Sheaf $\mathcal{F}$: $U \xrightarrow{\text{assignment}} \mathcal{F}(U)$ for any open subset $U \subset X$

$\mathcal{F}(U)$: Abelian gp. ($\sigma \in \mathcal{F}(U)$ is called a section of $\mathcal{F}$ over U)

$\sigma \mapsto r^V_U(\sigma) =: \sigma|_U$

and for any $U \subset V$, the restriction morphism $r^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$

satisfying
$$\begin{cases} (1) & r^U_U = Id \\ (2) & \text{For } U \subset V \subset W, \quad r^V_U \circ r^W_V = r^W_U \end{cases}$$
defines presheaf

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "r^W_V"] \arrow[dr, "r^W_U"'] & \mathcal{F}(V) \arrow[d, "r^V_U"] \\ & \mathcal{F}(U) \end{tikzcd}$$

Let $U = \bigcup_i U_i$ (covering by open subsets)

gluing property (uniqueness and existence)
$$\begin{cases} (3) & \sigma, s \in \mathcal{F}(U) \text{ satisfies } \sigma|_{U_i} = s|_{U_i} \ \forall i \Rightarrow \sigma = s \\ (4) & \text{If } \exists\, \sigma_i \in \mathcal{F}(U_i) \text{ satisfies } \sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j} \ \forall i, j \\ & \text{then } \exists\, \sigma \in \mathcal{F}(U) \text{ s.t. } \sigma|_{U_i} = \sigma_i \end{cases}$$

Ex:
- $A^{p,q}$ = sheaf of smooth (p,q)-forms
- A^k = sheaf of smooth k-forms
- G an Abelian gp. (Ex. $\mathbb{Z}, \mathbb{R}, \mathbb{C}$)
  $U \mapsto G(U) = G$ constant sheaf with value in G
- X complex mfd.
  $\mathcal{O}_X$ = sheaf of holomorphic functions
  $\mathcal{O}_X^*$ = sheaf of non-zero holomorphic functions.

- Morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection of morphism:
  For any U open subset, $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is a morphism.

$\ker(\varphi)(U) = \ker\{\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)\}$ a sheaf

$\mathcal{F}: U \mapsto \mathrm{Im}(U)$ is not a sheaf just a presheaf

<u>Ex</u>: $\mathcal{O}_{\mathbb{C}} \xrightarrow{\exp} \mathcal{O}^*_{\mathbb{C}}$ given by $\mathcal{O}_{\mathbb{C}}(U) \longrightarrow \mathcal{O}^*_{\mathbb{C}}(U)$

$$f \longmapsto \exp(f)$$

choose $U = \mathbb{C} \setminus \{0\}$, consider the holomorphic fct. $g(z) = z$

let $U_1 = U \setminus \mathbb{R}_{\geq 0}$ then $g|_{U_1} = \log z = \log|z| + i\cdot\theta \quad 0 < \theta < 2\pi$

$U_2 = U \setminus \mathbb{R}_{\leq 0}$ $\quad g|_{U_2} = \log z = \log|z| + i\theta', \ -\pi < \theta' < \pi$

$\Rightarrow \quad g|_{U_i} \in \mathrm{Im}(\exp|_{U_i})$ but $g \notin \mathrm{Im}(\exp|_U)$

- For any presheaf $\mathcal{F} \rightsquigarrow$ associated sheaf $\overline{\mathcal{F}}$

First ensure uniqueness: $\quad \mathcal{F}_1(U) = \dfrac{\mathcal{F}(U)}{\left\{\sigma \in \mathcal{F}(U) \;\middle|\; \begin{array}{l} \exists \text{ open covering } U = \bigcup V_i \\ \text{s.t. } \sigma|_{V_i} = 0 \end{array}\right\}}$

Next recover all glued sections: $\quad \overline{\mathcal{F}}(U) = \left\{ (\{V_i\}, \sigma_i \in \mathcal{F}_1(V_i)) \;\middle|\; \begin{array}{l} U = \bigcup V_i \\ \sigma_i|_{V_i \cap V_j} = \sigma_j|_{V_i \cap V_j} \end{array} \right\} \Big/ \sim$

$$(\{V_i\}, \sigma_i) \sim (\{W_k\}, s_k) \quad \text{iff} \quad \sigma_i|_{V_i \cap W_k} = s_k|_{V_i \cap W_k}.$$

- $\varphi: \mathcal{F} \to \mathcal{G} \quad \mathrm{Im}(\varphi) = \overline{\mathcal{F}}$ where $\mathcal{F}(U) = \mathrm{Im}(\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U))$.

