Σ Riem. surf. : closed complex mfd. of $\dim_{\mathbb{C}} = 1$

Hermitian metric h on Σ $\rightsquigarrow$ $\omega = \sqrt{-1}\, \underset{\overset{\vee}{0}}{h}\, dz \wedge d\bar{z} \Rightarrow d\omega = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; \partial\omega + \bar{\partial}\omega$

$\Rightarrow$ Kähler

$$H^1(\Sigma, \mathbb{C}) = \underbrace{H^{1,0}(\Sigma)}_{\substack{\| \\ H^0(\Sigma, \Omega^1_\Sigma) \\ \| \\ \{\text{hol. 1-forms}\}}} \oplus H^{0,1}(\Sigma)$$

$\downarrow$

$\dim_{\mathbb{C}} = 2g$ $\qquad$ $\dim H^0(\Sigma, \Omega^1_\Sigma) = g$.

$K_\Sigma = T^*\Sigma$ degree $\deg K_\Sigma = \int_\Sigma \underbrace{c_1(K_\Sigma)}_{\substack{\| \\ -\int_\Sigma \frac{1}{2\pi} S(g)\, dvol_g}} \overset{\text{Gauss-Bonnet}}{=} 2g-2$.

- Riemann–Hurwitz formula: $\phi : \Sigma_1 \longrightarrow \Sigma_2$ holomorphic, not constant.

$$\begin{array}{ccc} \Sigma_1 & \longrightarrow & \Sigma_2 \\ \cup & & \cup \\ p & \longmapsto & \phi(p) \\ (U, z) & \longmapsto & \phi(z) = z^k f(z) \quad f(0) \neq 0. \end{array}$$

$z^k f(z) \; \| \; \tilde{z}^k$, $\quad \phi(z) \; \| \; w$

k : ramification index of ϕ at p. $= \nu(p)$

If $k > 1$, then p is called a branch point.

branch locus of $\phi = \sum_{p \in \Sigma_1} (\nu(p) - 1) \cdot p$.

Choose a meromorphic 1-form ω on Σ_2

$$\omega = \frac{g(w)}{h(w)}\, dw$$

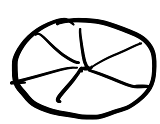
$$\phi^*(\omega) = \frac{g(z^k)}{h(z^k)} \cdot dz^k = \frac{g(z^k)}{h(z^k)} \cdot k z^{k-1} dz.$$

$$\Rightarrow \forall\ p \in \Sigma_1, \quad \mathrm{ord}_p(\underset{\substack{\uparrow \\ \text{meromorphic 1-form on } \Sigma_1}}{\phi^*\omega}) = k \cdot \mathrm{ord}_{\phi(p)}(\omega) + \underset{\substack{\shortparallel \\ (\nu(p)-1)\,p}}{(k-1)\,p}$$

$$\Rightarrow \quad \underset{\substack{\shortparallel \\ \sum_{p\in\Sigma_1} \mathrm{ord}_p(\phi^*\omega)\cdot p}}{K_{\Sigma_1}} = \underset{\substack{\shortparallel \\ \sum_{p\in\Sigma_1} \mathrm{ord}_{\phi(p)}\omega}}{\phi^* K_{\Sigma_2}} + \underset{\substack{\shortparallel \\ \sum_{p\in\Sigma_1} (\nu(p)-1)\,p}}{B}.$$

$$\Rightarrow \quad 2g_1 - 2 = n\cdot(2g_2-2) + \sum_{p\in\Sigma_1} (\nu(p)-1).$$

$$\Leftrightarrow \quad \boxed{\chi(\Sigma_1) = (\deg\phi)\,\chi(\Sigma_2) - \sum_{p\in\Sigma_1}(\nu(p)-1)}$$



