Cohomology of sheaves by Čech cohomology.

$\underline{U} = \{U_i\}$ an open covering.

$C^p := C^p(\underline{U}, \mathcal{F}) = \{\{\sigma_I\} : \sigma_I \in \mathcal{F}(U_I),\ U_I = U_{i_0} \cap \cdots \cap U_{i_p},\ I = \{i_0, \cdots, i_p\}\}$

$$C^p(\underline{U}, \mathcal{F}) \xrightarrow{\delta} C^{p+1}(\underline{U}, \mathcal{F})$$

$$\sigma \mapsto (\delta\sigma)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0 \cdots \hat{j} \cdots i_{p+1}} \Big|_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}}$$

$\{$cocycles$\}$

$$\delta^2 = 0 \Rightarrow \mathrm{Im}(\delta) \subseteq \ker(\delta) \Rightarrow \check{H}^p(\underline{U}, \mathcal{F}) = \frac{\ker(\delta : C^p \to C^{p+1})}{\mathrm{Im}(\delta : C^{p-1} \to C^p)}.$$

$\{$coboundaries$\}$

Ex $\check{H}^0(\underline{U}, \mathcal{F}) = \mathcal{F}(X)$ (global sections)

- Refinement: $\underline{U}' = \{U'_j\}_{j \in I'} < \underline{U} = \{U_i\}_{i \in I}$

  Refining map $\varphi : I' \to I$ satisfying $U'_j \subseteq U_{\varphi(j)}\ \forall j \in I'$

Then
$$\begin{array}{ccc} C^p(\underline{U}, \mathcal{F}) & \xrightarrow{\delta} & C^{p+1}(\underline{U}, \mathcal{F}) \\ \downarrow \rho = \rho_\varphi & & \downarrow \rho \\ C^p(\underline{U}', \mathcal{F}) & \xrightarrow{\delta} & C^{p+1}(\underline{U}', \mathcal{F}) \end{array}$$

$\rho(\sigma) = \rho_{\underline{U}\,\underline{U}'}(\sigma)_{j_0 \cdots j_p} = \sigma_{\varphi(j_0) \cdots \varphi(j_p)}$ $\qquad \rho\delta = \delta\rho$

$\rightsquigarrow \rho : \check{H}^p(\underline{U}, \mathcal{F}) \to \check{H}^p(\underline{U}', \mathcal{F})$

Take direct limit: $\check{H}^p(X, \mathcal{F}) = \varinjlim_{\underline{U}} \check{H}^p(\underline{U}, \mathcal{F}) = \{[\sigma] \in \check{H}^p(\underline{U}, \mathcal{F})\}/\sim$

$[\sigma] \sim [\sigma']$ if $\exists\ \underline{U}'' < \underline{U}$ s.t. $\rho_{\underline{U}\,\underline{U}''}([\sigma]) = \rho_{\underline{U}'\underline{U}''}([\sigma'])$

$\check{H}^p(\underline{U}', \mathcal{F})$ $\qquad \underline{U}'' < \underline{U}'$

To avoid taking direct limit, we have (${\underline{U}}$ is an acyclic cover)

Leray's Theorem: If $\underline{U}$ satisfies $\check{H}^p(U_I, \mathcal{F}) = 0$ $\forall I$

Then $\check{H}^p(X, \mathcal{F}) = \check{H}^p(\underline{U}, \mathcal{F})$.

Ex: $X = \mathbb{P}^1$, $U_1 = \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$ $U_1 \cap U_2 = \mathbb{C}^*$

$U_2 = \mathbb{P}^1 \setminus \{0\} \cong \mathbb{C}$

$\mathcal{F} = \mathcal{O}$ = sheaf of holomorphic functions

- $\check{H}^0(X, \mathcal{O}) = 0$ because of :

Fact: On a compact complex manifold, any (global) holomorphic function must be constant.

