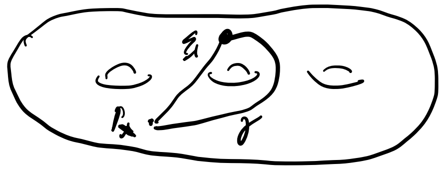
Abel-Jacobi: S cpt. Riem. surface of genus g.

$$\Rightarrow \quad H^0(S, \Omega_S^1) \cong \mathbb{C}^g = \{\omega_1, \omega_2, \cdots, \omega_g\}.$$

$$2h^{1,0} = h^{1,0} + h^{0,1} = h^1 = b_1 = 2g$$



$$D = \sum_\lambda (p_\lambda - q_\lambda) \in \mathrm{Div}^0(S) \quad \deg D = 0.$$

$$\mu_c(D) = \begin{pmatrix} \sum_\lambda \int_{p_\lambda}^{q_\lambda} \omega_1 \\ \sum_\lambda \int_{p_\lambda}^{q_\lambda} \omega_2 \\ \vdots \\ \sum_\lambda \int_{p_\lambda}^{q_\lambda} \omega_g \end{pmatrix} \in \mathbb{C}^g / \Lambda \cong \mathrm{Jac}(S)$$

$$\gamma = \sum_{k=1}^{2g} a_k \cdot \delta_k, \quad \{\delta_k\}_{k=1}^{2g} \text{ generates } H_1(S, \mathbb{Z}).$$

$$\mu_c - \mu_{c'} = \sum_{k=1}^{2g} a_k \cdot \int_{\delta_k} \omega_i = \sum_{k=1}^{2g} \Omega_{ik} a_k$$

$$(\Omega_{ik})_{\substack{1 \le i \le g \\ 1 \le k \le 2g}} = \begin{pmatrix} \int_{\delta_1} \omega_1 & \int_{\delta_2} \omega_1 & \cdots & \int_{\delta_{2g}} \omega_1 \\ \vdots & \vdots & & \vdots \\ \int_{\delta_1} \omega_g & \int_{\delta_2} \omega_g & \cdots & \int_{\delta_{2g}} \omega_g \end{pmatrix}$$

$g \times 2g$

Period matrix

$\Lambda = \mathbb{Z}\cdot$span of columns of Ω.

$\mathbb{C}^g/\Lambda = \mathrm{Jac}(S)$ diffeomorphic to $(S^1)^{2g}$.

$$\mu: \mathrm{Div}^0(S) \longrightarrow \mathrm{Jac}(S) = \mathbb{C}^g/\Lambda.$$

$$\begin{array}{ccc} \cup & & \\ \{\text{principal divisors}\} & \longrightarrow & [0] \in \mathbb{C}^g/\Lambda \end{array}$$

$\mathrm{div}(f) = (f)_0 - (f)_\infty$

$$\rightsquigarrow \mu: \mathrm{Div}^0(S)/\{\text{principal div}\} \to \mathbb{C}^g/\Lambda.$$

$$\begin{array}{c} \shortparallel \\ \mathrm{Pic}^0(S) = \{\text{hol. lines bundles with } c_1(L)=0\} \end{array}$$

$$\left(\begin{array}{l} H^0(S,\mathcal{O}^*) \to H^1(S,\mathbb{Z}) \to H^1(S,\mathcal{O}) \to H^1(S,\mathcal{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \\ \mathrm{Pic}^0(S) \cong \dfrac{H^1(S,\mathcal{O}) \cong \mathbb{C}^g}{\mathrm{Im}(H^1(S,\mathbb{Z}) \to H^1(S,\mathcal{O}))} \end{array} \right)$$

Thm (Abel) μ is injective

(Jacobi) μ is surjective.

