Riemann-Roch: $D = \sum_i a_i P_i$, $a_i \in \mathbb{Z}$. $\deg D = d = \sum_i a_i$.

$$\wr$$

$$[D] = L, \quad \mathcal{O}(L)$$

$$h^0(S, \mathcal{O}(D)) - \underbrace{h^1(S, \mathcal{O}(D))}_{\parallel} = d - g + 1$$

$$h^0(S, \Omega^1(-D))$$

$$h^0(S, \mathcal{O}(D)) = d - (g - h^0(S, \Omega^1(-D))) + 1$$

$$\parallel$$

$$\dim\{f \in \mathcal{M}(S) : (f) + D \geq 0\}$$

$$\parallel \qquad D = \sum_i P_i \quad P_i \neq P_j$$

$$\dim\{\eta \text{ meromorphic 1-forms, poles of order} \leq 2, \text{ no residue, integral periods}\}$$

$$\left(\frac{a_i}{z^2} + O(1)\right) dz,$$

reprocity law for meromorphic forms.

- E holomorphic vector bundle $\longrightarrow M^n$

$$\chi(E) = \sum_{p=0}^{n} (-1)^p h^p(M, \mathcal{O}(E)). \quad \text{hol. Euler char.}$$

$$\chi(\mathcal{O}_S) = h^0(S, \mathcal{O}_S) - \underbrace{h^1(S, \mathcal{O}_S)}_{\substack{\parallel \\ h^0(S, \Omega^1_S)}} = 1 - g = \underbrace{\deg \mathcal{O}_S}_{\substack{\parallel \\ 0}} - g + 1.$$

$L \to S$ holomorphic line bdle.

$$\chi(L) = h^0(\mathcal{O}(L)) - h^1(\mathcal{O}(L)) = d - g + 1$$

$$\parallel$$

$$\boxed{\chi(L) = c_1(L) + \chi(\mathcal{O}_S)}$$

- $0 \to \mathcal{O}(E_1) \to \mathcal{O}(E_2) \to \mathcal{O}(E_3) \to 0.$

$$\rightsquigarrow \quad \chi(\mathcal{O}(E_2)) = \chi(\mathcal{O}(E_1)) + \chi(\mathcal{O}(E_3))$$

- $0 \to \mathcal{O}(D) \to \mathcal{O}(D+P) \to \mathbb{C}_p \to 0.$

$$\Rightarrow \quad \underline{\chi(\mathcal{O}(D+P))} = \underline{\chi(\mathcal{O}(D)) + 1}$$
$$\underline{\chi(\mathcal{O}(D))} = \underline{\chi(\mathcal{O}(D+P)) - 1.}$$

by induction $\Longrightarrow$ RR for all divisor.

- $\underline{\text{genus } 0}$ $\qquad D = P$

$$h^0(D) - \underbrace{h^1(D)}_{} = 1 - 0 + 1 = 2 \Rightarrow H^0(S, \mathcal{O}(P)) = \{1, f\}$$

$$h^1(D) = h^0(\Omega^1(-D)) \leq h^0(\Omega^1) = 0$$

$f: S \to \mathbb{P}^1$ bihol.

$\Rightarrow S \cong \mathbb{P}^1$

$\underline{\text{genus} = 1}$: $\qquad$ Abel-Jacobi: $S \cong \mathrm{Jac}(S) = \mathbb{C}/\Lambda.$

$\underline{\overset{g}{\overset{\shortparallel}{\text{genus}}} \geq 2}$: $\underline{\text{Canonical map}}$, $\underline{H^0(S, \Omega^1_S) = \{\omega_1, \cdots, \omega_g\}}$

$$\iota_K : S \longrightarrow \mathbb{P}^N$$
$$p \longmapsto [\omega_0(p), \cdots, \omega_g(p)].$$

ι_K is an embedding $\Leftrightarrow$ $\forall p, q \in S, \exists\, \omega \in H^0(S, \Omega^1)$ s.t. $\omega(p) = 0, \omega(q) \neq 0$

