X cpt. Kähler mfd.

Kähler identity: $\Delta_d = 2\Delta_{\bar\partial} = 2\Delta_\partial$

$\Delta_d : A^{p,q} \to A^{p,q}$

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

$$\begin{array}{ccc} H^k(X,\mathbb{C}) & = & \bigoplus_{p+q=k} H^{p,q}(X) \\ \wr\wr & & \wr\wr \\ \mathcal{H}^k(X) & = & \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X) \cong \mathcal{H}^{p,q}_{\bar\partial}(X) \cong H^q(X,\Omega^p). \end{array}$$

$\mathcal{H}^k(X) = \{\eta : \Delta_d \eta = 0\} = \sum \eta^{p,q}$

$\mathcal{H}^{p,q}(X) = \{\eta^{p,q} \in A^{p,q} : \Delta_d \eta^{p,q} = 0\}$

$$H^k(X,\mathbb{C}) = \bigoplus_{r\geq 0} L^r P^{k-2r}, \quad P^{k-2r} = \ker(\Lambda) \cap H^{k-2r}$$

$\Lambda : \Lambda^{k-2r} \to \Lambda^{k-2r-2}$

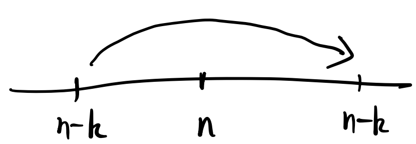
$\rightsquigarrow$ Hard Lefschetz Thm:

$\Lambda = L^*$, $L = \omega\wedge. : \Lambda^k \to \Lambda^{k+2}$

$$\boxed{\Delta \text{ commutes with } L, \Lambda, *}.$$

$$\begin{array}{ccc} L : \mathcal{H}^k & \to & \mathcal{H}^{k+2} \\ \| & & \| \\ \bigoplus \mathcal{H}^{p,q} & & \bigoplus \mathcal{H}^{r,s} \end{array}$$

$k \geq 0$: $L^k : H^{n-k}(X) \to H^{n+k}(X)$ is an isomorphism.



Cor: For $i < n$, $L: H^i \xrightarrow{L} H^{i+2}$ is injective.

$$\begin{tikzcd} H^i \arrow[r, "L"] \arrow[dr, "L^{n-i}", "\cong"'] & H^{i+2} \arrow[d, "L^{n-i-1}"] \\ & H^{2n-i} \end{tikzcd}$$

$\Rightarrow$ $b_i \le b_{i+2}$. for $i < n$.

- Hodge-Riemann Bilinear relation:

$$B: H^k(X) \times H^k(X) \to \mathbb{C}$$
$$(\eta, \varphi) \mapsto \int_X \eta \wedge \varphi \wedge \omega^{n-k}$$

$P^{r,s} = \ker(\Lambda) \cap H^{r,s}$ for $s = k - r$.

- $B(P^{r,s}, P^{r',s'}) = 0$ if $r + r' \neq k$.

- $\exists C_r$, s.t. $C_r B(\xi, \bar{\xi}) > 0$ (positive definite).

$\xi \in P^{r,s} = \ker(\Lambda) \cap H^{r,s}$. $\Lambda: H^{r,s} \to H^{r-1,s-1}$.

Ex: $P^{r,0} = H^{r,0} = \{\text{holomorphic } r\text{-forms}\}$

$$\begin{array}{ccc} H^{r,0} & & \{\text{holomorphic } r\text{-forms}\} \\ \downarrow \Lambda & & \Downarrow \\ 0 & & \xi \end{array}$$

$$C \cdot \int \xi \wedge \bar{\xi} \wedge \omega^{n-r} = \|\xi\|^2_{L^2} > 0.$$

$n=1$, $H^{1,0} = \{\text{holomorphic 1-forms}\} = H^0(\Omega^1)$

Riem. surf.

$$H^0(\Omega^1) = \operatorname{Span}\{\eta_1, \dots, \eta_g\}$$

$$\sqrt{-1} \int \eta \wedge \bar{\eta} = \|\eta\|^2_{L^2} > 0$$

· X complex mfd.