Proof of Abel: Need to show:

$$\mu(D) \in \Lambda \Rightarrow \underset{\substack{\| \\ \operatorname{div}(f)}}{D} \text{ principal}$$

$$\underset{\substack{\| \\ \operatorname{div}(f)}}{\boxed{D = \sum_\lambda (p_\lambda - q_\lambda)}} = \sum_i a_i p_i + \sum_j b_j q_j \qquad a_\lambda > 0, \ b_\lambda < 0.$$

$$\left(\sum_i a_i + \sum_j b_j = 0. \right).$$

$\eta = \frac{1}{2\pi\sqrt{-1}} \frac{df}{f}$ meromorphic 1-form with simple poles.

(1) $\operatorname{div}(\eta) = -\sum_\lambda (p_\lambda + q_\lambda)$.

$f = z^k h(z) \quad h(0) \neq 0$

$\frac{df}{f} = k \cdot \frac{dz}{z} + \frac{dh}{h}$

(2) $\operatorname{Res}_{p_i}(\eta) = \frac{a_i}{2\pi\sqrt{-1}}$, $\operatorname{Res}_{q_i}(\eta) = \frac{b_i}{2\pi\sqrt{-1}}$, $\sum_i a_i + b_i = 0$.

(3) $\int_\gamma \eta \in \mathbb{Z}$ for any closed loop γ on $S - \{p_i, q_j\}_{i,j}$

Find η satisfying (1)–(3).

- One can find η with simple poles satisfying $\underline{(1)-(2)}$. by using

$$0 \to \Omega_S^1 \to \Omega_S^1\Big(\sum_\lambda (P_\lambda + Q_\lambda)\Big) \xrightarrow{\mathrm{Res}} \sum_\lambda (\mathbb{C}_{P_\lambda} \oplus \mathbb{C}_{Q_\lambda}) \to 0$$

$$\to H^0\Big(\Omega_S^1\Big(\sum_\lambda (P_\lambda + Q_\lambda)\Big)\Big) \to \bigoplus (\mathbb{C}_{P_\lambda} \oplus \mathbb{C}_{Q_\lambda}) \to \underline{H^1(S, \Omega_S^1)}$$

$$\eta \longmapsto \Big(\sum_i a_i + b_i = 0\Big) \qquad H^1(S,\Omega_S^1) \cong \mathbb{C}$$

- 

$$\int_\gamma \eta = \mathrm{Res}_{P_i} \eta = a_i \in \mathbb{Z}.$$

$$\int_{\underline{\delta_k}} \eta \in \mathbb{Z} \quad \forall\, k = 1, \ldots, 2g.$$

$\delta_1, \ldots, \delta_g, \delta_{g+1}, \ldots, \delta_{2g}$.

Change a basis for $H^0(S, \Omega_S^1)$ over $\mathbb{C}$

$$\boxed{\int_{\delta_k} \omega_i = \delta_{ik} \text{ for } 1 \le k \le g.}$$

$$\Rightarrow \Omega = \left(\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & \int_{\delta_{g+1}} \omega_1 & \cdots & \int_{\delta_{2g}} \omega_1 \\ 0 & 1 & & \vdots & \vdots & & \vdots \\ \vdots & 0 & \ddots & 0 & & & \\ 0 & 0 & & 1 & \int_{\delta_{g+1}} \omega_g & \cdots & \int_{\delta_{2g}} \omega_g \end{array}\right)$$

$$\int_{\delta_i} \Big(\underbrace{\eta - \Big(\sum_{k=1}^{g} \int_{\delta_k} \eta\Big) \omega_k}_{} \Big) = 0 . \quad \underbrace{1 \le i \le g}_{} .$$

Still need $\underline{\int_{\delta_{g+k}} \eta} \in \mathbb{Z}$. $\quad 1 \le k \le g$. $\qquad \mu(D) \in \Lambda$.

