- Integrable Complex structures

X : differentiable manifold of real dim. $2n$

An almost complex structure : $J: T_{\mathbb{R}}X \to T_{\mathbb{R}}X$ s.t. $J^2 = -Id$.

$$\Rightarrow \quad T_{\mathbb{C}}X = T_{\mathbb{R}}X \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X, \quad T^{1,0}X = \ker(J - \sqrt{-1}\, Id)$$
$$T^{0,1}X = \ker(J + \sqrt{-1}\, Id) = \overline{T^{1,0}}$$

- If X is a complex mfd. with local holomorphic coordinates $\{z_i\}$ then
$$z_i = x_i + \sqrt{-1}\, y_i$$

$J: T_{\mathbb{R}}X \to T_{\mathbb{R}}X$ given by $\begin{cases} J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i} \\ J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i} \end{cases}$

$T^{1,0}X = \mathrm{Span}\{\frac{1}{2}(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i})\} = \mathrm{Span}\{\frac{\partial}{\partial z_i}\}$ is then involutive:

$[T^{1,0}X, T^{1,0}X] \subseteq T^{1,0}X$ because $[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}] = 0$. $\forall i, j$.

<u>Def</u>: An almost complex structure J is integrable if there exist coordinates $\{z_i\}$, $z_i = x_i + \sqrt{-1}\, y_i$, such that $T^{1,0}_J X = \mathrm{Span}\{\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i})\}$.

(Ex) If J is integrable, then such coordinates make X a complex mfd. i.e. the transition functions are holomorphic.

Conversely, we have the important:

Theorem (Newlander - Nirenberg) An almost complex structure is integrable if and only if $T^{1,0}_J X$ is involutive. (i.e. $[T^{1,0}_J X, T^{1,0}_J X] \subseteq T^{1,0}_J X$)

<u>Rmk</u>: $T^{1,0}_J X$ is involutive $\Leftrightarrow$ $\forall\, V, W \in T_{\mathbb{R}}X$. $[V - iJV, W - iJW] \in T^{1,0}_J X$

$$[V - iJV, W - iJW] = ([V,W] - [JV, JW]) - i([V, JW] + [JV, W])$$

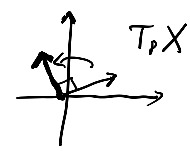
$$\Leftrightarrow \underbrace{J[V, JW] + J[JV, W] + [V,W] - [JV, JW]}_{N(V,W)} = 0$$

$N(V,W)$ ← Nijenhuis tensor

Example: X: smooth real 2-dim mfd. orientable

A conformal structure $[g]$ represented by a Riem. metric

$\rightsquigarrow$ rotation by 90° $= J : TX \to TX$



$T^{1,0}_J X$ is 1-dim $\Rightarrow$ $T^{1,0}_J X$ is involutive

$\Rightarrow$ J is integrable $\rightsquigarrow$ holomorphic coordinate $\{z\}$

Non Example: (Topological Fact)

The only spheres admitting an almost complex structures are S^2 and S^6

$S^2 = \{x \in \operatorname{Im}\mathbb{H} : |x|^2 = 1\}$ (quaternion)

standard $J : TS^m \to TS^m$ $\quad m = 2, 6$

$S^6 = \{x \in \operatorname{Im}\mathbb{O} : |x|^2 = 1\}$ (octonion)

$J_p(v) = p \times v := \frac{1}{2}(p \cdot v - v \cdot p)$ cross product

$m = 2$: integrable $S^2 = \mathbb{P}^1$

$m = 2$: The standard J is not integrable

Open question: Is there an integrable complex structure on S^6?

Expect/attempt: No / Chern, Atiyah, ...

