$\mathscr{F}$: sheaf on X

$$\check{H}^q(X,\mathscr{F}) = \varinjlim_{\underline{U}} \check{H}^q(\underline{U},\mathscr{F}) = \varinjlim_{\underline{U}} \frac{\ker(\delta: C^q \to C^{q+1})}{\operatorname{Im}(\delta: C^{q-1}\to C^q)}$$

Ex: $A^{p,q}$ are fine sheaves.

- $\mathscr{F}$ is called a fine sheaf if for any open covering $\underline{U}=\{U_i\}$, there exist morphisms $\eta_i : \mathscr{F}(U_i) \to \mathscr{F}(U_i)$ s.t. $\operatorname{supp}(\eta_i) \subset\subset U_i$ and $\sum_i \eta_i = Id$

$\left(\begin{array}{l} \text{i.e. } \sum_i \eta_i \circ r_{UU_i} = Id \\ \text{for any open } U\subset X \end{array}\right)$

Prop: If $\mathscr{F}$ is a fine sheaf, then $H^q(X,\mathscr{F})=0$ for $q>0$.

Pf: Let $\sigma = \{\sigma_{i_0\cdots i_q}\}$ be a cocycle.

Set $\tau_{i_1\cdots i_q} = \sum_i \eta_i \sigma_{i i_1\cdots i_q}$

Then
$$(\delta\tau)_{i_0\cdots i_q} = \sum_{j=0}^{q} (-1)^j \tau_{i_0\cdots \hat{j}\cdots i_q}$$
$$= \sum_{j=0}^{q}(-1)^j \sum_i \eta_i \sigma_{i i_0\cdots\hat{j}\cdots i_q}$$
$$= \sum_i \eta_i \sum_{j=0}^{q}(-1)^j \sigma_{i i_0\cdots\hat{j}\cdots i_q} \overset{\delta\sigma=0}{=} \sum_i \eta_i \sigma_{i_0\cdots i_q} = \sigma_{i_0\cdots i_q}$$

$\Rightarrow \delta\tau = \sigma \Rightarrow [\sigma]=0$.

- Dolbeault Theorem: $\check{H}^q(X,\Omega^p) = H^{p,q}_{\bar\partial}(X)$

sheaf of holomorphic p-forms.

Pf. Consider the resolution

$$0\to \Omega^p \to A^{p,0} \xrightarrow{\bar\partial} A^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} A^{p,q} \xrightarrow{\bar\partial} A^{p,q+1} \to \cdots$$

sheaf of smooth (p,q)-forms

set: $Z^k = \ker(\bar\partial: A^{p,k}\to A^{p,k+1}) = \operatorname{Im}(\bar\partial: A^{p,k-1}\to A^{p,k})$.

Then we get short exact sequences:

$$0 \to \underset{\overset{\|}{Z^0}}{\Omega^p} \to A^{p,0} \to Z^1 \to 0 \rightsquigarrow H^{q-1}(A^{p,0}) \to H^{q-1}(Z^1) \to H^q(\Omega^p) \to H^q(A^{p,0})$$

$$H^q(A^{p,0}) \overset{\|}{0} \text{ if } q \geq 1.$$

$$0 \to Z^1 \to A^{p,1} \to Z^2 \to 0$$

$$\cdots\cdots$$

$$0 \to Z^{q-2} \to A^{p,q-2} \to Z^{q-1} \to 0$$

$$0 \to Z^{q-1} \to A^{p,q} \to Z^q \to 0$$

$H(A^q) \overset{\|}{0}$

Last one gives: $H^0(A^{p,q-1}) \xrightarrow{\bar\partial} H^0(Z^q) \to H^1(Z^{q-1}) \to 0$

$$\Rightarrow \quad H^1(Z^{q-1}) \cong \frac{H^0(Z^q)}{\bar\partial(H^0(A^{p,q-1}))} = \frac{\{\bar\partial\text{-closed } (p,q)\text{-forms}\}}{\{\bar\partial\text{-exact } (p,q)\text{-forms}\}} = H^{p,q}_{\bar\partial}(X)$$

Going upward: $H^1(Z^{q-1}) = H^2(Z^{q-2}) = \cdots = H^{q-1}(Z^1) \underset{q\geq 2}{=} H^q(\Omega^p)$

$\Big($ $q=1$: $H^0(A^{p,0}) \to H^0(Z^1) \to H^1(\Omega^p) \to 0$

$$\Rightarrow H^1(\Omega^p) = \frac{\{\bar\partial\text{-closed } (p,1)\text{-forms}\}}{\{\bar\partial\text{-exact } (p,1)\text{-forms}\}} = H^{p,1}_{\bar\partial}(X) \Big)$$

- Lerays Thm (for Ω^p): If $\underline{U} = \{U_i\}$ is acyclic for Ω^p (i.e. $H^q(U_I, \Omega^p) = 0$, for all $q \geq 1$ and any I)

  Then $\check{H}^q(\underline{U}, \Omega^p) = \check{H}^q(X, \Omega^p)$.