Reciprocity Law: ω holomorphic 1-form

η meromorphic 1-form with simple poles.

$$\sum_{i=1}^{g} \Big(\Pi^{i} N^{g+i} - \Pi^{g+i} N^{i} \Big) = 2\pi\sqrt{-1} \sum_{\lambda} \underline{\mathrm{Res}_{s_\lambda}(\eta)} \underline{\int_{s_0}^{s_\lambda} \omega}$$

$\Pi^i = \int_{\delta_i} \omega$, $\quad N^{g+i} = \int_{\delta_{g+i}} \eta$, $\quad \Pi^{g+i} = \int_{\delta_{g+i}} \omega$, $\quad N^i = \int_{\delta_i} \eta$

$$\sum_{i=1}^{g} \Big(\Big(\int_{\delta_i} \omega^k\Big) \int_{\delta_{g+i}} \eta - \int_{\delta_{g+i}} \omega^k \cdot \int_{\delta_i} \eta \Big) = \underline{\Big(\underline{a_\lambda} \int_{p_0}^{p_\lambda} \omega_k + \underline{b_\lambda} \int_{p_0}^{q_\lambda} \omega_k \Big)}$$

$\int_{\delta_i} \omega^k = \delta_{ik}$, $\quad \int_{\delta_i} \eta = 0$

LHS $= \underline{\int_{\delta_{g+k}} \eta}$

$\frac{1}{2\pi\sqrt{-1}}$ $\quad \underline{\sum_\lambda a_\lambda + \sum_\lambda b_\lambda = 0}$

$\sum_\lambda a_\lambda = -\sum \mathrm{Res}(\eta)$

RHS $= \sum_\lambda \int_{q_\lambda}^{p_\lambda} \omega_k = \underline{\sum_\lambda \int_{\alpha_\lambda} \omega_k}$

$$\int_{\delta_{g+k}} \eta = \sum_\lambda \int_{\partial_\lambda} \omega_k .$$

$$3P_1 - P_2 - 2P_3$$
$$\shortparallel$$
$$(P_1 - P_2) + 2(P_1 - P_3)$$

$$\mu(D) = \begin{pmatrix} \sum_\lambda \int_{\partial_\lambda} \omega_1 \\ \vdots \\ \sum_\lambda \int_{\partial_\lambda} \omega_g \end{pmatrix} \in \Lambda$$

$$= \sum_{k=1}^{2g} m_k \cdot \Omega_{ik} = \sum_{k=1}^{2g} m_k \cdot \int_{\delta_k} \omega_i$$

$$\Rightarrow \int_{\sum_\lambda \partial_\lambda} \omega_i = \boxed{\int_{\sum_{k=1}^{2g} m_k \cdot \delta_k} \omega_i}$$

$$\operatorname{Span}\{\omega_i, \overline{\omega_i}\} = H^1(S, \mathbb{C}).$$

$$\shortparallel \qquad\qquad \shortparallel$$

$$\int_{\delta_{g+i}} \eta \qquad \sum_{k=1}^{2g} m_k \int_{\delta_k} \omega_i$$

$$\eta \rightsquigarrow \underline{\eta - \sum_{k=1}^{g} m_{g+k} \omega_k}$$

$$\eta = \omega_i, \quad \omega = \omega_k$$

$$\int_{\delta_{g+i}} \eta - \sum_{k=1}^{g} m_{g+k} \int_{\delta_{g+i}} \omega_k$$

$$\begin{pmatrix} \Omega_1 \\ \shortparallel \\ I & \Omega_2 \end{pmatrix}$$

$$\shortparallel$$

$$\sum_{k=1}^{g} + \sum_{k=g+1}^{2g} \quad \sum_{k=1}^{2g} m_k \cdot \int_{\delta_k} \omega_i - \sum_{k=1}^{g} m_{g+k} \cdot \int_{\delta_{g+k}} \omega_i$$

$$= m_i \in \mathbb{Z}$$

$$\underline{\int_{\delta_i}} \left(\eta - \sum_{k=1}^{g} m_{g+k} \omega_k\right) = 0 - m_{g+i} = \underline{-m_{g+i}} \in \mathbb{Z} \quad i = 1, \cdots, g.$$

$$\mu : \mathrm{Div}^0(S) \to \mathbb{C}^g / \Lambda = \mathbb{Z} \text{ span of periods of } \omega_1, \dots, \omega_g.$$