- Differential forms on complex manifold X, $\{z_i\}$

(p,q)-form: $\varphi = \sum_{\substack{I,J \subset \{1,\dots,n\} \\ |I| = p \\ |J| = q}} \varphi_{IJ}\, dz_I \wedge d\bar{z}_J$

$I = \{i_1, \dots, i_p\}$, $J = \{j_1, \dots, j_q\}$

$dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$

$d\bar{z}_J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$

$U \subset X$ open set

$A^{p,q}(U) = \{$ smooth (p,q)-forms on $U\}$

$A^k(U) = \{$ smooth complex valued k-forms on $U\}$

$$\begin{array}{ccc} A^k & \xrightarrow{\quad d \quad} & A^{k+1} \\ \parallel & & \parallel \\ \bigoplus_{p+q=k} A^{p,q} & & \bigoplus_{r+s=k+1} A^{r,s} \end{array}$$

$$d|_{A^{p,q}} : A^{p,q} \longrightarrow A^{p+1,q} \oplus A^{p,q+1}$$

$$\varphi_{IJ}\, dz_I \wedge d\bar{z}_J \mapsto \underbrace{\frac{\partial \varphi_{IJ}}{\partial z_i} dz_i \wedge dz_I \wedge d\bar{z}_J}_{= \partial \varphi} + \underbrace{\frac{\partial \varphi_{IJ}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J}_{\bar{\partial} \varphi}$$

$$d = \partial + \bar{\partial}, \quad 0 = d^2 = (\partial + \bar{\partial})(\partial + \bar{\partial}) = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$$

$$\Rightarrow \begin{cases} \partial^2 = 0 \\ \partial\bar{\partial} + \bar{\partial}\partial = 0 \\ \bar{\partial}^2 = 0. \end{cases}$$

$$H^k_{dR}(X) = \frac{\ker(d : A^k(X) \to A^{k+1}(X))}{\mathrm{Im}(d : A^{k-1}(X) \to A^k(X))}$$

$$H^{p,q}_{Dol}(X) = H^{p,q}_{\bar{\partial}}(X) = \frac{\ker(\bar{\partial} : A^{p,q}(X) \to A^{p,q+1}(X))}{\mathrm{Im}(\bar{\partial} : A^{p,q-1}(X) \to A^{p,q}(X))} = \frac{\{\bar{\partial}\text{-closed } (p,q) \text{ forms}\}}{\{\bar{\partial}\text{-exact } (p,q)\text{-forms}\}}$$

Lemma ($\bar{\partial}$-Poincaré Lemma) $H^{p,q}_{\bar{\partial}}(\Delta^l) = 0$ for any $q > 0$. $\Delta = \{z \in \mathbb{C} : |z| < r\} = \Delta_r$

Pf: step 1: $\varphi = \sum_{I,J} \varphi_{IJ}\, dz_I \wedge d\bar{z}_J$ $\bar{\partial}$-closed

$\Rightarrow \bar{\partial}\varphi = \pm \sum_{I,J} dz_I \wedge \bar{\partial}(\varphi_{IJ}\, d\bar{z}_J) = 0 \Rightarrow \sum_J \varphi_{IJ}\, d\bar{z}_J$ is $\bar{\partial}$-closed

Just need show $\sum_J \varphi_{IJ}\, d\bar{z}_J = \bar{\partial}\eta_I$ since then $\varphi = \pm\bar{\partial}(\sum_I dz_I \wedge \eta_I)$.

so can assume $I = \phi$ and $\varphi = \sum_J \varphi_J\, d\bar{z}_J$.

step 2: Set $k = \max\{j : \exists J \text{ s.t. } j \in J \text{ and } \varphi_J \neq 0\} \geq 1$

Prove stronger: $\bar{\partial}\varphi = 0 \Rightarrow \varphi = \bar{\partial}\eta$. Moreover if φ depends smoothly (resp. holomorphically) on $w_1, \dots, w_r$, then η also depends smoothly (resp. hol.) on $w_1, \dots, w_r$.