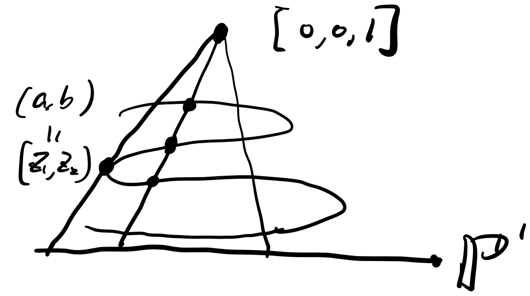
Application: sm. projective plane curve on $\mathbb{P}^2$ of degree d.

$$C = \{ F[z_0 : z_1 : z_2] = 0 \}$$

Project from: $[0,0,1]$ to $[1,*,*] \cong \mathbb{P}^1$

$$\deg \pi = \deg F = d$$

$$f(z_1,z_2) = F(1,z_1,z_2).$$



$L = \{ \frac{\partial f}{\partial z_1}(z_1-a) + \frac{\partial f}{\partial z_2}(z_2-b) = 0 \}$ passes through $[0,0,1]$

$$\Leftrightarrow \quad \frac{\partial f_1}{\partial z_2} = 0 \ \text{ at } (z_1,z_2)$$

$\{$ branched points $\} = \# \{ \underbrace{f(z_1, z_2) = 0 = \frac{\partial f_1}{\partial z_2}} \} = d(d-1)$

$$2g - 2 = d \cdot (2 \cdot 0 - 2) + d(d-1)$$

$$\Rightarrow 2g = d^2 - 3d + 2 = (d-1)(d-2) \Rightarrow g = \frac{(d-1)(d-2)}{2}.$$

<u>Another way</u>: $(K_{\mathbb{P}^2} \otimes [C])|_C = K_C$

$$\Rightarrow \int_C c_1(H^{\otimes(-3)} \otimes [C]) = \int_C (-3 \cdot c_1(H) + d \cdot c_1(H))$$

$$2g - 2 = -3d + d \cdot d = d^2 - 3d$$

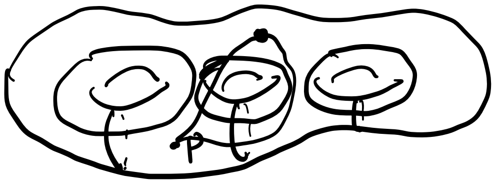
$$\Rightarrow g = \frac{d^2 - 3d + 2}{2} = \frac{(d-1)(d-2)}{2}$$

Abel-Jacobian Theorem

<u>Abel-Jacobian map</u>: S Riemann surface.

$$H^0(S, \Omega_S^1) = \mathrm{Span}\{\omega_1, \cdots, \omega_g\}$$

S



basis for $H_1(S, \mathbb{Z}) \cong \mathbb{Z}^{2g}$: $\{\delta_1, \cdots, \delta_g, \delta_{g+1}, \cdots, \delta_{2g}\}$.

$$S \xrightarrow{\phi} \mathbb{C}^g/\Lambda = \mathrm{Jac}(S)$$ Jacobian of Riem. surf. S.

$$q \longmapsto \underbrace{\left(\int_p^q \omega_1, \int_p^q \omega_2, \ldots, \int_p^q \omega_g\right)^t}$$

$$\int_\gamma \omega_i - \int_{\gamma'} \omega_i = \int_{\sum a_k \delta_k} \omega_i = \sum_k a_k \int_{\delta_k} \omega_i$$

Periods.

Period matrix

$$\Omega = \begin{bmatrix} \int_{\delta_1}\omega_1 & \int_{\delta_2}\omega_1 & \cdots & \int_{\delta_{2g}}\omega_1 \\ \int_{\delta_1}\omega_2 & \int_{\delta_2}\omega_2 & \cdots & \int_{\delta_{2g}}\omega_2 \\ \vdots & & & \vdots \\ \int_{\delta_1}\omega_g & \int_{\delta_2}\omega_g & \cdots & \int_{\delta_{2g}}\omega_g \end{bmatrix} = \Omega_{ik} \cdot a_k$$

$$\left(\phi_\gamma(q) - \phi_{\gamma'}(q)\right)_i = \sum_{k=1}^{2g} \Omega_{ik} \, \underset{\mathbb{Z}}{\underset{\cap}{a_k}}, \quad 1 \le i \le g.$$

