}) = H^{2m}_c(X,\mathbb{R})^*$

$$[Y]: H^{2m}_c(X,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$[\varphi] \longmapsto \int_{Y_{reg}} \varphi$$

Need the estimates:
- $\int_{Y_{reg}} \varphi < +\infty$
- $\int_{Y_{reg}} d\eta = 0$ for a smooth η (with cpt. support)

These follow from the structure result:

<u>Fact</u>: Y is locally a finite covering over the polydisk $\Delta^m = \{ \{z_i\}_{i=1}^m \in \mathbb{C}^m : |z_i| < 1 \}$ branched along an analytic hypersurface.

Use this fact, one can prove

$\forall p \in Y$ $\exists$ nbhd. U s.t. $\int_{U \cap Y_{reg}} \varphi \le C \int_{\Delta^m} |dz_I \wedge d\bar{z}_I|$

$dz_I = dz_1 \wedge \cdots \wedge dz_n$

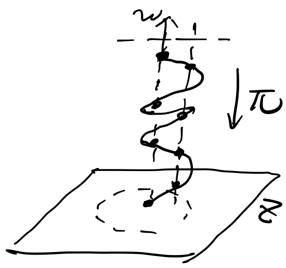
Moreover $\exists$ small nbhd. U_ε of Y_{sing} s.t.

$$\left| \int_{Y_{reg}} d\eta \right| = \lim_{\varepsilon \to 0} \left| \int_{Y \setminus U_\varepsilon} d\eta \right| = \lim_{\varepsilon \to 0} \left| \int_{\partial U_\varepsilon} \eta \right| \le C \cdot Vol_{2m-1}(\partial U_\varepsilon) \xrightarrow{\varepsilon \to 0} 0$$

Sketch of proof in the case Y is a hypersurface: $Y=\{f(z,w)=0\}$

$(z_1,\cdots,z_{n-1},w)\in\mathbb{C}^{n-1}\times$

By a generic choice of (z,w), $f(0,w)\not\equiv 0$ and $Y\cap\{z=0\}\subset\Delta=\{|w|<1\}$.



$\Rightarrow$ for fixed z with $|z|<\varepsilon<<1$, $w\mapsto f(z,w)\not\equiv 0$

$\Rightarrow$ assume $Y\cap\pi^{-1}(z)=\{\underbrace{b_1(z),\cdots,b_d(z)}_{\text{can repeat with multiplicities}}\}$

then

$$b_1(z)^l+\cdots+b_d(z)^l=\frac{1}{2\pi i}\int_{|w|=1}\frac{w^l\frac{\partial f}{\partial w}(z,w)}{f(z,w)}dw \quad \text{is holomorphic in } z$$

So

$$\sigma_k(z)=\sigma_k(b_1(z),\cdots,b_d(z))=\sum_{1\le i_1<\cdots<i_k\le d} b_{i_1}(z)\cdots b_{i_k}(z),$$ which is a polynomial of $\{b_1^l+\cdots+b_d^l,\ l\ge 1\}$

is also holomorphic in z

Set $g(z,w)=w^d-\sigma_1(z)w^{d-1}+\sigma_2(z)\cdot w^{d-2}-\cdots+(-1)^d\sigma_d(z)$

with $\sigma_i(0)=0$ (such polynomials are called Weierstrass polynomials)

Then $\{g(z,w)=0\}=Y$

$\Rightarrow$ $\dfrac{f(z,w)}{g(z,w)}=:h(z,w)$ is holomorphic and nonzero near $0\in\mathbb{C}^n$.

$\pi : (z,w) \mapsto z$ is branched at z if there are repeated roots.
or equivalently if the discriminant

$$D(z) = \prod_{i<j} (b_i(z) - b_j(z))^2 \quad \text{is zero.}$$

$D(z)$ is symmetric w.r.t. $\{b_i\} \Rightarrow D(z)$ is a polynomial of $\{\sigma_k(z)\}_k$

$\Rightarrow D(z)$ is holomorphic in z. Set $S = \{z : D(z) = 0, |z_i| \ll 1\}$

Then $\pi|_Y : Y \to \Delta_\varepsilon^m$ is a finite covering along the analytic subvariety $S \cap \Delta_\varepsilon^m$. ∎

- Intersection of analytic subvarieties.

