X complex mfd.

$$0 \to \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i)} \mathcal{O}^* \longrightarrow 0$$

$$f \longmapsto e^{2\pi i f}$$

$$\rightsquigarrow \quad H^1(X,\mathcal{O}) \to \underbrace{H^1(X,\mathcal{O}^*)}_{\cong} \xrightarrow{c_1} H^2(X,\mathbb{Z}) \xrightarrow{\phi} H^2(X,\mathcal{O}).$$

$\{$holomorphic line bundles$\}$

$$\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta) : g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1\}$$

$\{g_{\alpha\beta}\}$ $\downarrow c_1$

$$\left\{\left(\frac{1}{2\pi i}\log g_{\alpha\beta}\right) + \left(\frac{1}{2\pi i}\log g_{\beta\gamma}\right) + \left(\frac{1}{2\pi i}\log g_{\gamma\alpha}\right) \in \mathbb{Z}(U_\alpha \cap U_\beta \cap U_\gamma)\right\}$$

$$\mathrm{Image}(c_1) = \ker(\phi : H^2(X,\mathbb{Z}) \xrightarrow{\phi} H^2(X,\mathcal{O})).$$

$$H^2(X,\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus \underbrace{H^{0,2}(X)}_{\cong\, H^2(\mathcal{O})}.$$

Kähler

$\rightsquigarrow$ Thm (Lefschetz (1,1)-thm).

$$\boxed{\mathrm{Image}(c_1 : H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z})) = H^{1,1}(X) \cap H^2(X,\mathbb{Z})}$$

$$H^2(X,\mathbb{C})$$

Hodge Conj for projective mfds:

The cycle map $\{$analytic cycles of codim $p\} \longrightarrow H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X)$

$$H^{p,p}(X) \subset H^{2p}(X,\mathbb{C})$$

is surjective $\quad \sum_i a_i Z_i, \quad a_i \in \mathbb{Q}$

irred. analytic subvar. of codim p.

$b_{\alpha\beta} - b_{\beta\gamma} + b_{\gamma\alpha}$

$$\{a_{\alpha\beta\gamma} \in \mathbb{Z}\} = \underline{a} \in \underbrace{H^2(X,\mathbb{Z})}_{} \xrightarrow{\phi} H^2(X,\mathcal{O}).$$

$\{a_{\alpha\beta\gamma}\} \to \mathbb{R}$, $\quad H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}) \to H^2(X,\mathbb{C})$

$\bar\partial b_{\alpha\beta} = \omega_\alpha^{0,1} - \omega_\beta^{0,1}$

$\bar\partial \omega_\alpha^{0,1} = \bar\partial \omega_\beta^{0,1}$

$d b_{\alpha\beta} = \omega_\alpha - \omega_\beta$

$$\underline{H^2_{dR}(X,\mathbb{C})} = \underbrace{H^2(X,\mathbb{C}) = H^{2,0} \oplus \boxed{H^{1,1}} \oplus H^{0,2}}$$

$d\omega_\alpha = d\omega_\beta$

$$0 \to \mathbb{R} \to A^0 \to A^1 \to A^2 \to \cdots$$

$$\underline{0 \to \mathbb{R} \to A^0 \to \mathcal{Z}^1 \to 0}, \quad \mathcal{Z}^1 = \ker(d : A^1 \to A^2).$$

$$\Rightarrow \underline{H^1(X,\mathcal{Z}^1)} \overset{\cong}{\Rightarrow} H^2(X,\mathbb{R}) \xrightarrow{\kappa} \underline{H^2(X,A^0) = 0}$$

A^0 is fine sheaf

$H^1(X,A) \to H^1(X,\mathcal{Z}^1)$

$\overset{\shortparallel}{0}$

$$(a_{\alpha\beta\gamma}) \in H^2(X,\mathbb{R}) \Rightarrow \underline{a_{\alpha\beta\gamma} = b_{\alpha\beta} - b_{\beta\gamma} + b_{\gamma\alpha}}, \; b_{\alpha\beta} \in A^0(U_\alpha \cap U_\beta)$$

$a_{\alpha\beta\gamma} \in A^0$

$$\kappa((a_{\alpha\beta\gamma})) = 0 \in H^2(X,A^0)$$

$$\Rightarrow \quad 0 = d a_{\alpha\beta\gamma} = d b_{\alpha\beta} - d b_{\beta\gamma} + d b_{\gamma\alpha} \Rightarrow \underline{(d b_{\alpha\beta})} \in Z^1(A^1).$$

$\Rightarrow$ because $H^1(A^1) = 0$, $\; d b_{\alpha\beta} = \omega_\alpha - \omega_\beta$ with $\omega_\alpha \in A^1(U_\alpha)$.

