S cpt. Riem. surface genus g.

$H^0(S,\Omega_S^1) = \mathbb{C}\{\omega_1,\cdots,\omega_g\}$, $\{\delta_1,\cdots,\delta_{2g}\}$ basis for $H_1(S,\mathbb{Z}) \cong \mathbb{Z}^{2g}$.

$$\mu: \mathrm{Div}^0(S) \longrightarrow \mathbb{C}^g/\Lambda \qquad \boxed{\left(\int_{\delta_i}\omega_k = \delta_{ik},\ 1\le i,k\le g\right)}$$

$$\sum_\lambda (p_\lambda - q_\lambda) \longmapsto \left[\begin{pmatrix} \sum_\lambda \int_{q_\lambda}^{p_\lambda}\omega_1 \\ \vdots \\ \sum_\lambda \int_{q_\lambda}^{p_\lambda}\omega_g \end{pmatrix}\right] \qquad \Lambda = \sum_{i=1}^{2g}\mathbb{Z}\begin{pmatrix}\int_{\delta_i}\omega_1 \\ \vdots \\ \int_{\delta_i}\omega_g\end{pmatrix}$$

Thm (Abel). $\mu(D) \equiv 0 \pmod{\Lambda} \Leftrightarrow D$ is principal i.e. $D = \mathrm{div}(f) = (f)_0 - (f)_\infty$.

$$\Rightarrow \quad \mu: \mathrm{Div}^0(S)/\{\text{principle divisors}\} \longrightarrow \mathbb{C}^g/\Lambda_S = \mathrm{Jac}(S).$$

Thm (Jacobi inversion thm) μ is surjective.

$$\mu: \mathrm{Sym}^g(S) \longrightarrow \mathrm{Jac}(S)$$

$\mathrm{Sym}^g(S) = X$, $\mathrm{Sym}^g(S) = \underbrace{S\times S\times\cdots\times S}_{g \text{ copies}}$

$$(p_1,\cdots,p_g) \longmapsto \left(\sum_\lambda \int_{p_0}^{p_\lambda}\omega_i\right)_{i=1,\cdots,g} \in \mathbb{C}^g/\Lambda.$$

- μ is hol. between compact complex mfd. } $\Rightarrow$ μ is surjective
  $d\mu$ surjective at general points

Consequence of Proper Mapping Thm.

Easier proof:

If μ is not surjective. then $q \notin Im(\mu)$ $\quad \dim_{\mathbb{R}} M = m$.

$H^m(J \setminus \{q\}, \mathbb{R}) = 0$ $\qquad H^k(M) \overset{\text{dual}}{\leftrightarrow} H_c^{m-k}(M)$

$\Updownarrow \quad \Uparrow$

$\left(H_c^0(J \setminus \{q\}, \mathbb{R}) = 0 \right)$

Volume form $\Omega \overset{m\text{-form}}{=} d\eta > 0$, ($(m-1)$-form)

$$\int_X \mu^* \Omega = \int_X \mu^* d\eta = \int_X d(\mu^* \eta) = 0.$$

$\int_X \mu^*\Omega > 0$

Ex: $g = 1$. $\mu : Sym^1(S) \longrightarrow Jac(S)$

$Sym^1(S) = S$, $\quad p \longmapsto \left[\int_{p_0}^{p} \omega \right] \in Jac(S)$

$H^0(S, \Omega_S^1) = \mathbb{C} \cdot \{\omega\}$.

$\deg \Omega_S^1 = 2g - 2 = 0$.

$\Rightarrow$ Abel-Jacobi thm: μ is injective and surj.

injective $\uparrow$ Abel $\qquad$ surj. $\uparrow$ Jacobi.

$$\begin{pmatrix} \mu(p) = \mu(p') \\ \Updownarrow \\ p - p' = div(f) \\ \Updownarrow \\ p = p' \end{pmatrix}$$

μ is isomorphism: $S \cong \mathrm{Jac}(S) = \mathbb{C}/\Lambda$.

<u>Reciprocity Law</u>: ω holomorphic 1-form

η meromorphic 1-form.

Choose a local coordinate z around sing. point P of η.