$$X^n \supset Y^m \longrightarrow [Y] \in H^{2n-2m}(X)$$

$$X^n \supset Z^{n-m} \longrightarrow [Z] \in H^{2m}(X)$$

$$H^{2n-2m}(X) \times H^{2m}(X) \to H^{2n}(X, \mathbb{R}) \cong \mathbb{R}$$

$$([Y], [Z]) \mapsto [Y] \cdot [Z]$$

$\|$ Assume Y and Z intersect at isolated points

$$\sum_{P \in Y \cap Z} m(P; Y \cap Z) \cdot P$$

local multiplicity

<u>Fact:</u> $m(P; Y\cap Z)\geq 0$ (not true for real submfds)

$\Rightarrow$ If Y and Z intersect at isolated points then $\#\{Y\cap Z\}\leq [Y]\cdot[Z]$

- (co) homology of $\mathbb{P}^n = \mathbb{C}^n\cup\mathbb{C}^{n-1}\cup\mathbb{C}^{n-2}\cup\cdots\cup\mathbb{C}\cup pt$

$$\mathbb{P}^n = \mathbb{C}^n\cup\mathbb{P}^{n-1} = \mathbb{C}^n\cup\mathbb{C}^{n-1}\cup\mathbb{P}^{n-2}$$

$$\Rightarrow \quad H_m(\mathbb{P}^n,\mathbb{Z}) = \begin{cases}\mathbb{Z} & m \text{ even}, 0\leq m\leq 2n\\ 0 & \text{otherwise}\end{cases}$$

$$\Rightarrow \quad H^m(\mathbb{P}^n,\mathbb{Z}) = \begin{cases}\mathbb{Z} & m \text{ even } 0\leq m\leq 2n\\ 0 & \text{otherwise}\end{cases}$$

- If $L^m\cong\mathbb{P}^m$ is an m-dim linear subspace, then

$$H_{2m}(\mathbb{P}^n,\mathbb{Z})\cong\mathbb{Z}\cdot[L^m] = H^{2n-2m}(\mathbb{P}^n,\mathbb{Z})$$

Application: If $Z^m\subset\mathbb{P}^n$ is a closed analytic subvariety then $[Z] = d\cdot[L^m]$ for some $d>0$.

$\hookrightarrow \deg(Z) = d$

Choose an $(n-m)$-dim linear subspace L^{n-m} of $\mathbb{P}^n$

then $[Z]\cdot[L^{n-m}] = d\cdot[L^m]\cdot[L^{n-m}] = d \in \mathbb{Z}_{>0}$

$\|$ if Z and L intersect at isolated points.

$\sum_{P\in Z\cap L} m(P; Z\cap L)$

· If Z is a hypersurface with $\deg(Z)=1$ i.e. $\underset{[Z]=[L^{n-1}]}{\overset{H}{\|}}$

then for any two pts $p, q \in Z$, the line^L containing p an q must be contained in Z

$\Rightarrow$ Z must be a $(n-1)$-dim linear subspace, or equivalently a hyperplane H $\left(\begin{array}{l}\text{since otherwise}\\ 2 \geq \#(L\cap Z) > [L]\cdot[Z]\end{array}\right)$

· $\mathrm{Aut}(\mathbb{P}^n) = \{$ biholomorphism from $\mathbb{P}^n$ to $\mathbb{P}^n\}$

$\cup$

$\sigma: \mathbb{P}^n \to \mathbb{P}^n$ holomorphic, bijective (and $\sigma^{-1}: \mathbb{P}^n \to \mathbb{P}^n$ also holomorphic)

<u>Thm:</u> $\mathrm{Aut}(\mathbb{P}^n) = PGL(n+1, \mathbb{C})$

Sketch of proof: $\sigma \in \mathrm{Aut}(\mathbb{P}^n)$, Fix any hyperplane H

Then

$$[\sigma \cdot H] = d \cdot [H] \Rightarrow d \in \mathbb{Z} \Rightarrow d = 1 \Rightarrow \sigma \cdot H = H' \text{ is a hyperplane}$$

$$\Downarrow \qquad\qquad d^{-1} \in \mathbb{Z}$$

$$[\sigma^{-1} H] = d^{-1} \cdot [H]$$

Set $\forall i \quad z_i = \dfrac{Z_i}{Z_0}$ which is a meromorphic fct. on $\mathbb{P}^n$

$$(z_i = 0) = (Z_i = 0) \cap \underset{\{Z_0 \neq 0\}}{\underset{\parallel}{\mathbb{C}^n}} =: H_i \cap \mathbb{C}^n$$

$$\sigma(H_i) = H_i' = \{a_0 Z_0 + \cdots + a_n Z_n = 0\}$$

$\Rightarrow \dfrac{\sigma^* z_i}{a_0 + a_1 z_1 + \cdots + a_n z_n}$ is holomorphic on $\mathbb{P}^n$, hence a constant

$\Rightarrow \sigma$ is linear.