[Ex: Prove the fact by using maximum principal for $|f|^2$)

- To calculate $\check{H}^1(X, \mathcal{O})$

We will use the fact that $\underline{U} = \{U_1, U_2\}$ is acyclic (to be proved later)

and use Leray's thm.

$$C^1(\underline{U}, \mathcal{O}) = \{ \sigma_{12} \in \mathcal{O}(U_1 \cap U_2) \}$$

$$\mathcal{O}(U_1 \cap U_2) = \mathcal{O}(\mathbb{C}^*)$$

$$\sigma_{12} = \sum_{k=1}^{\infty} a_{-k} \cdot z^{-k} + \sum_{j=0}^{+\infty} b_j \cdot z^j = \delta \tau$$

$$\sum_{k=1}^{\infty} a_{-k} \cdot z^{-k} = \sum_{k=1}^{\infty} a_{-k} \cdot w^k \in \mathcal{O}(U_2), \qquad \sum_{j=0}^{+\infty} b_j \cdot z^j \in \mathcal{O}(U_1)$$

$\left(\begin{array}{l} w: \text{coordinate on } U_2 \\ \Rightarrow z = w^{-1} \end{array} \right)$

$$\tau = \{\tau_1, \tau_2\}, \quad \tau_1 = \sum_j b_j z^j, \quad \tau_2 = -\sum_{k=1}^{\infty} a_{-k} \cdot w^k$$

$$\Rightarrow \check{H}^1(\mathbb{P}^1, \mathcal{O}) = \check{H}^1(\underline{U}, \mathcal{O}) = 0$$

- Ex: $\mathcal{F} = \Omega^1$ = sheaf of holomorphic 1-forms

$$\Omega^1(U) = \{ \omega \in A^1(U) : \omega = f(z)\, dz \text{ locally with } f(z) \text{ holomorphic} \}$$

$$\Omega^1(U_1) = \{ f(z)\, dz : f \text{ hol.} \}$$

$$\Omega^1(U_2) = \{ g(w)\, dw : g \text{ hol. in } w \}$$

$$\Omega^1(U_1 \cap U_2) = \{ f(z) dz : z \in \mathbb{C}^*, f \text{ hol.} \}, \qquad U_1 \cap U_2 = \mathbb{C}^*$$

(Use Laurent series to prove)

$$\check{H}^0(\underline{U}, \Omega^1) = 0, \quad \check{H}^1(\underline{U}, \Omega^1) = \mathbb{C} \cdot \left[\frac{dz}{z}\right] \cong \mathbb{C}, \qquad \frac{dz}{z} = -\frac{dw}{w}$$

- Short exact sequence of sheaves:

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{G} \to 0$$

<u>Exact</u>: • α is injective; β is surjective

• $\ker\beta = \operatorname{Im}\alpha$.

$\rightsquigarrow$ long exact sequence:

$$0 \to H^0(X,\mathcal{E}) \to H^0(X,\mathcal{H}) \to H^0(X,\mathcal{G}) \xrightarrow{\delta^*} H^1(X,\mathcal{E}) \to H^1(X,\mathcal{H})$$

$$\xrightarrow{\delta^*} H^1(X,\mathcal{G})$$

$$\cdots \leftarrow H^3(X,\mathcal{E}) \leftarrow H^2(X,\mathcal{G}) \leftarrow H^2(X,\mathcal{H}) \leftarrow H^2(X,\mathcal{E}) \leftarrow H^1(X,\mathcal{G})$$