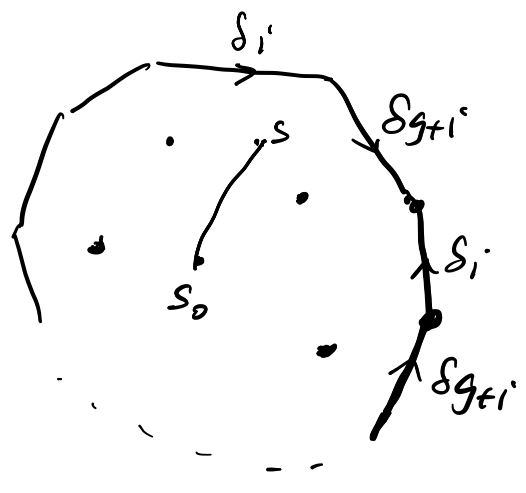
$$\eta(z) = \left(a_{-n}^{P} z^{-n} + \cdots + a_{-1}^{P} z^{-1} + a_0 + a_1 z + \cdots\right) dz.$$

$$\omega(z) = \left(b_0^P + b_1^P z + \cdots\right) dz.$$

$$\left(\begin{array}{ll} a_{-1}^P = \mathrm{Res}_P(\eta) = \frac{1}{2\pi i}\int_{\partial P} \eta. & \sum_P \mathrm{Res}_P(\eta) = 0. \end{array} \right)$$

$$\Pi^i = \int_{\delta_i} \omega, \quad N^i = \int_{\delta_i} \eta. \quad i = 1, \ldots, 2g.$$

$\delta_1, \ldots, \delta_{2g}$: cycles on S representing a canonical basis



$$\pi(s) = \int_{s_0}^{s} \omega$$

holomorphic on Δ

continuous on $\bar{\Delta}$

$$\int_{\partial \Delta} \pi \cdot \eta = 2\pi\sqrt{-1} \cdot \sum_{\lambda} \mathrm{Res}_{s_\lambda}(\pi \cdot \eta)$$

p singularity of η.

$$\pi(z) = \int_{s_0}^{z} \omega = \int_{s_0}^{p} \omega + \int_{p}^{z} (b_0^p + b_1^p z + b_2^p z^2 + \cdots) dz$$

$$\pi(z) = C + b_0^p z + \frac{1}{2} b_1^p z^2 + \cdots + \frac{b_j^p z^{j+1}}{j+1} + \cdots$$

$$\eta = (a_{-n}^p z^{-n} + \cdots + a_{-1} z^{-1} + a_0 + a_1 z + \cdots) dz$$

$$\mathrm{Res}_p(\pi \cdot \eta) = \sum_{j=2}^{n} \frac{a_{-j}^p \cdot b_{j-2}^p}{j-1} \qquad a_{-j}^p z^{-j} \cdot \frac{b_{j-2}^p}{j-1} \cdot z^{+(j-1)}$$

$\mathrm{Res}_p(\eta) = 0 . \ \forall p \in S$, η : meromorphic 1-form of 2nd type.

$$\sum_{i=1}^{g} (\Pi^i N^{g+i} - \Pi^{g+i} N^i) = 2\pi\sqrt{-1} \sum_{p,j} \frac{a_{-j}^p b_{j-2}^p}{j-1}.$$

ω hol. η meromorphic 1-form of 2nd type

($\mathrm{Res}_p \eta = 0$, $\forall p \in S$)

Weil Reciprocity: f, g meromorphic fct. on S. $\mathrm{div}(f) \cap \mathrm{div}(g) = \phi$.

$$\prod_{p \in S} f(p)^{\mathrm{ord}_p(g)} = \prod_{p \in S} g(p)^{\mathrm{ord}_p(f)}.$$

$$\omega = \frac{df}{f}, \quad \eta = \frac{dg}{g} . \quad \int_{\partial \Delta} (\log f) \cdot \frac{dg}{g}$$

$\frac{df}{f} = d \log f$

Riemann-Roch: $D = \sum_i a_i P_i$ divisor $\leftrightarrow [D]$ line bundle $= L$

$$\chi(\mathcal{O}(L)) = h^0(S, \mathcal{O}(L)) - h^1(S, \mathcal{O}(L)) = d - g + 1$$

$$d = \deg D = \sum_i a_i = \int_S c_1(L), \quad g = \text{genus of } S$$

Pf: $H^0(S, \mathcal{O}(L)) = \{ f \in \mathcal{M}(S), \text{ s.t. } \boxed{(f) + D \geq 0} \}$.

$L = [D] = (f_\alpha)$ $\quad g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ $\quad h_\alpha = g_{\alpha\beta} h_\beta = \frac{f_\alpha}{f_\beta} \cdot h_\beta$

$\Downarrow$

$\left(\frac{h_\alpha}{f_\alpha} = \frac{h_\beta}{f_\beta} = f. \quad (f) + (f_\alpha) = (h_\alpha) \geq 0. \right)$

Assume $D = \sum_\lambda P_\lambda$ ($\mathbb{N} \ni 0$), $a_\lambda = 1$ ($\in \mathbb{Z}$)

$\mathcal{M}(S) \ni f, \rightsquigarrow$ df meromorphic 1-form, holomorphic on $S - \{P_\lambda\}_\lambda$.

near P_λ, f has pole at most a_λ.