Pf: Because of Dolbeault's Thm ($\check{H}^q(X,\Omega^p) \cong H^{p,q}_{\bar\partial}(X)$)
We just need to prove $\check{H}^q(\underline{U}, \Omega^p) \cong H^{p,q}_{\bar\partial}(X)$.

Consider the double complex

$$D = \left\{ D^{r,s} = C^r(\underline{U}, A^{p,s}) \;:\; \begin{matrix} \delta: D^{r,s} \to D^{r+1,s} \\ \bar\partial: D^{r,s} \to D^{r,s+1} \end{matrix}, \; r,s \geq 0 \right\}$$

and the associated single complex:

$$K^q = \bigoplus_{r+s=q} D^{r,s} \quad \text{with} \quad d = \bar\partial + (-1)^q \delta : K^q \to K^{q+1}$$

$$\pm d^2 = (\bar\partial + \delta)(\bar\partial - \delta) = \bar\partial^2 - \bar\partial\delta + \delta\bar\partial - \delta^2 = 0.$$

<u>Claim I</u>: $H^q(K^\bullet) := \dfrac{\operatorname{Ker}(d : K^q \to K^{q+1})}{\operatorname{Im}(d : K^{q-1} \to K^q)}$ is always isomorphic to $H^{p,q}_{\bar\partial}(X)$

To see this, we construct a map from $H^q(K^\bullet) \xrightarrow{\alpha} H^{p,q}_{\bar\partial}(X)$ as follows:

Pick any $\underset{(\sigma_{r,s})}{\underset{\shortparallel}{[\sigma]}} \in H^q(K^\bullet)$ with $\sigma_{r,s} \in C^r(\underline{U}, A^{p,s})$

$$\begin{aligned}
&\sigma_{0,q} \to \delta\sigma_{0,q} - \bar\partial\sigma_{1,q} = 0 \\
&\qquad\qquad \uparrow \\
&\qquad\quad \sigma_{1,q-1} \\
&\qquad\qquad\qquad \ddots \\
&\qquad\qquad\qquad\qquad \sigma_{q-1,1} \to \delta\sigma_{q-1,1} - \bar\partial\sigma_{q,0} = 0 \\
&\qquad\qquad\qquad\qquad\qquad\quad \uparrow \\
&\qquad\qquad\qquad\qquad\qquad \sigma_{q,0} \xrightarrow{\delta} \delta\sigma_{q,0} = 0
\end{aligned}$$

Because $\check{H}^r(A^{p,s}) = 0$, for any $r > 0$, we can inductively solve:

$$\begin{aligned}
\sigma_{q,0} &= \delta \cdot \tau_{q-1,0} \\
\pm\bar\partial\tau_{q-1,0} + \sigma_{q-1,1} &= \delta\tau_{q-2,1} \\
&\cdots \\
\pm\bar\partial\tau_{1,q-2} + \sigma_{1,q-1} &= \delta\tau_{0,q-1}
\end{aligned}
\quad \rightsquigarrow \quad \sigma - d\{\tau_{r',s'}\} = \overset{\tilde\sigma_{0,q}}{\overset{\shortparallel}{\underset{D^{0,q}}{\underset{\Cap}{\sigma_{0,q} - \bar\partial\tau_{0,q-1}}}}}$$

Global

Then

$d\sigma = 0 \Rightarrow d\tilde\sigma_{0,q} = 0 \Rightarrow \tilde\sigma_{0,q}$ is a $\bar\partial$-closed (p,q)-form

$$d\tilde\sigma_{0,q} = 0 \;\boxed{\Leftrightarrow}\; \begin{cases} \delta\tilde\sigma_{0,q} = 0 \\ \bar\partial\tilde\sigma_{0,q} = 0 \end{cases}$$

Define $[\sigma] \overset{\partial}{\longmapsto} [\tilde{\sigma}_{0,q}]$

(Ex) prove $\partial : H^q(K^\bullet) \to H^{p,q}_{\bar\partial}(X)$ is an isomorphism

- Claim II: If $\underline{U}$ is acyclic for Ω^p, then there is an isomorphism $\beta : H^q(K^\bullet) \to H^q(\Omega^p)$.

