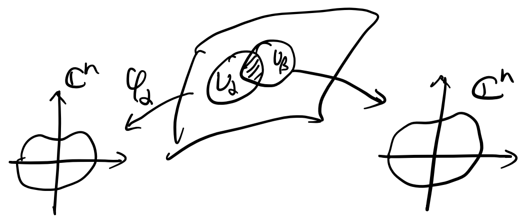
Complex Geometry studies Complex manifolds

- Complex mfd = differentiable mfd with an integrable complex structure
  = mfds admitting holomorphic coordinate charts

$X = \bigcup_\alpha U_\alpha$, $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$



$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$

$(\varphi_{\alpha\beta}^1, \cdots, \varphi_{\alpha\beta}^n)$ $\qquad \cap\ \mathbb{C}^n \qquad \cap\ \mathbb{C}^n$

is holomorphic : $\varphi_{\alpha\beta}^i$ is holomorphic for $i = 1, \cdots, n$

- $f : U \longrightarrow \mathbb{C}$ is holomorphic if $\frac{\partial f}{\partial \bar{z}_i} = 0$ at any point in U, $i = 1, \cdots, n$

$U \subset \mathbb{C}^n$, $f(z) = f(z_1, \cdots, z_n)$ $\qquad \frac{\partial f}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right) f$

$z^{\vec{m}} = z_1^{m_1} \cdots z_n^{m_n}$

Fact: f is holomorphic around $0 \in \mathbb{C}^n$ if $f(z) = \sum_{\vec{m} \in N} a_{\vec{m}} z^{\vec{m}}$ for $|z| << 1$.

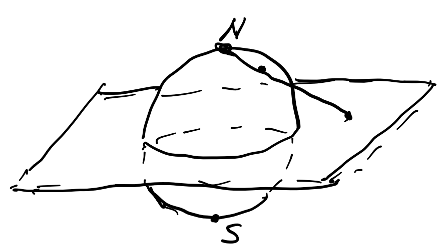
Ex: Complex mfds are always orientable (as real mfds).

- Example: $\dim_{\mathbb{C}} X = 1$: Rieman surfaces = 1-dim complex mfd.

classification of closed Riemann surfaces. (closed = compact and without bdry)

. $g = 0$: Rieman sphere $\mathbb{P}^1 \cong S^2 \subset \mathbb{R}^3$

Steographic Projection



$U_1 = S^2 \setminus N \xrightarrow{\varphi_1} \mathbb{C} \qquad (u_1, u_2, u_3) \mapsto \dfrac{u_1 + i u_2}{1 - u_3} = z$

$U_2 = S^2 \setminus S \xrightarrow{\varphi_2} \mathbb{C} \qquad (u_1, u_2, u_3) \mapsto \dfrac{u_1 - i u_2}{1 + u_3} = z' = \dfrac{(u_1 - i u_2)\cdot(1 - u_3)}{1 - u_3^2} = \dfrac{1 - u_3}{u_1 + i u_2} = \dfrac{1}{z}$, $\quad 1 - u_3^2 = u_1^2 + u_2^2$

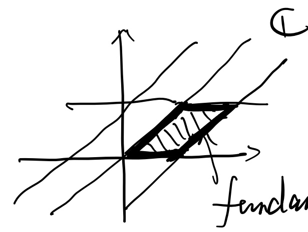
$\varphi_2(U_1 \cap U_2) \longrightarrow \varphi_1(U_1 \cap U_2)$ holomorphic

$\mathbb{C}^* \ni z' \longmapsto (z')^{-1} = z$

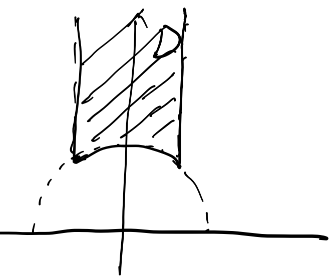
- $g = 1$: torus $\quad \mathbb{C}/\Lambda \qquad \Lambda = \mathbb{Z}\cdot w_1 + \mathbb{Z}\cdot w_2 \cong \mathbb{Z} \oplus \mathbb{Z}$

$\wr$ normalize

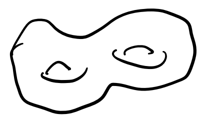
$\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot \tau$ with $\mathrm{Im}(\tau) > 0$



fundamental domain



$\tau \in D$ parametrizes the set of complex structures on the torus

- $g \geq 2$:  $\Sigma_g = B_1 / \pi_1(\Sigma_g) \qquad B_1 = \{ z \in \mathbb{C} : |z| < 1 \}$

complex structures depend on how $\pi_1(\Sigma_g)$ acts on B_1

Fact: there are $\underbrace{3g-3}$ parameters parametrizing complex structures.

dimension of quadratic differentials.

Rieman, Teichmuller

Kodaira–Spencer $\qquad \left(\text{Riem. metric } g_1 \overset{\text{conf.}}{\sim} g_2 \Leftrightarrow g_2 = c \cdot g_1 \right)$

Rmk: Riemann surface $\Longleftrightarrow$ oriented surface with a conformal structure

- higher dim:

Complex Projective Space $\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / \{z \sim \lambda \cdot z,\ \lambda \neq 0\}$

$\quad = \{[z] : z \neq 0\}$

parametrizes the set of lines passing through $0 \in \mathbb{C}^{n+1}$.

$z = (z_1, \cdots, z_{n+1})$ homogeneous coordinates.