$$\Rightarrow \quad 0 = d\omega_\alpha - d\omega_\beta \Rightarrow [d\omega_\alpha] \in \frac{H^0(X,\mathcal{Z}^2)}{dH^0(X,A^1)} \in \underline{H^2_{dR}(X)}.$$

$$\uparrow$$

$$(a_{\alpha\beta\gamma}) \in \underline{H^2(X,\mathbb{R})}$$

$$H^2(X,\mathcal{O}) \ni (a_{\alpha\beta\gamma}) \rightsquigarrow a_{\alpha\beta\gamma} = b_{\alpha\beta} - b_{\beta\gamma} + b_{\gamma\alpha} \quad b_{\alpha\beta} \in A^0(U_\alpha \cap U_\beta)$$

$$0 = \bar\partial a_{\alpha\beta\gamma} = \bar\partial b_{\alpha\beta} - \bar\partial b_{\beta\gamma} + \bar\partial b_{\gamma\alpha}, \; (\bar\partial b_{\alpha\beta}) \in Z^1(A^{0,1})$$

$$\Rightarrow \bar\partial b_{\alpha\beta} = \omega_\alpha^{0,1} - \omega_\beta^{0,1} \Rightarrow \bar\partial \omega_\alpha^{0,1} = \bar\partial \omega_\beta^{0,1} \Rightarrow [\bar\partial \omega_\alpha] \in H^{0,2}_{\bar\partial}(X)$$

Kodaira Embedding: X compact complex mfd.

Then X embeds into projective space iff there exists a positive line bundle $L \to X$.

$\uparrow$

$\exists$ a Hermitian metric h on L whose Chern curvature

$\sqrt{-1}\sum (g_{i\bar{j}}) dz_i \wedge d\bar{z}_j = \omega_h = \sqrt{-1}\,\bar{\partial}\partial \log|s|_h^2$ where s is local trivializing hol. section is a positive $(1,1)$-form.

$= \sqrt{-1}\,\bar{\partial}\partial \log|s|_h^2 + \underbrace{\sqrt{-1}\,\bar{\partial}\partial \log|f|^2}_{=0}$

positive: $(g_{i\bar{j}}) > 0$.

- embedding $X \xhookrightarrow{\iota_X} (\mathbb{P}^N, \underset{= H}{\mathcal{O}(1)})$ $\quad \underset{= \mathcal{O}(-1)^\vee}{\mathcal{O}(1)_z} = (\mathbb{C}\cdot(z))^*$

$(\underbrace{\iota_X^* \mathcal{O}(1)}_{= L \to X}, \iota_X^* h_{FS})$ $\quad \underbrace{\sqrt{-1}\,\bar{\partial}\partial \log h_{FS}}_{= \omega_F} > 0$

- Start with $\omega_h = \sqrt{-1}\,\bar{\partial}\partial \log h > 0$.

$H^0(X, L^k) \ni \{s_0, s_1, \cdots, s_N\}$

$X \xrightarrow{\iota_k} \mathbb{P}^N$

$x \longmapsto [\underset{= f_0(x)\cdot s}{s_0(x)}, \cdots, \underset{= f_N(x)\cdot s}{s_N(x)}]$

Need to show that when k is sufficiently large.

$$H^0(X, L^k) \twoheadrightarrow L^k|_p \oplus L^k|_q \iff \iota_k \text{ separate points.}$$

$$\downarrow$$

$$H^1(X, L^k \otimes I_{p,q}) \cong H^1(\widetilde{X}, \pi^* L^k \otimes \mathcal{O}(-E_p - E_q)) = 0 \quad \text{when } k \gg 1$$

$$0 \to L^k \otimes I_{p,q} \to L^k \longrightarrow L^k \otimes (\mathcal{O}_p \oplus \mathcal{O}_q) \to 0$$

$$\mathcal{O}_p = \mathcal{O}_X / I_p \cong \mathbb{C}$$

$$0 \to L^k \otimes I_{2p} \longrightarrow L^k \to L^k \otimes \mathcal{O}_{2p} \to 0$$

$$\mathcal{O}_{2p} = \mathcal{O}_X / I_p^2$$

$$f = f(p) + \sum_i a_i z_i + h.o.t.$$

$$\rightsquigarrow \quad H^0(X, L^k) \twoheadrightarrow H^0(X, L^k \otimes \mathcal{O}_{2p}) \to H^1(L^k \otimes I_{2p}) = H^1(\widetilde{X}, \pi^* L^k \otimes \mathcal{O}(-2E_p)) = 0 \quad \text{when } k \gg 1.$$

$$\Downarrow$$

$\iota_k : X \to \mathbb{P}^N$ separate 1st. jets.

$\Rightarrow$ ι_k is an immersion.

Thm (Nakano-Kodaira) L positive line bundle over X^n.

Then $H^q(X, \Omega^p(L)) = 0$ for $p+q > n = \dim X$.

$$\Rightarrow \text{Kodaira vanishing} \quad H^q(X, \underbrace{\overset{\Lambda^n T^*X}{\overset{\parallel}{\Omega^n}}(L)}_{\overset{\parallel}{H^q(X, K_X \otimes L)}}) = 0, \quad \text{when } q>0.$$

($\rightsquigarrow$ Kawamata-Viehweg vanishing:

For L semipositive and big (i.e. $\int_X c_1(L)^n > 0$), $H^q(K_X \otimes L) = 0$.)