$\forall p \in S, \exists\, \omega$ vanishing exactly to order 1 at p

$$\Leftrightarrow \boxed{h^0(K - p - q) \leq h^0(K - p)}, \quad \forall p, q.$$

$$h^0(K-p) - \underbrace{h^1(K-p)}_{\substack{\| \\ h^0(p)=1}} = 2g-3-g+1 = g-2. \Rightarrow h^0(K-p) = g-1.$$

$$h^0(K-p-q) - \underbrace{h^1(K-p-q)}_{\substack{\| \\ h^0(p+q)}} = 2g-2-2-g+1 = g-3.$$

$$g-3+h^0(p+q) \overset{\le}{<} g-1 \Leftrightarrow \underset{1 \le}{h^0(p+q)} \overset{=}{<} 2 \Leftrightarrow h^0(p+q) = 1.$$

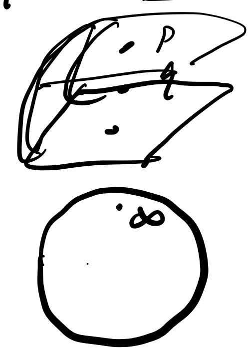
$\Updownarrow$

ι_K is an embedding

ι_K is not an embedding $\Leftrightarrow$ $h^0(p+q) = 2$.

$\Leftrightarrow$ $\exists$ meromorphic fct. f with exactly 2 poles

$\Leftrightarrow$ $\exists$ holomorphic map $f: S \to \mathbb{P}^1$ which is a 2-sheeted branched cover.



$\Leftrightarrow$ S is hyperelliptic

when $g \ge 2$, $\forall\, p \in S$.

$$h^0(\Omega^1_S) = g > h^0(\Omega^1_S(-p)) = g-1$$

$$h^0(\Omega^1_S(-p)) - \underbrace{h^1(\Omega^1_S(-p))}_{\substack{\| \\ h^0(p)=1}} = 2g-2-1-g+1 = g-2 \Rightarrow h^0(\Omega^1_S(-p)) = g-1$$

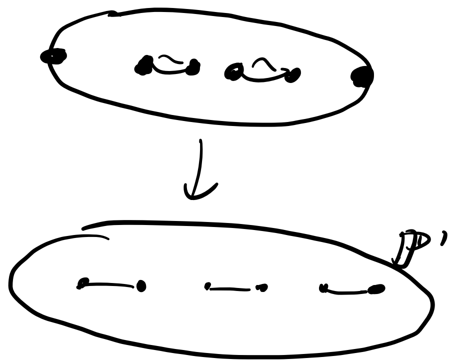
$\iota_K: S \to \mathbb{P}^{g-1}$ well-defined

ι_K is an not an embedding $\iff \exists f \in H^0(P+Q)$ merom. fct with exactly 2 poles

$\iff \iota_K$ is not injective.

Prop: Any genus 2 Riem. surf. is hyperelliptic.

Pf: $\underline{\iota_K : S \to \mathbb{P}^{g-1} = \mathbb{P}^1}$



Riemann-Hurwitz

$$\begin{array}{ccc} 2g-2 & = m\cdot(-2) & + b \\ \| & \| & \\ +2 & -2m + b & \end{array}$$

$m=2$, $b = 2+2m = 6$.

genus 3: If ι_K is 2:1, then hyperelliptic

ι_K: 1:1, $\iota_K : S \to \mathbb{P}^{g-1} = \mathbb{P}^2$

$\boxed{\frac{(d-1)(d-2)}{2} = \frac{3\cdot 2}{2} = 3}$ $\quad d = \deg C = \int c_1(K) = 2g-2 = 4.$

canonical curve genus 3 $\longleftrightarrow$ smooth quartic curve on $\mathbb{P}^2$.

Fact: C curve in $\mathbb{P}^{g-1}$ of degree $2g-2$

non-degenerate: C is not contained in any hyperplane.

Then C is a canonical curve: $C = \iota_K(S)$.