For $\alpha = 1, \cdots, n+1$, $U_\alpha = \{[z] : z_\alpha \neq 0\}$

$$\varphi_\alpha : U_\alpha \to \mathbb{C}^n \quad [z] \mapsto \left(\frac{z_1}{z_\alpha}, \cdots, \widehat{\frac{z_\alpha}{z_\alpha}}, \cdots, \frac{z_{n+1}}{z_\alpha}\right) = (w_1, \cdots, w_n)$$

For example:

$$\varphi_1 : U_1 \to \mathbb{C}^n \quad [z] \mapsto \left(\frac{z_2}{z_1}, \cdots, \frac{z_{n+1}}{z_1}\right) = (u_1, \cdots, u_n)$$

transition from U_1 to U_α:

$$(u_1, \cdots, u_n) \xrightarrow{\varphi_1^{-1}} [1, u_1, \cdots, u_n] \xmapsto{\varphi_\alpha} \left(\frac{1}{u_\alpha}, \frac{u_1}{u_\alpha}, \cdots, \frac{u_{\alpha-1}}{u_\alpha}, \widehat{\frac{u_\alpha}{u_\alpha}}, \frac{u_{\alpha+1}}{u_\alpha}, \cdots, \frac{u_n}{u_\alpha}\right)$$

holomorphic away from $\{u_\alpha = 0\} = U_1 \cap U_\alpha$

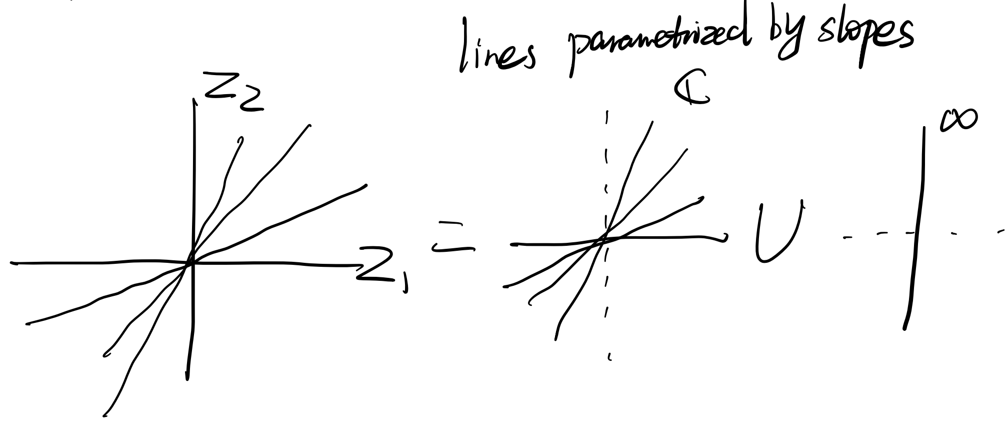
- $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$ $\qquad$ $\mathbb{P}^{n-1} = \{[0, u_1, \dots, u_n]\}$

$$\mathbb{C}^n = \{[1, u_1, \dots, u_n]\} = \{[z] : z_1 \neq 0\} = U_1$$

$$\mathbb{P}^{n-1} = \{[z] : z_1 = 0\}$$

<u>n=1</u>.

$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$



- Complex submfds $\quad M^m \subset X^n$

M is a m-dim complex submfd if $\forall p \in M$,
$\exists$ hol. chart $\quad (U, (z_1, \dots, z_n))$ s.t. $M \cap U = \{z_{m+1} = \dots = z_n = 0\}$

<u>Def</u>: A projective mfd. X is a closed complex submfd. of $\mathbb{P}^N$.

<u>Fact</u> (Chow's Theorem) Any projective mfd. is <u>globally</u> realized as the zero set of a collection of homogeneous polynomials.

- A polynomial $F: \mathbb{C}^{N+1} \xrightarrow{\neq 0} \mathbb{C}$ is $\underbrace{\text{homogeneous}}_{\text{of degree } d}$ if

$$F(\lambda z) = \lambda^d \cdot F(z) \text{ for any } \lambda \in \mathbb{C} \text{ and } z \in \mathbb{C}^{N+1}$$

so $F = \sum_{|\vec{I}|=d} a_{\vec{I}} z^{\vec{I}}$.

- $\{[z] : F(z) = 0\} \subset \mathbb{P}^N$ is well defined (hypersurface)

- $\{[z] : F_1(z) = 0, \cdots, F_r(z) = 0\} = Y$

$$Y \cap \underset{\{z_1 \neq 0\}}{\underset{\parallel}{U_1}} = \left\{ u \in \mathbb{C}^N : \begin{array}{l} \overset{= f_1(u)}{F_1(1, u_1, \cdots, u_N)} = 0 \\ \cdots \\ \underset{f_r(u)}{F_r(1, u_1, \cdots, u_N)} = 0 \end{array} \right\}$$

$Y \cap U_1$ is a complex $\underbrace{\text{submfd}}_{\text{of dim } N-c}$ if $\operatorname{rank}\left(\frac{\partial F_q}{\partial u_i}\right)_{\substack{1 \le q \le r \\ 1 \le i \le n}} = \underset{c}{\underset{\parallel}{\text{constant}}}$

(complex version of the Implicit Function Theorem)

Ex: $\{[z] : F(z)=0\}$ is a smooth complex mfd. if (and only if) the system $\left\{\frac{\partial F}{\partial z_i}=0,\ i=1,\cdots,N+1\right\}$ has only 0 solution.