$c_k \in \mathbb{R}$

$$\forall i, \ \sum_k c_k \Omega_{ik} = 0 \Rightarrow \sum_k c_k \int_{\delta_k} \overline{\omega_i} = 0 \Rightarrow \sum_k c_k \delta_k = 0$$

$$\sum_k c_k \int_{\delta_k} \omega_i$$

in $H_1(S, \mathbb{R})$

$\{\omega_i, \overline{\omega_i}\}_i$ span $H^1(S, \mathbb{C})$

$\Rightarrow$ Column vectors linearly independent over $\mathbb{R}$.

$$\Rightarrow \Lambda = \{ m_1 \Pi_1 + \cdots + m_{2g} \Pi_{2g} \ ; m_i \in \mathbb{Z} \}$$

lattice generated by column vectors of Period matrix Ω.

$$\mathrm{Div}(S) = \left\{ \sum_i a_i p_i - \sum_k b_k q_k \ : \ \begin{array}{l} a_i > 0,\ b_k > 0 \\ p_i \neq p_j,\ q_k \neq p_\ell \end{array} \right\}$$

$$\mathrm{Div}_0(S) = \left\{ \sum_\lambda (p_\lambda - q_\lambda),\ p_\lambda, q_\lambda \in S \right\}$$

$$\mathrm{Div}_0(S) \longrightarrow \mathrm{Jac}(S) = \mathbb{C}^g / \Lambda \cong \mathbb{Z}^{2g}$$

$$\sum_\lambda (p_\lambda - q_\lambda) \longmapsto \begin{pmatrix} \sum_\lambda \int_{q_\lambda}^{p_\lambda} \omega_1 \\ \sum_\lambda \int_{q_\lambda}^{p_\lambda} \omega_2 \\ \vdots \\ \sum_\lambda \int_{q_\lambda}^{p_\lambda} \omega_g \end{pmatrix} \mod \Lambda$$

$$\left(\int_{\gamma} \omega_i \right) = \sum_k \Omega_{ik} a_k = \Omega \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_{2g} \end{pmatrix}.$$

$$\gamma = \sum_k a_k \delta_k$$

$f \in \mathcal{M}(S)$ meromorphic fct. on S

$$(f) = \mathrm{div}(f) = (f)_0 - (f_\infty) = \sum_{p \in S} \mathrm{ord}_p(f) \cdot p$$

<u>Lem</u>: $AJ((f)) = 0 \in \mathbb{C}^g / \Lambda$

<u>Pf</u>: $t_0 f + t_1 \cdot 1 = f_t \qquad t = [t_0, t_1] \in \mathbb{P}^1$

$$[t_0, t_1] \longmapsto \sum_\lambda (P_{t,\lambda} - q_{t,\lambda}) = \mathrm{div}(f_t).$$

$$\downarrow$$

$$\sum_\lambda \int_{q_{t,\lambda}}^{P_{t,\lambda}} \omega_i \mod \Lambda$$

$$\overset{\curvearrowleft}{\mathbb{P}^1} \xrightarrow{\phi} \underline{\mathrm{Jac}(S) = \mathbb{C}^g/\Lambda} \quad \{dz_1, \cdots, dz_g\}$$

$$\phi^* dz_i \in H^0(\mathbb{P}^1, \Omega^1) = 0 \Rightarrow \phi^* dz_i = 0.$$

$$\Rightarrow \phi \text{ is constant} \Rightarrow \phi([1,0]) = \phi([0,1]) = AJ(\overset{\overset{\text{0-div}}{\shortparallel}}{(1)})$$

$$\underset{0 \in \mathbb{C}^g/\Lambda.}{\shortparallel}$$

$$\begin{array}{ccc} \mathrm{Div}^0(S) & \xrightarrow{\mu} & \underline{\mathrm{Jac}(\mathbb{C}^g/\Lambda)} \\ \cup & & \\ \{\text{principal divisors}\} & \longrightarrow & 0 \end{array}$$

$$\Rightarrow \quad \mathrm{Div}^0(S) / \{\text{principal divisors}\} \xrightarrow{AJ} \mathrm{Jac}(\mathbb{C}^g/\Lambda).$$

$$\shortparallel$$

$$\mathrm{Pic}^0(S) \cong \{\text{holomorphic line bundles with } c_1(L) = 0\}$$

Thm: AJ is an isomorphism.