Pf of this fact: $\begin{matrix} H \\ \downarrow \\ C \end{matrix}$ $\deg(H) = 2g-2 \Rightarrow \underline{\deg(K-H)=0}$.

$$\overset{g}{\underset{}{h^0(H)}} - h^0(K-H) = (2g-2) - g + 1 = g-1.$$

$h^0(K-H) \overset{?}{\neq} 0$, $h^0(K-H) = h^1(H)$

$\Rightarrow \boxed{h^0(K-H)=1 \Rightarrow H=K}$

$$\left(\begin{matrix} h^0(K-H) \neq 0 \\ \Updownarrow \\ K = H \end{matrix} \right)$$

$$\boxed{\text{If } \deg D = 0, \text{ then } h^0(D) \neq 0 \Leftrightarrow \underline{D=0}.}$$

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genus 4 canonical curve: $C \subset \mathbb{P}^{g-1} = \mathbb{P}^3$. degree $2g-2 = 6$.

must be a complete intersection of X_1, X_2.

$X_1 \uparrow$ quadric, $X_2 \uparrow$ cubic

$2 \cdot 3$

$$h^0(\mathbb{P}^3, \underline{\mathcal{O}(2)}) = \binom{3+2}{2} = \frac{5 \cdot 4}{2} = 10.$$

$$h^0(C, 2K) = \underline{h^1(2K)} + 2(2g-2) - g + 1 = 3g-3$$

$h^1(2K) = h^0(-K) = 0$

$3g-3 = 3 \cdot 4 - 3 = 9$.

$$\overset{\neq 0}{\ker} \to H^0(\mathbb{P}^3, \mathcal{O}(2)) \to H^0(C, \mathcal{O}(2)|_C)$$

$\mathcal{O}(2)|_C = 2K|_C$

$\ker = \mathbb{C} \cdot \{F\}$. $\Rightarrow C \subset \{F=0\}$

$\{F=0\} = Q = X_1$

$$h^0(\mathbb{P}^3, \mathcal{O}(3)) = \binom{3+3}{3} = \frac{6\times5\times4}{3\times2\times1} = 20.$$

$$h^0(C, \mathcal{O}(3)) = 3\cdot(2g-2) - g + 1 = 5g-5 = 15.$$

cubics that contains Q is

$$h^0(\mathbb{P}^3, \mathcal{O}(3H-Q)) - 1 = h^0(\mathbb{P}^3, \mathcal{O}(1)) - 1 = 4.$$

$\Rightarrow$ $\exists$ cubic hypersurface X_2 containing C.

<u>genus 5</u>: $\quad h^0(\mathbb{P}^4, \mathcal{O}(2)) = \binom{4+2}{2} = \frac{6\cdot5}{2} = 15$

$\Downarrow$

$\mathbb{P}^{5-1} = \mathbb{P}^4 \quad h^0(C, \underbrace{\mathcal{O}(2)}_{= 2K}) = 2(2g-2) - g + 1 = \underbrace{3g-3}_{= 3\cdot5-3 = 12}$

degree $= 2\cdot5-2 = 8 = 2^3$

$\Rightarrow$ $\exists$ 3 quadrics X_1, X_2, X_3 s.t. $\boxed{C = X_1 \cap X_2 \cap X_3.}$

$c_1(\mathcal{O}(D)) = 0$

<u>Prop</u>: If $\deg D = 0$, then $h^0(D) \neq 0 \Leftrightarrow D =$ principle.

<u>pf</u>: $D = 0 \Rightarrow h^0(D) = h^0(\mathcal{O}_S) = 1.$

$h^0(D) \neq 0 \Leftrightarrow$ $\exists$ meromorphic fct. f s.t. $(f) + D \geq 0$.

$\Downarrow$

$s \qquad (s)_0 = D. \quad \deg D = \#\{P_i\} = 0. \Rightarrow \boxed{\mathcal{O}(D) \cong \mathcal{O}}$

$\quad \| $

$\sum_i P_i \qquad\qquad \Rightarrow D = \operatorname{div}(f).$