similar construction: Look at:

$$\begin{tikzcd}
& \vdots & \vdots & \\
\cdots \arrow[r] & C^r(\underline{U}, A^{p,s+1}) \arrow[r] \arrow[u] & C^{r+1}(\underline{U}, A^{p,s+1}) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^r(\underline{U}, A^{p,s}) \arrow[r] \arrow[u] & C^{r+1}(\underline{U}, A^{p,s}) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^r(\underline{U}, A^{p,s-1}) \arrow[r] \arrow[u, "\bar\partial"] & C^{r+1}(\underline{U}, A^{p,s-1}) \arrow[r] \arrow[u, "\bar\partial"] & \cdots \\
& \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

The vertical cohomology is $\bigoplus_{|I|=r} H^{p,s}_{\bar\partial}(U_I) = \bigoplus_{|I|=r} H^s(U_I, \Omega^p)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \; 0$

(The horizontal cohomology is $H^r(A^{p,s})$)

acyclic hypothesis

$\Longrightarrow$ We can solve inductively

$$
\begin{tikzcd}
0 \\
\sigma_{0,q} \arrow[u] \\
\tau'_{0,q-1} \arrow[u, "\bar\partial"] \arrow[r] & \sigma_{1,q-1} \\
 & {} \arrow[u] \arrow[r] & \cdots \\
 & & & \sigma_{q-1,1} \\
 & & & {} \arrow[u] \arrow[r] & \sigma_{q,0}
\end{tikzcd}
$$

$$\sigma_{0,q} = \bar\partial \tau'_{0,q-1}$$

$$\pm \delta \tau'_{0,q-1} + \sigma_{1,q-1} = \bar\partial \tau'_{1,q-2}$$

$$\cdots$$

$$\pm \delta \tau'_{q-2,1} + \sigma_{q-1,1} = \bar\partial \tau'_{q-1,0}$$

$$\rightsquigarrow \quad \sigma - d\{\tau'_{r',s'}\} = \underbrace{\sigma_{q,0} \pm \delta \tau'_{q-1,0}}_{= \tilde\sigma'}$$

$d\tilde\sigma' = 0 \Rightarrow \tilde\sigma'$ is a cocycle in $C^q(A^{p,0})$ satisfying

$$\bar\partial \tilde\sigma' = 0$$

$\Rightarrow \tilde\sigma'$ represents a cohomology class in $\check{H}^q(\underline{U}, \Omega^p)$.

Ex $\quad \underset{H^q(K^\bullet)}{\underset{\cap}{[\sigma]}} \longmapsto \underset{\check{H}^q(\underline{U}, \Omega^p)}{\underset{\cap}{[\tilde\sigma']}}$ is an isomorphism.

so we conclude $$\begin{array}{c} H^{p,q}_{\bar\partial}(X) \cong H^q(K^\bullet) \cong \check{H}^q(\underline{U}, \Omega^p). \\ \wr\| \\ H^q(X, \mathcal{H}). \end{array}$$

More details about such type of construction can be found in the book: Bott–Tu: Diff. Forms in Alg. Topology, GTM 82.

- A more conceptual Proof (following Griffiths–Harris)

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & C^r(\underline{U}, A^{p,s+1}) \arrow[u] \arrow[r] & C^{r+1}(\underline{U}, A^{p,s+1}) \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^r(\underline{U}, A^{p,s}) \arrow[u] \arrow[r] & C^{r+1}(\underline{U}, A^{p,s}) \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^r(\underline{U}, A^{p,s-1}) \arrow[u, "\bar\partial"] \arrow[r] & C^{r+1}(\underline{U}, A^{p,s-1}) \arrow[u, "\bar\partial"] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Set
$$C^r(\underline{U}, Z^s) = \ker(\bar\partial : C^r(\underline{U}, A^{p,s}) \to C^r(\underline{U}, A^{p,s+1})).$$

Then we have the exact sequences of complexes:

$$0 \to C^\bullet(\underline{U}, Z^s) \to C^\bullet(\underline{U}, A^{p,s}) \xrightarrow{\bar\partial} C^\bullet(\underline{U}, Z^{s+1}) \to 0$$

argue as in the proof of Dolbeault thm that:

$$\check{H}^q(\underline{U}, \Omega^p) \cong \check{H}^{q-1}(\underline{U}, Z^1) \cong \cdots \cong \check{H}^1(\underline{U}, Z^{q-1})$$

$$\cong \frac{\check{H}^0(\underline{U}, Z^q)}{\bar{\partial}\check{H}^0(\underline{U}, A^{0,q-1})} = \frac{H^0(X, Z^q)}{\bar{\partial}H^0(X, A^{0,q-1})} = H^{p,q}_{\bar{\partial}}(X).$$