- Construction of δ^* by diagram-chasing:

$$\begin{tikzcd}
0 \arrow[r] & C^{p-1}(\underline{U},\mathcal{E}) \arrow[r, "\alpha"] \arrow[d] & C^{p-1}(\underline{U},\mathcal{H}) \arrow[r, "\beta"] \arrow[d] & C^{p-1}(\underline{U},\mathcal{G}) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C^{p}(\underline{U},\mathcal{E}) \arrow[r, "\alpha"] \arrow[d] & C^{p}(\underline{U},\mathcal{H}) \arrow[r, "\beta"] \arrow[d, "\delta"] & C^{p}(\underline{U},\mathcal{G}) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C^{p+1}(\underline{U},\mathcal{E}) \arrow[r, "\alpha"] \arrow[d] & C^{p+1}(\underline{U},\mathcal{H}) \arrow[r, "\beta"] \arrow[d] & C^{p+1}(\underline{U},\mathcal{G}) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C^{p+2}(\underline{U},\mathcal{E}) \arrow[r, "\alpha"] & C^{p+2}(\underline{U},\mathcal{H}) \arrow[r, "\beta"] & C^{p+2}(\underline{U},\mathcal{G}) \arrow[r] & 0
\end{tikzcd}$$

Assume for suff refinement of the open covering, the horizontal sequences are exact.

- Fix $[\sigma] \in \check{H}^p(\underline{U}, \mathcal{G})$ represented by a cocycle $\sigma \in \mathrm{Ker}(\delta)$

β surjective $\Rightarrow$ $\sigma = \beta\tau$, for $\tau \in C^p(\mathcal{F})$

$\Rightarrow \beta(\delta\tau) = \delta \cdot \beta\tau = \delta\sigma = 0$

$\xRightarrow{\mathrm{Ker}\beta = \mathrm{Im}\alpha}$ $\delta\tau = \alpha\mu$ for $\mu \in C^{p+1}(\mathcal{E})$

$\Rightarrow \alpha\delta\mu = \delta\alpha\mu = \delta\delta\tau = 0 \xRightarrow{\alpha \text{ inj.}} \delta\mu = 0$

i.e. $\mu \in \mathrm{Ker}(\delta)$

Define: $\delta^*[\sigma] = [\mu] \in \check{H}^{p+1}(\underline{U}, \mathcal{E})$.

- show $\mathrm{Ker}(\delta^*) \subseteq \mathrm{Im}(\beta)$

If $\mu = \delta\nu$ for $\nu \in C^p(\mathcal{E})$. Then

$\delta(\tau - \alpha\nu) = \delta\tau - \alpha\delta\nu = \alpha\mu - \alpha\mu = 0$

and $\beta(\tau - \alpha\nu) = \beta\tau - \beta\alpha\nu = \sigma$

$\Rightarrow [\sigma] = \beta[\tau - \alpha\nu] \in \mathrm{Im}(\beta)$.

(Ex) Check exactness at other terms in the sequence.

- Dolbeault's theorem.

  X complex mfd.

  Ω^p: sheaf of holomorphic p-forms

  $$\Omega^p(U) = \{ \omega \in A^{p,0} : \omega = \sum_I f_I(z)\, dz_I \quad |I| = p \}$$

  $f_I(z)$ holomorphic.

Thm: $\check{H}^q(X, \Omega^p) \cong H^{p,q}_{\bar\partial}(X)$ smooth (p,q)-forms.

$$= \frac{\operatorname{Ker}(\bar\partial : A^{p,q}(X) \to A^{p,q+1}(X))}{\operatorname{Im}(\bar\partial : A^{p,q-1}(X) \to A^{p,q}(X))}$$

Ex: $H^1(\mathbb{C}^*, \mathcal{O}) = H^{0,1}_{\bar\partial}(\mathbb{C}^*) = 0$, $H^1(\mathbb{C}^*, \Omega^1) = H^{1,1}_{\bar\partial}(\mathbb{C}^*) = 0$.

$\Rightarrow \{ \underset{\mathbb{C}}{\underset{\shortparallel}{U_1}}, \underset{\mathbb{P}^1 \setminus \{0\}}{\underset{\shortparallel}{U_2}} \}$ is a cyclic covering for $\mathcal{O}, \Omega^1$

- Resolution of the sheaf Ω^p:

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \mathcal{A}^{p,2} \xrightarrow{\bar\partial} \cdots$$

Will show $H^r(\mathcal{A}^{p,q}) = 0$, $\forall r > 0$.