Recall:

<u>Leray's Thm</u>: X a complex mfd. $\underline{U}=\{U_i\}_{i\in I}$.

If $\underline{U}$ is acyclic for Ω^P (i.e. $H^q(U_I,\Omega^P)=0,\ \forall I,q$), then $\check{H}(\underline{U},\Omega^P)=\check{H}(X,\Omega^P)$.

(Ω^P ↑ sheaf of holomorphic P-forms)

<u>Pf.</u> Consider the double complex $D^{r,s}=C^r(\underline{U},A^{p,s})$ and the associated single complex $K^t=\bigoplus_{r+s=t} D^{r,s}$. $d: K^t \to K^{t+1}$, $d = \delta+(-1)^t\bar\partial$, $d^2=0$.

Claim I: $H^q(K^\bullet)\cong H^{p,q}_{\bar\partial}(X)$ (always true)

$$
H^{p,*}_{\bar\partial}(X)\left\{\begin{array}{c} \uparrow\bar\partial \\ H^0(A^{p,2}) \\ \uparrow\bar\partial \\ H^0(A^{p,1}) \\ \uparrow\bar\partial \\ H^0(A^{p,0}) \end{array}\right.
\quad
\begin{array}{ccccccc}
\uparrow\bar\partial & & \uparrow\bar\partial & & \uparrow\bar\partial & & \\
C^0(\underline{U},A^{p,2}) & \xrightarrow{\delta} & C^1(\underline{U},A^{p,2}) & \xrightarrow{\delta} & C^2(\underline{U},A^{p,2}) & \to & \\
\uparrow\bar\partial & & \uparrow\bar\partial & & \uparrow\bar\partial & & \\
C^0(\underline{U},A^{p,1}) & \xrightarrow{\delta} & C^1(\underline{U},A^{p,1}) & \xrightarrow{\delta} & C^2(\underline{U},A^{p,1}) & \to & \cdots \\
\uparrow\bar\partial & & \uparrow\bar\partial & & & & \\
C^0(\underline{U},A^{p,0}) & \xrightarrow{\delta} & C^1(\underline{U},A^{p,0}) & \xrightarrow{\delta} & C^2(\underline{U},A^{p,0}) & \to & \cdots \\
\hline
\bigoplus_i H^0(U_i,\Omega^p) & & \bigoplus_{ij} H^0(U_{ij},\Omega^p) & & \bigoplus_{ijk} H^0(U_{ijk},\Omega^p) & & \\
\| & & \| & & \| & & \\
C^0(\underline{U},\Omega^p) & \xrightarrow{\delta} & C^1(\underline{U},\Omega^p) & \xrightarrow{\delta} & C^2(\underline{U},\Omega^p) & &
\end{array}
$$

$$\underbrace{C^0(\underline{U},\Omega^p) \xrightarrow{\delta} C^1(\underline{U},\Omega^p) \xrightarrow{\delta} C^2(\underline{U},\Omega^p)}_{\check{H}^*(\underline{U},\Omega^p)}$$

Claim II: If $\underline{U}$ is acyclic for Ω^p, then $\check{H}^q(\underline{U},\Omega^p)=H^q(K^\bullet)$

<u>Rmk:</u> If $\underline{U}$ is not acyclic, then we have the E_1-page:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\bigoplus H^{p,2}_{\bar\partial}(U_i) & \xrightarrow{d_1} & \bigoplus_{ij} H^{p,2}(U_{ij}) & \xrightarrow{d_1} & \bigoplus_{ijk} H^{p,2}_{\bar\partial}(U_{ijk}) \\
\bigoplus_i H^{p,1}_{\bar\partial}(U_i) & \xrightarrow{d_1} & \bigoplus_{ij} H^{p,1}_{\bar\partial}(U_{ij}) & \xrightarrow{d_1} & \bigoplus_{ijk} H^{p,1}_{\bar\partial}(U_{ijk}) \\
\bigoplus_i H^0(U_i,\Omega^p) & \xrightarrow{d_1} & \bigoplus_i H^0(U_{ij},\Omega^p) & \xrightarrow{d_1} & \bigoplus_{ijk} H^0(U_{ijk}) \\
\hline
\end{array}
$$

$$E_1^{r,s} = \bigoplus_{|I|=r+1} H_{\bar\partial}^{p,s}(U_I)$$

$d_1 : E_1^{r,s} \to E_1^{r+1,s}$ is induced by δ, $d_1^2 = 0$

$$\rightsquigarrow \quad E_2^{r,s} = H_{d_1}^r(E^{\bullet,s}) \qquad D^{r,s}$$

$\begin{matrix} \Psi \\ [x] \end{matrix}$ is represented by $x \in C^r(\underline{U}, A^{p,s})$ $\left(= D^{r,s}\right)$ satisfying