$$\mu^{-1}(0) = \{\text{principal divisors}\}$$

- Surjectivity: $\mu : S^g \to \mathbb{C}^g/\Lambda$

$$S^g = \sum_{k=1}^{g} (p_k - p_0)$$

$$\Updownarrow$$

$$\mathrm{Sym}^g S = \{ (p_1, \dots, p_g) \in \underbrace{S \times S \times \dots \times S}_{g \text{ times}} \} / S_g$$

$$(z_1, \dots, z_g) \rightsquigarrow (\sigma_1(z_1, \dots, z_g), \sigma_2(z_1, \dots, z_g), \dots, \sigma_g(z_1, \dots, z_g))$$

$$\mu((p_1, \dots, p_g)) = \begin{pmatrix} \sum_k \int_{p_0}^{p_k} \omega_1 \\ \vdots \\ \sum_k \int_{p_0}^{p_k} \omega_g \end{pmatrix}$$

$$\frac{\partial \mu}{\partial z_i} = \frac{\partial}{\partial z_i} \begin{pmatrix} \int_{p_0}^{p_i} \omega_1 \\ \vdots \\ \int_{p_0}^{p_i} \omega_g \end{pmatrix} = \begin{pmatrix} \frac{\omega_1}{dz_i} \\ \vdots \\ \frac{\omega_g}{dz_i} \end{pmatrix}$$

$$\underset{\substack{\| \\ d\mu}}{\mathrm{Jac}(\mu)} = \begin{pmatrix} \frac{\omega_1}{dz_1} & \cdots & \frac{\omega_1}{dz_g} \\ \vdots & & \vdots \\ \frac{\omega_g}{dz_1} & \cdots & \frac{\omega_g}{dz_g} \end{pmatrix}$$

Choose P_1 s.t. $\underline{\omega_1(P_1) \overset{=1}{\neq} 0}$

$\underbrace{\omega_2(P_1) = \omega_3(P_1) = \cdots = 0}$.

Choose P_2 s.t.

$\underbrace{\omega_2(P_2) \neq 0,\ \omega_3(P_2) = \cdots = \omega_g(P_2) = 0}$

$\vdots$

at $(\underbrace{P_1, \cdots, P_g}_{\text{general}})$

$$\begin{pmatrix} 1 & & * & \\ 0 & 1 & & \\ \vdots & 0 & \ddots & \\ 0 & \vdots & & 1 \end{pmatrix}$$

rank $= g$.

$\mathrm{rank}(d\mu) = g \Longrightarrow \mu$ is surjective.

$\mu : \underset{M}{\mathrm{Sym}^g(S)} \to \underset{N}{\mathrm{Jac}(S)}$ holomorphic

Choose a volume form ψ on $\mathrm{Jac}(S)$. $\int_N \psi > 0$.

$\Rightarrow \int_M \mu^* \psi > 0$

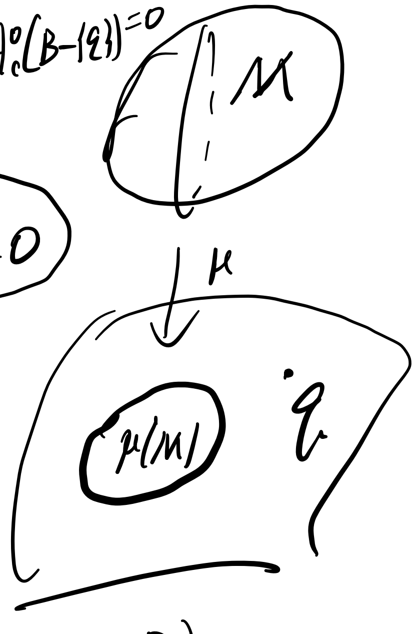
$H^0_c(N - \{q\}) = H^0_c(B - \{q\}) = 0$

$\updownarrow$

$\forall q \in N \setminus \mathrm{Im}(\mu)$, $H^{2g}(N - \{q\}, \mathbb{R}) = 0$

$\Rightarrow \psi = d\varphi$, $\underline{\varphi \in A^{2g-1}(N - \{q\})}$

$$\int_M \mu^* d\varphi = \int_M d(\mu^* \varphi) = 0.$$



$H^2(S \setminus q, \mathbb{R}) = 0$