A divisor $D = \sum_i a_i D_i$ locally finite formal linear combination of analytic hypersurfaces. with $a_i \in \mathbb{Z}$.

$$X = \bigcup_\alpha U_\alpha, \quad D_i \cap U_\alpha = \{ f_{i\alpha} = 0 \}$$

$$\rightsquigarrow [D] = \left\{ g_{\alpha\beta} = \prod_i \left(\frac{f_{i\alpha}}{f_{i\beta}} \right)^{a_i} \right\}$$

$$\| $$

$$\bigcup_\alpha (U_\alpha \times \mathbb{C}) \Big/ (p, f) \sim (p, g_{\alpha\beta} f).$$

$$\underset{U_\beta \times \mathbb{C}}{\cap} \qquad \underset{U_\alpha \times \mathbb{C}}{\cap}$$

$$\underline{\{\text{divisors}\} \cong H^0(X, \mathcal{M}^*/\mathcal{O}^*).}$$

$$D \cap U_\alpha = (f_\alpha = 0) = (f_\alpha \cdot g_\alpha = 0)$$

f_α: meromorphic, $g_\alpha \in \mathcal{O}^*(U_\alpha)$.

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$$

$$0 \to \underset{\substack{\| \\ \{\text{global meromorphic fcts on } X\}}}{\underline{H^0(X, \mathcal{M}^*)}} \to \underset{\substack{\| \\ \{\text{divisors}\}}}{\underline{H^0(X, \mathcal{M}^*/\mathcal{O}^*)}} \to \underset{\substack{\| \\ \underline{\{\text{hol. line bundles}\}}}}{\underline{H^1(X, \mathcal{O}^*)}} \quad (\{g_{\alpha\beta} : U_\alpha \cap U_\beta \neq \emptyset\})$$

$$(f) \longmapsto \underset{\substack{\uparrow \\ \text{principal divisors}}}{\{f = 0\}} \longmapsto \text{trivial line bundle.}$$

$D = \sum_i a_i D_i$ is effective if $a_i \geq 0$.

$$D \cap U_\alpha = \{\underset{\substack{\uparrow \\ \text{holomorphic}}}{f_\alpha} = 0\} \rightsquigarrow \text{transition fction } g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$$

global sections of $\overset{\substack{L \\ \|}}{[D]} \Leftrightarrow \underline{\{(h_\alpha) : h_\alpha = g_{\alpha\beta} h_\beta\}}$.

$$\rightsquigarrow \underline{f_\alpha = g_{\alpha\beta} f_\beta}$$

$\Rightarrow$ global section $\underset{\substack{\| \\ \{f_\alpha \in \mathcal{O}(U_\alpha)\}}}{\underline{s_D \in H^0(X, L)}}$

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

$$H^1(X,\mathcal{O}) \to \underbrace{H^1(X,\mathcal{O}^*)}_{\substack{\| \\ \{\text{hol. line bdles}\} \\ \cup}} \xrightarrow{c_1} H^2(X,\mathbb{Z})$$

$$\downarrow$$

$$D = \sum_i a_i D_i \rightsquigarrow [D] \qquad H^2(X,\mathbb{R})$$

$$\Downarrow$$

cycle class $\eta_D \in H^2(X,\mathbb{R}) \cong H_{2n-2}(X,\mathbb{R})$

$$\eta_D = c_1([D]) \in \underline{H^2(X,\mathbb{R})}$$

holomorphic line bundle. Hermitian metric h on L.

$$R(h) = \frac{\sqrt{-1}}{2\pi}\bar\partial\partial \log|s|_h^2$$ where s is a local nowhere-vanishing holomorphic section

Chern curv. form.

$$= \frac{\sqrt{-1}}{2\pi}\bar\partial\partial \log|s'|_h^2 + \underbrace{\sqrt{-1}\,\bar\partial\partial \log|f|^2}_{=0} \qquad s' = f\cdot s$$

$R(h)$ is a closed $(1,1)$-form. $[R(h)]_{DR} = \eta_D$.

Show $\boxed{[R(h)]_{DR} \overset{\frac{\sqrt{-1}}{2\pi}\bar\partial\partial\log|s|_h^2}{=} c_1(L) \in H^2(X,\mathbb{R})}$.

$c_1(L) =$ Image of $[L] = [D]$ under the $H^1(X,\mathcal{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z})$.