Thm A: If $D \in Div^0(S)$ satisfies $\mu(D) = 0 \in Jac(S)$

then D is principal.

Thm J: μ is surjective.

Proof of Thm A: If $f \in m(S)$, $\eta = \frac{1}{2\pi i}\frac{df}{f}$ meromorphic 1-form. with simple poles.

$f(z) = z^k h(z)$, $\frac{df}{f} = k\frac{dz}{z} + \frac{dh}{h}$

$(\eta)_\infty = -\sum_\lambda (P_\lambda + Q_\lambda)$, $\underline{div(f) = \sum_\lambda (P_\lambda - Q_\lambda)}$

For any loop γ on $S - \{P_i, Q_i\}$,

$$\int_\gamma \eta = \frac{1}{2\pi i}\int_\gamma \frac{df}{f} = \frac{1}{2\pi i}\int_\gamma d\log f \in \mathbb{Z}$$

$$\log f \Big|_P^P$$

$\sum_\lambda (P_\lambda - Q_\lambda)$

$\shortparallel$

$D \in Div^0(S)$, If $\underline{\mu(D) = 0 \in Jac(S)}$, then $\exists f \in m(S)$

s.t. $D = div(f)$.

First find η meromorphic 1-form with simple poles

at $\{P_\lambda, q_\lambda\} = \mathrm{supp}(D) = -(\eta)_\infty$.

$\mathrm{Res}_{P_\lambda}(\eta) = a_\lambda$, $\mathrm{Res}_{q_\lambda}(\eta) = b_\lambda$.

and $\int_\gamma \eta \in \mathbb{Z}$ for any loop in $S \setminus \{P_\lambda, q_\lambda\}$.

$$\Rightarrow f = \exp\left(2\pi i \int_{P_0}^{P} \eta\right)$$

Step 1: For any $\{P_\lambda\}$ finite subset of S,

For any $\{a_\lambda \in \mathbb{C}\}$ s.t. $\sum_\lambda a_\lambda = 0$. $(\longleftrightarrow D \in \mathrm{Div}^0)$

$\exists$ meromorphic 1-form η with simple poles, s.t.

$$(\eta)_\infty = -\sum P_\lambda, \quad \mathrm{Res}_{P_\lambda}(\eta) = a_\lambda$$

Pf: $0 \to \Omega^1 \to \Omega^1\left(\sum_\lambda P_\lambda\right) \xrightarrow{\mathrm{Res}} \sum_\lambda \mathbb{C}_{P_\lambda} \to 0$

$$H^0\left(\Omega^1_S\left(\sum_\lambda P_\lambda\right)\right) \xrightarrow{\mathrm{Res}} \bigoplus_\lambda \mathbb{C}_{P_\lambda} \xrightarrow{\partial} \underline{H^1(S, \Omega^1)} \to H^1\left(S, \Omega^1\left(\sum_\lambda P_\lambda\right)\right)$$

$$H^1(S,\Omega^1) \cong \underline{H^0(S, \mathcal{O}) \cong \mathbb{C}}.$$

$\dim \mathrm{Im}(\mathrm{Res}) = \dim \ker \partial$ has $\underline{\mathrm{codim}} \leq 1$.

$\mathrm{codim} = \text{rank of } \partial$.

$\dim \underline{\mathrm{Im}(\mathrm{Res})}$ $\mathrm{codim} \geq 1$ $\Rightarrow \mathrm{Im}(\mathrm{Res}) = \{(a_\lambda) : \sum_\lambda a_\lambda = 0\}$

$\dim \mathrm{Im}(\mathrm{Res}) \neq \mathrm{codim}$

$$\sum_\lambda \mathrm{Res}_{P_\lambda}(\eta) = 0$$