Pf: $(L, h) \rightsquigarrow$ $D_h' : \Gamma(X, L) \to \Gamma(X, A^{1,0} \otimes L)$
$\bar\partial : \Gamma(X, L) \to \Gamma(X, A^{0,1} \otimes L)$.

$$D = D_h' + \bar\partial, \quad D_h^2 = \underline{D_h' \bar\partial + \bar\partial D_h' = \omega_h}$$

$$H^q(X, \Omega^p(L)) \cong \underset{\overset{\cup}{\phi}}{H^{p,q}_{\bar\partial}}(L) \cong \mathcal{H}^{p,q}_{\bar\partial}(L)$$

Kähler identities: $\underline{[\Lambda, \bar\partial] = \pm D_h'^*}$, $[\Lambda, D_h'] = \pm \bar\partial^*$

$$\Lambda = L^*, \quad L = \omega_h \Lambda \quad = \bar\partial D'$$

$$0 = ((\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\phi, \phi) = (\underbrace{\bar\partial\bar\partial^*\phi}_{\substack{\shortparallel \\ \bar\partial[\Lambda, D']\phi}}, \phi)$$

$$\bar\partial[\Lambda, D']\phi = \underbrace{\bar\partial\Lambda D'}_{\substack{\shortparallel \\ [\bar\partial,\Lambda] + \underbrace{\Lambda\bar\partial D'}_{\substack{\shortparallel\\ \Lambda\cdot L}}}}\phi - \underbrace{\bar\partial D'}_{\substack{\shortparallel\\ \omega}}\Lambda\phi$$

$$D'^* = [\bar\partial, \Lambda]$$

$$\Rightarrow \quad 0 = \underbrace{(D'^* D'\phi, \phi)}_{\substack{\vee\!\shortparallel\\ 0}} + \underbrace{([\Lambda, L]\phi, \phi)}_{\substack{\shortparallel \\ \underbrace{((p+q-n)\phi, \phi)}_{\substack{\vee\!\shortparallel \\ 0}}}} \Rightarrow \|\phi\|^2_{L^2} = 0$$

$p+q>n$.

$$\Downarrow$$

$$\phi = 0.$$

Ex: $n=1$. Riemann surface.

$L \to X$ holomorphic line bundle.

$$0 \to L(-p-q) \to L \to \mathcal{O}_p \oplus \mathcal{O}_q \to 0$$

$$\Rightarrow \quad H^0(X, L) \to \mathbb{C}_p \oplus \mathbb{C}_q \to \underbrace{H^1(L(-p-q))}_{\substack{\shortparallel\\ 0}}$$

$$0 \to L(-2p) \to L \to \mathcal{O}_{2p} \to 0$$

$$\Rightarrow \quad H^0(X, L) \to \mathbb{C}^2 \to \underbrace{H^1(L-2p)}_{\substack{\shortparallel\\ 0}}$$

$$H^1(L(-p-q)) \xrightarrow[\text{Serre duality}]{\text{Kodaira}} \underline{H^0(X, K_X \otimes L^{-1}(p+q))}$$

$$H^0(X, L) \neq 0 \Rightarrow \deg L \geq 0.$$

$\deg L = \sum c_i$

$\Downarrow$ $\qquad \Nearrow$

$\exists$ effective divisor $D = \sum_i c_i p_i$ with $c_i \geq 0$

$$\deg(K_X \otimes L^{-1}(p+q)) = \deg(K_X) + 2 - \deg(L)$$

$\Rightarrow$ If $\deg(L) > \deg(K_X) + 2$, then

$\iota_L : X \to \mathbb{P}^N$ is an embedding.

<u>Fact</u>: $\deg(K_X) = 2g - 2$ where g is the genus of X.

$$\deg(K_X) = \int c_1(K_X)$$

Hermitian metric h on X.

$h = \lambda |dz|^2$, $\lambda > 0$. $\quad \lambda = \left|\frac{\partial}{\partial z}\right|^2$

$K_X = \mathcal{O}(dz)$. $\quad |dz|_h^2 = \lambda^{-1}$

$$\omega_h = \frac{\sqrt{-1}}{2\pi} \bar\partial\partial \log |dz|_h^2 = \frac{\sqrt{-1}}{2\pi} \bar\partial\partial \log \lambda^{-1} = \frac{\sqrt{-1}}{2\pi} \left(\lambda \frac{-\partial^2}{\partial z \partial \bar z} (\log \lambda)\right) \lambda^{-1} dz \wedge d\bar z$$

$- S\, dvol$.

$$\frac{1}{2\pi} \int S\, dv_g \underset{\uparrow}{=\!=} \chi(X) = 2 - 2g.$$

Gauss-Bonnet

$$-\frac{1}{\lambda} \Delta \log \lambda = S$$

$$\deg K_X = -\chi(X) = 2g - 2.$$

$$\lambda|dz|^2 = \frac{|dz|^2}{(1+|z|^2)^2}, \Rightarrow \lambda = (1+|z|^2)^{-2} \Rightarrow S = -\frac{1}{\lambda}\Delta\log\lambda = 1.$$