$s_\alpha = g_{\alpha\beta}\cdot s_\beta$

$|s_\alpha|_h^2 = |g_{\alpha\beta}|^2\cdot|s_\beta|_h^2$

$\log|s_\alpha|_h^2 = \log|g_{\alpha\beta}|^2 + \log|s_\beta|_h^2$

$[L] \ni \{(g_{\alpha\beta})\}$

$\frac{\sqrt{-1}}{2\pi}\bar\partial\partial\log|s|_h^2$

$\delta(\log g_{\alpha\beta})$

$=$

$\mathbb{R}(U_{\alpha\beta\gamma}) \supset \mathbb{Z}(U_{\alpha\beta\gamma}) \ni \sigma_{\alpha\beta\gamma} = \frac{1}{2\pi i}\left(\log g_{\alpha\beta} - \log g_{\alpha\gamma} + \log g_{\beta\gamma}\right)$

$= \log\left(\underline{g_{\alpha\beta}\, g_{\alpha\gamma}^{-1}\, g_{\beta\gamma}}\right)$

$= 1$

$$\boxed{H^2_{DR}(X,\mathbb{C}) \cong H^2(X,\mathbb{C})}$$

<u>Ex</u>: $\mathbb{P}^n = \{[z_0 : z_1 : \cdots : z_n] : z \neq 0\}$

$L = \mathcal{O}(-1) = \{([z], v) : v \in \mathbb{C}\cdot z\}$

$U_0 = \{z_0 \neq 0\}$, $s_0(z) = (1, z) \in \mathbb{C}\cdot(1,z)$.

$U_0 = \{z_1 = \frac{z_1}{z_0}, \ldots, z_n = \frac{z_n}{z_0}\}$, $(1,z) = [1,z]$

$U_1 = \{z_1 \neq 0\} = \{u_1 = \frac{z_0}{z_1}, u_2 = \frac{z_2}{z_1}, \ldots, u_n = \frac{z_n}{z_1}\}$

$$s_0(z) = (1, z) = (1, z_1, z_2, \cdots, z_n)$$

$$= \left(1, \frac{z_1}{z_0}, \frac{z_2}{z_0}, \cdots, \frac{z_n}{z_0}\right)$$

$$= \left(1, \frac{1}{u_1}, \frac{u_2}{u_1}, \cdots, \frac{u_n}{u_1}\right)$$

$$= \frac{1}{u_1}\underline{(u_1, 1, u_2, \cdots, u_n)} = \frac{1}{u_1} \cdot s_1$$

$$\Rightarrow \operatorname{div}(s_0)\big|_{U_1} = -(u_1 = 0) = -H\big|_{U_1} \quad H = (z_0 = 0)$$

$$\Rightarrow \operatorname{div}(s_0) = -H \Rightarrow L = \mathcal{O}(-1) = [-H].$$

Fact: meromorphic section s of holomorphic line bundle L. Then $[\underset{\substack{\| \\ \text{zero}(s) - \text{pole}(s)}}{\operatorname{div}(s)}] \cong L$.

$$s \longleftrightarrow f_\alpha = g_{\alpha\beta} \cdot f_\beta \Rightarrow g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$$

$\operatorname{div}(s) \cap U_\alpha = (f_\alpha)$. $\qquad [\operatorname{div}(s)] = L$.

$D \longrightarrow [D]$ + meromorphic section

$D \geq 0 \longrightarrow [D]$ + holomorphic section.

- Canonical line bundle of $\mathbb{P}^n$.

$$s = \underbrace{dz_1 \wedge \cdots \wedge dz_n}$$

$$z_1 = \frac{Z_1}{Z_0} = u_1^{-1}$$

$$z_2 = \frac{Z_2}{Z_0} = \frac{u_2}{u_1}$$

$$\parallel$$

$$-u_1^{-2} du_1 \wedge u_1^{-1} du_2 \wedge \cdots \wedge u_1^{-1} du_n$$

$$\parallel$$

$$-u_1^{-(n+1)} du_1 \wedge du_2 \wedge \cdots \wedge du_n$$

$$\Rightarrow \quad (\mathrm{div}(s))|_{U_1} = -(n+1)\cdot(u_1 = 0)|_{U_1}$$

$$\Rightarrow \quad \underline{[\mathrm{div}(S)] = [-(n+1)\cdot H]}$$

$$\parallel \qquad\qquad \parallel$$

$$\Lambda^n T^* \mathbb{P}^n = K_{\mathbb{P}^n}$$

$-K_{\mathbb{P}^n} = [(n+1) H]$ anticanonical line bundle.