$\delta x = \bar\partial y$ with $y \in D^{r,s-1}$

$d_2 : E_2^{r,s} \to E_2^{r+2,s-1}$ is given by

$$[x] \mapsto [\delta y]$$

$$\begin{tikzcd} x \arrow[r] & \delta x = \bar\partial y \\ & y \arrow[u, "\bar\partial"] \arrow[r] & \delta y \end{tikzcd}$$

Inductively we get a spectral sequence $\{E_k, d_k\}$

$$d_k : E_k^{r,s} \longrightarrow E_k^{r+k, s-k+1}$$

s.t. $\quad E_{k+1}^{r,s} = H^{r,s}(E_k^{\bullet,\bullet})$

This process terminates: For some $m \in \mathbb{Z}_{>0}$, s.t.

$$E_m = E_{m+1} = \cdots = \cdots =: E_\infty \rightsquigarrow H^\bullet(K^\bullet)$$

(Spectral sequence degenerates at E_m)

- Cycle class

X: a complex mfd.

$Y \subset X$ is an analytic subvariety if $\forall p \in Y$, there exists an open nbhd. $U \ni p$ s.t.

$$U \cap Y = \{f_1 = \cdots = f_\ell = 0\} \text{ for } f_i \in \mathcal{O}(U), \quad i=1,\cdots,\ell$$

$$U \cap Y_{reg} = \left\{ z \in U \cap Y : \operatorname{rk}\left\{\frac{\partial f_i}{\partial z_k}\right\}_{\substack{i=1,\cdots,\ell \\ k=1,\cdots,n}} \text{ is locally constant} \right\}$$

$$Y_{sing} = Y \setminus Y_{reg}.$$

Ex: $X = \mathbb{C}^n$, $f \in \mathcal{O}(\mathbb{C}^n)$, $Y = \{f = 0\}$.

$Y = Y_{reg} \cup X_{sing}$ where

$$Y_{reg} = \left\{ z \in \mathbb{C}^n ;\ \operatorname{rk}\left\{\frac{\partial f}{\partial z_i}\right\}_{\cancel{1\times} n} = 1 \right\}$$

$$Y_{sing} = Y \setminus Y_{reg}$$

General fact:

- $Y = Y_1 \cup \cdots \cup Y_r$ decomposition into irreducible analytic subvarieties

- Y is irreducible $\Leftrightarrow$ Y_{reg} is connected.

Define: $\dim Y = \dim Y_{reg}$

- Assume Y is irreducible, $\dim Y = m$

For a generic choice of local coordinates $\{z_i\}$ on X.

the projection of $(z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_m)$ realizes

Y locally as a finite covering over a polydisk $\{\{z_j\} \in \mathbb{C}^m : |z_j| < \varepsilon\}$

branched along an analytic hypersurf.

(Consequence of Weierstrass Preparation Thm)

Now define $[Y] \in H^{2(n-m)}(X, \mathbb{R})$.

By Poincaré duality given by the bilinear pairing

$$H_c^{2m}(X, \mathbb{R}) \times H^{2n-2m}(X, \mathbb{R}) \to \mathbb{R}$$

cptly supported (de Rham) cohomology

$$([\eta], [\omega]) \mapsto \int_X \eta \wedge \omega$$

$$H^{2n-2m}(X, \mathbb{R}) \cong H_c^{2m}(X, \mathbb{R})^*$$

So we just need to define $[Y]$ as a linear functional on $H_c^{2m}(X, \mathbb{R})$.

- If Y is smooth, then for any $[\eta] \in H_c^{2m}$ define

$$[Y] \; [\eta] \longmapsto \int_Y \eta$$

In general, set $[\eta] \longmapsto \int_{Y_{reg}} \eta$

<u>Fact</u>: - The right hand side is finite.

- If $\eta_2 - \eta_1 = d\chi$ for $\chi \in C_c^\infty(X, \mathbb{R})$ then

$$\int_{Y_{reg}} d\chi = 0$$

(consequence of estimate by using the finite covering picture)

Alternatively, can define:

$$\begin{array}{ccc} [Y]_{sing} \in H_{2m}(X,\mathbb{R})_{sing} & \cong & (H_c^{2m}(X,\mathbb{R}))^* \\ \wr\wr & & \wr\wr \; PD \\ [Y] \in & & H^{2n-2m}(X,\mathbb{R}) \end{array}$$

Rmk: For the constant sheaf $\mathbb{R}$, $\check{H}^*(X,\mathbb{R}) \cong H^*_{sing}(X,\mathbb{R})$ singular cohomology

(by constructing isomorphic complexes computing the two sides [see Griffiths-Harris])