$V = \left\{ \boxed{\begin{array}{l} \eta = df \text{ has no residues} \\ \qquad \text{no periods} \\ \mathrm{ord}_{P_\lambda}(df) \geq -2, \ \forall P_\lambda \in D \end{array}} \right\}$

$f = a_{-n} z^{-n} + \cdots + a_{-1} z^{-1} + a_0 + a_1 z + \cdots$

$df = -n a_{-n} z^{-n} + \cdots + a_{-1} (-1) z^{-2} dz + a_1 dz$

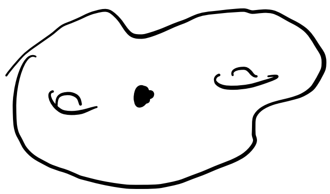
$\rightsquigarrow f(P) = \int_{P_0}^{P} \eta.$

$$\dim H^0(S, \mathcal{O}(D)) = \dim V + 1$$

($\mathcal{O}(D) = \mathcal{O}(L)$) $\quad df = df' \Rightarrow f = f' + \lambda, \ \lambda \in \mathbb{C}$.

$$\eta = \left(\frac{b_{-2}^{P_\lambda}}{z^2} + b_0 + b_1 z + \cdots \right) dz. \quad \text{near } P_\lambda, \ \lambda = 1, \cdots, d.$$

$\forall P \in S$, $\exists$ meromorphic 1-form ω, hol. on $S - \{P\}$ with double pole at P.



$$0 \to \Omega^1(P) \to \Omega^1(2P) \to \mathbb{C}_P \to 0$$

$$0 \to H^0(\Omega^1(P)) \to H^0(\Omega^1(2P)) \to \mathbb{C}_P \to H^1(S, \Omega^1(P)) = 0$$

$H^0(\Omega^1(P)) \cong H^0(\Omega^1) \cong \mathbb{C}^g$; $\quad H^0(\Omega^1(2P)) \neq 0$; $\quad H^1(S, \Omega^1(P)) = 0$

$|H| = 2 > \dim S$

$[P] = L > 0$.

$$P_\lambda \rightsquigarrow \omega_{P_\lambda}$$

$$(a_1, \cdots, a_d) \in \mathbb{C} \mapsto \eta_a = \sum_\lambda a_\lambda \cdot \omega_{P_\lambda} = (a_\lambda \cdot z_\lambda^{-2} + [0])\,dz_\lambda$$

$$\int_{\delta_i} \eta_a = 0. \qquad \eta_a \rightsquigarrow \eta_a - \sum_{i=1}^{g} c_i \omega_i$$

$i = 1, \cdots, g$.

$$V \subset W = \{ \boxed{\varphi_a} : \text{meromorphic 1-form. } \int_{\delta_i} \varphi_a = 0,\ i = 1, \cdots, g.\ \ \varphi_a(z_\lambda) = [a_\lambda \cdot z_\lambda^{-2} + O(1)]\,dz \}$$

$W \cong \mathbb{C}^d$, $\quad a = (a_1, a_2, \cdots, a_d)$.

$$\Psi : W \to \mathbb{C}^g \qquad \ker(\Psi) = V.$$

$$\varphi_a \longmapsto \left(\int_{\delta_{g+1}} \varphi_a, \cdots, \int_{\delta_{g+i}} \varphi_a, \cdots, \int_{\delta_{2g}} \varphi_a \right)$$

$$\int_{\delta_{g+i}} \varphi_a = 2\pi\sqrt{-1} \cdot \sum_\lambda a_\lambda \left(\frac{\omega_i}{dz_\lambda}\right)(P_\lambda).$$

$\omega = \omega_i$

$\eta = \varphi_a$

$$h^0(D) = \dim \ker(\psi) + 1 = \boxed{d - \text{rank}(\psi) + 1}$$

$h^0(D) = \dim H^0(S, \mathcal{O}(D))$

$\text{rank}(\psi) = g - \dim H^0(S, \Omega^1(-D))$, where $\Omega^1(-D) = K - D$

$$h^0(D) - h^0(K-D) = d - g + 1$$

$h^0(K-D) = h^1(D)$.

$\blacksquare$