<u>Ex</u>: $\quad \mathcal{F} := \{U \mapsto \mathrm{Im}(\exp: \mathcal{O}(U) \to \mathcal{O}^*(U))\}$ Then $\overline{\mathcal{F}} = \mathcal{O}^*$

(e.g. $z \in \overline{\mathcal{F}}(\mathbb{C} \setminus \{0\})$)

- Stalks: Fix $x \in X$. Then

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U) \qquad U_1 \subset U_2 \Rightarrow r^{U_2}_{U_1} : \mathcal{F}(U_2) \to \mathcal{F}(U_1)$$

$$\parallel$$

$$\{(U, \sigma \in \mathcal{F}(U)) : x \in U\} / \sim$$

$$(U_1, \sigma_1) \sim (U_2, \sigma_2) \Leftrightarrow \exists\, W \subset U_1 \cap U_2,\ \text{s.t. } \sigma_1|W = \sigma_2|W.$$

morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G} \rightsquigarrow$ morphism of stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$.

Def: $\varphi : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective)

Ex: φ is injective $\Leftrightarrow \ker(\varphi) = 0 \Leftrightarrow \varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for any U open.

surjective $\Leftrightarrow \operatorname{Im}(\varphi) = \mathcal{G}$.

Warning: φ surjective $\not\Rightarrow \varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for any U

(e.g. $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ surj).

- If $\varphi : \mathcal{F} \to \mathcal{G}$ is injective, the quotient sheaf $\mathcal{G}/\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{G}(U)/\mathcal{F}(U)$

Def:

- A complex of sheaves: sequence of morphisms

$$\mathcal{F}_\bullet: \cdots \to \mathcal{F}_{i-1} \xrightarrow{\partial_{i-1}} \mathcal{F}_i \xrightarrow{\partial_i} \mathcal{F}_{i+1} \to \cdots$$

s.t. $\partial_i \circ \partial_{i-1} = 0$, $\forall i$.

- Exact at $\mathcal{F}_i$ if $\ker(\partial_i) = \mathrm{Im}(\partial_{i-1})$

(• Cohomology sheaves: $\mathcal{H}^i(\mathcal{F}_\bullet) = \dfrac{\ker(\partial_i)}{\mathrm{Im}(\partial_{i-1})}$)

- Short exact sequence: $0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$ satisfies

$$\begin{cases} \beta \circ \alpha = 0,\ \ker(\beta) = \mathrm{Im}(\alpha) \\ \alpha \text{ is injective } (\ker(\alpha) = 0) \\ \beta \text{ is surjective } (\mathrm{Im}(\beta) = \mathcal{G}) \end{cases}$$

$\rightsquigarrow$ left exact not right exact in general and need higher cohomology to get:

$$0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \to H^1(U, \mathcal{E}) \to H^1(U, \mathcal{F}) \to H^1(U, \mathcal{G}) \to H^2(U, \mathcal{E}) \to \cdots$$

- Čech cohomology Fix an open covering $\{U_\alpha\}_{\alpha \in S}$ s.t. $X = \bigcup_\alpha U_\alpha$

p-th Čech gp. of cochains

$$C^p(\underline{U}, \mathcal{F}) = \left\{ \{\sigma_I\} : \begin{array}{l} I = \{i_1, \cdots, i_p\} \text{ ordered},\ |I| = p,\ \sigma_I \in \mathcal{F}(U_I) \\ \text{where } U_I = \bigcap_{i \in I} U_i \end{array} \right\}$$

$$\delta: C^p(\underline{U}, \mathcal{F}) \longrightarrow C^{p+1}(\underline{U}, \mathcal{F})$$

$$\sigma \mapsto (\delta\sigma)_{i_1 \cdots i_{p+1}} = \sum_{j=1}^{p+1} (-1)^j \, \sigma_{i_1 \cdots \hat{j} \cdots i_{p+1}} \big|_{U_{\{i_1 \cdots \hat{j} \cdots i_{p+1}\}}}$$

Ex:

$$C^0(\underline{U}, \mathcal{F}) \to C^1(\underline{U}, \mathcal{F})$$

$$\sigma \mapsto (\delta\sigma)_{ij} = \sigma_i - \sigma_j$$

$$C^1(\underline{U}, \mathcal{F}) \to C^2(\underline{U}, \mathcal{F})$$

$$\sigma \longmapsto (\delta\sigma)_{ijk} = -\sigma_{jk} + \sigma_{ik} - \sigma_{ij}$$

Ex $\delta^2 = 0$, so $\mathrm{Im}(\delta) \subseteq \ker(\delta) \subseteq C^p(\underline{U}, \mathcal{F})$

Def:

$$\check{H}^p(\underline{U}, \mathcal{F}) = \frac{\ker(\delta: C^p \to C^{p+1})}{\mathrm{Im}(\delta: C^{p-1} \to C^p)}$$

Example: $\check{H}^0(\underline{U}, \mathcal{F}) = \mathcal{F}(X)$ global sections.