$k=1$: $\varphi = \varphi_1 \, d\bar{z}_1$ $\bar\partial$-closed $\Rightarrow$ φ_1 is holomorphic in $z_2, \cdots, z_n$

set $$\eta = \frac{1}{2\pi i} \iint_\Delta \frac{\varphi_1(\zeta, \bar\zeta, z_2, \cdots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar\zeta$$

Then $\bar\partial \eta = \varphi$ by using $\bar\partial\left(\frac{d\zeta}{\zeta - z}\right) = 2\pi i \cdot \delta_z$

$$\left(\iint_\Delta \underset{\substack{\uparrow \\ \text{test fct.}}}{\chi} \cdot \bar\partial\left(\frac{d\zeta}{\zeta - z}\right) = -\iint \bar\partial \chi \wedge \frac{d\zeta}{\zeta - z} = -\iint_{\Delta \setminus \Delta_z(\varepsilon)} \bar\partial \chi \wedge \frac{d\zeta}{\zeta - z} \right.$$
$$\left. = \int_{\partial\Delta_z(\varepsilon)} \chi \cdot \frac{d\zeta}{\zeta - z} \sim \chi(z) \int_0^{2\pi} \frac{d(\varepsilon e^{i\theta})}{\varepsilon \cdot e^{i\theta}} = 2\pi i \chi(z) \right)$$

$k>1$: $$\varphi = \sum_{J \subset \{1, \cdots, k-1\}} \varphi_J \, d\bar{z}_J + \sum_{J' \subset \{1, \cdots, k-1\}} \varphi_{J'k} \, d\bar{z}_{J'} \wedge d\bar{z}_k$$

$$\bar\partial \varphi = \sum_{J \subset \{1, \cdots, k-1\}} \bar\partial' \varphi_J \, d\bar{z}_J + \sum_{J \subset \{1, \cdots, k-1\}} \frac{\partial \varphi_J}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_J + \sum_{J' \subset \{1, \cdots, k\}} \bar\partial'(\varphi_{J'k} \, d\bar{z}_{J'}) \wedge d\bar{z}_k$$

$$\bar\partial' = \sum_{i \neq k} d\bar{z}_i \wedge \frac{\partial}{\partial \bar{z}_i}$$

$\bar\partial \varphi = 0 \Rightarrow \bar\partial'(\varphi_J \, d\bar{z}_J) = 0 \Rightarrow$ φ_J depends smoothly on z_k, holomorphically on $z_{k+1}, \cdots, z_n$

By induction $\sum_{J \subset \{1, \cdots, k-1\}} \varphi_J \, d\bar{z}_J = \bar\partial' \eta$ $\quad$ η depends smoothly on $z_k, z_{k+1}, \cdots, z_n$ and holomorphically on $z_{k+1}, \cdots, z_n$

$J \subset \{1, \cdots, k-1\}$

Then $\varphi - \bar\partial\eta = \sum_{J\subset\{1,\cdots,k-1\}} \varphi_J\, d\bar z_J \wedge d\bar z_k + \sum_{J'\subset\{1,\cdots,k-1\}} \varphi_{J'k}\, d\bar z_{J'} \wedge d\bar z_k$

$$- \bar\partial'\eta - \sum_{i=k}^{n} d\bar z_i \wedge \frac{\partial}{\partial \bar z_i}\eta$$

$$= \underbrace{\sum_{J'\subset\{1,\cdots,k-1\}} \varphi_{J'k}\, d\bar z_{J'}\wedge d\bar z_k - d\bar z_k \wedge \frac{\partial\eta}{\partial\bar z_k}}_{\parallel}$$

$$= d\bar z_k \wedge \psi$$

$$\bar\partial(\varphi - \bar\partial\eta) = 0 \Rightarrow \bar\partial'\psi = 0 \xRightarrow{\text{by induction}} \psi = \bar\partial'\rho$$

$$\Rightarrow d\bar z_k \wedge \psi = \bar\partial(d\bar z_k \wedge \rho) \Rightarrow \varphi = \bar\partial(\eta + d\bar z_k \wedge \rho).$$ induction completed.

Ex Check the proof works to show $H^{p,q}_{\bar\partial}((\Delta^*)^k \times \Delta^l) = 0$ for any $q>0$ for any $k, l \in \mathbb{Z}_{\geq 0}$.

Analytic case:

$$z^k \bar z^l d\bar z = \bar\partial\left(z^k \cdot \frac{\bar z^{l+1}}{l+1}\right) \quad l \neq 1, \ k \in \mathbb{Z}$$

$$z^k \cdot \bar z^{-1} d\bar z = \bar\partial(z^k \cdot \log|z|^2). \quad l = 1, \ k \in \mathbb{Z}$$