- $Y \subset X$ sm. complex hypersurface

$Y \cap U_\alpha = \underline{(f_\alpha = 0)}$. $[Y] = \left\{ \frac{f_\alpha}{f_\beta} = g_{\alpha\beta} \right\}$

$\underline{df_\alpha \in N_Y^* |_{(Y \cap U_\alpha)}}$, $df_\alpha | TY \equiv 0$ $(f_\alpha \equiv 0 \text{ on } Y)$

$$df_\alpha|_Y = d\left(\frac{f_\alpha}{f_\beta} f_\beta\right) = \underbrace{\frac{f_\alpha}{f_\beta}}_{g_{\alpha\beta}} df_\beta|_Y + d\left(\frac{f_\alpha}{f_\beta}\right) \cdot \overset{=0}{f_\beta|_Y}$$

$(g_{\alpha\beta}^{-1})$

$$(df_\alpha)|_Y = g_{\alpha\beta} \cdot (df_\beta)|_Y \Rightarrow N_Y^* = [Y]^{-1} = [-Y]$$

$$s_\alpha \in H^0(U_\alpha, L) \longrightarrow g_{\alpha\beta} = \frac{s_\beta}{s_\alpha} \qquad \begin{matrix} g_{\alpha\beta} f_\beta \\ \| \end{matrix}$$

$$s \in H^0(X, L) \qquad s = f_\alpha \cdot s_\alpha = f_\beta \cdot s_\beta \Rightarrow f_\alpha = \frac{s_\beta}{s_\alpha} f_\beta$$

$$N_Y^* = [-Y]|_Y \Leftrightarrow N_Y = [Y]|_Y$$

$$0 \to TY \to TX|_X \to N_Y \to 0$$

$$0 \leftarrow T^*Y \leftarrow T^*X|_X \leftarrow N_Y^* \leftarrow 0$$

$$\Lambda^n T^*X|_X \cong \Lambda^n T^*Y \otimes N_Y^*$$

$$K_X|_X = K_Y \otimes N_Y^* = K_Y \otimes [-Y].$$

$$\Leftrightarrow \quad K_Y = K_X|_X \otimes [Y].$$

Ex: $Y \subset \mathbb{P}^n$ hypersurface degree d

$$\eta_Y = d \cdot \eta_H$$

$$\boxed{[Y] \cong [d \cdot H]}$$

$$\begin{matrix} h^{0,1} & & & & h^{0,2} \\ \| & & & & \| \\ H^1(\mathbb{P}^n, \mathcal{O}) \to & H^1(\mathbb{P}^n, \mathcal{O}^*) & \xrightarrow{c_1} & H^2(\mathbb{P}^n, \mathbb{Z}) \to & H^2(\mathbb{P}^n, \mathcal{O}) \\ \| & & & & \| \\ 0 & & c_1(Y) = [\eta_Y] & & 0 \end{matrix}$$

$$\{Y\} \in H_{2n-2}(\mathbb{P}^n) \cong H^2(\mathbb{P}^n)$$

$$\begin{array}{ccc} \{Y\} & & \\ \| & \in & \\ \{d\cdot H\} & & \end{array}$$

$$K_Y = K_{\mathbb{P}^n}|_Y \otimes [Y]$$

$$= [-(n+1)H] \otimes [dH] = [-(n+1-d)H]$$

$$-K_Y = [\underline{(n+1-d)} H] \quad \begin{cases} 1 \le d \le n & \text{Fano hyp.} \\ d = n+1 & \text{Calabi-Yau} \\ d > n+1 & \text{general type} \end{cases}$$

$n = 2.$

$$\begin{cases} 1 \le d \le 2 & \text{rational curve} \\ d = 3 & \text{sm. elliptic curve} \\ d > 3 & \text{sm. genus} \ge 1. \end{cases}$$